Nonlinear Anisotropic Diffusion Equations

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Outline

1. Linear and nonlinear parabolic equations
2. Anisotropic diffusion equations
3. Self-similar solutions and variables
4. Existence and uniqueness of self-similar fundamental solutions
5. Anisotropic p-Laplacian
6. Tools and Proofs
7. Asymptotic behaviour and some properties of the supports

collaboration with Filomena Feo, Napoli, and Bruno Volzone, Milano (2021 to 2025), plus ongoing work with Marcos Llorca, Madrid

The heat equations

In the modeling of diffusion processes with PDEs the Heat Equation is the Mother Equation

$$\boxed{u_t = \Delta u} \tag{1}$$

• Next to it, there are the two typical parabolic families of related linear PDEs

$$\boxed{u_t = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + cu + f}$$

$$\boxed{u_t = \sum_{ij} \partial_i(a_{ij}\partial_j u) + \sum_i \partial_i(b_i u) + cu + f}$$

where (a_{ij}) is a positive definite matrix, it possibly varies with space and time. They are a powerful tool in PDES and advanced mathematics.

The coefficients can account for all kinds of effects, in particular inhomogeneities or anisotropies.

► The HE and the linear Parabolic Equation Models have produced a huge number of concepts, techniques and connections for pure and applied science, like the Gaussian function, separation of variables, spectral decomposition, Dirichlet forms, Maximum Principles, functional inequalities, ...

Nonlinear equations

- Let us take a step forward and expand the family of diffusive models to include nonlinearities. Nonlinearities are basic in science, since the time of Newton and before. But developing a relevant nonlinear PDE theory is a difficult step not taken until the (last part of the) 20th century.
- Indeed, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very non-linear. A quite general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)}$$

  with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin, Ladyzhenskaya et al., after the seminal work of DeGiorgi and Nash,...)
- In this generality the mathematical theory is too rich to admit a simple description. Reference works. Books by Ladyzhenskaya-Solonnikov-Uraltseva, Friedman, Smoller,... But they are only basic reference. Deep progress has relied on studying specific relevant examples, like Obstacle Problem, Stefan Problem and others we see next.

Porous Medium and p-Laplacian

- In the family of the Heat Equation there are two nice nonlinear diffusion equations that are tightly connected in the mathematical analysis. They are the Porous Medium Equation, the simplest really nonlinear heat equation of this type,

$$u_t = \Delta(u^m), \quad m \neq 1,$$

and the p-Laplacian Equation which in its evolution form reads

$$u_t = \Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad 1 < p \neq 2,$$

with its more famous elliptic counterpart

$$\Delta_p(u) + f = 0.$$

Note the usual notation $\Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, widely accepted in the community of PDEs and Calc of Variations.

- They have become two of the nonlinear heat equations more studied, their properties are in some sense similar but in other senses they are different.

Porous Medium and p-Laplacian

- The study of those two equations has shown that they generate a flow in time starting from data in the typical space $u_0 \in L^1(\mathbb{R}^n)$. The flow has the form of a nonlinear semigroup in $X = L^1(\mathbb{R}^n)$, $u(t) \in C([0,\infty) : X)$. The p-Laplacian accepts $X = L^p(\mathbb{R}^n)$ for every $p \geq 1$.

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  ► If $m > 1$ for the PME or $p > 2$ for the PLE we have finite propagation of the support and free boundaries. Free boundaries was the topic where Maestro Caffarelli contributed wonderful results to Mathematics. The equations are said to be in the Slow Propagation regime.

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  ► On the contrary if $m < 1$ in the PME or $p < 2$ in the PLE we have the Fast Diffusion range that has extra strong infinite propagation of positivity and a chaotic richness of mathematical analysis.
- The analysis and the applicable results have not much in common with the Heat Equation, i.e., the case $m = 1$ (or $p = 2$ resp.).

Asymptotics. Related Equations

- I was interested in the long time evolution of these nonlinear diffusion equations, mainly in free space, following the ideas of Zeldovich, Barenblatt, and Kamin; Caffarelli ; Brezis, Benilan and Crandall (and many others).

G. Barenblatt, Prikl. Mat. Meh. 1952 (early work on PME and PLE)

JLV, The Porous Medium Equation 2007 (Oxford UP), Smoothing ... 2006 (OUP, Book on FDE)

JLV, Lecture Notes in Maths 2018 (Survey on local and nonlocal equations, Summer Course in Cetraro, Italy, 2016)

Related types of equations

- Nonlinear diffusion models on manifolds. Specially hyperbolic space. Collaboration with Grillo, Bonforte, and Muratori. This work started in the 2000s still continues
- Nonlinear diffusion models with nonlocal operators. In particular PME and PLE models with fractional Laplacians. Main reference is Caffarelli
- Works with nonlinear diffusion and reaction, drift or interaction effects.
- Works in nonhomogeneous media with singular weights.

Anisotropic diffusion equations

In this talk we will study another issue: anisotropic nonlinear evolution equations. The are several classes of anisotripis that have been considered. We chose as main example the following variation of the PME:

$$\text{(APME)} \qquad u_t = \sum_{i=1}^{N} (u^{m_i})_{x_i x_i} \quad \text{in} \quad Q := \mathbb{R}^N \times (0, +\infty)$$

with $u \geq 0$, $N \geq 2$ and $m_i > 0$ for $i = 1, ..., N$. Used in applications of the PME or FDE as model of motion of water or gases in some anisotropic media.

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• For $m_i = m > 0$ for all i we get the isotropic case

$$u_t = \Delta u^m$$

Written in divergence form: $u_t = \mathrm{div}(d(u)\nabla u)$. The diffusion coefficient is $d(u) = mu^{m-1}$. We find two different nonlinear types of behaviour:

- if $u \to 0$ then $d(u) \to 0$ when $m > 1$ (slow diffusion)
- if $u \to 0$ then $d(u) \to \infty$ when $0 < m < 1$ (fast diffusion)

$m > 1$ Porous media equation (PME),

$0 < m < 1$ Fast diffusion equation (FDE)

Moreover, for $m = 1$ we get the linear heat equation

Anisotropic diffusion equations: our assumptions

(APME) $$u_t = \sum_{i=1}^{N} (u^{m_i})_{x_i x_i} \quad \text{in} \quad Q := \mathbb{R}^N \times (0, +\infty),$$

with $N \geq 2$, $u(x,0) \geq 0$ $(\Rightarrow u \geq 0)$.

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with $N \geq 2, \quad u(x,0) \geq 0 \quad (\Rightarrow u \geq 0)$.

▶ When we consider fast diffusion exponents $0 < m_i < 1$ we require an joint anisotropic lower bound of the form $\sum_{i=1}^{N} m_i > N - 2$. We can also write it as

$$\text{(H2)} \qquad \overline{m} > m_c \quad \left(m_c := \frac{N-2}{N} \text{ for } N > 2, m_c = 0 \text{ for } N = 2\right),$$

Thus, the usual lower condition on m in the isotropic fast diffusion deals now only with the arithmetic mean of the exponents:

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We will see why this happens.

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- Control of large anisotropies is also needed in anisotropic theories

(H3) $$m_i \leq \frac{2}{N} + \overline{m} \qquad \forall i$$

Anisotropy limitation

People studying anisotropy in the Calculus of Variations are concerned about the amount of separation among the anisotropic exponents as a strong difficulty in the theory (see reference below). In that sense we have

$$(H1) + (H2) \Rightarrow (H3)$$

So there is no extra problem in fast diffusion. Of course, no (H3) appears in the isotropic case.

► We have to take (H3) into account here only for slow diffusions $m_i > 1$. There is a real problem there.

Remarks: For $N = 2$ and $m_1 < m_2$, (H3) means $m_2 < (m_1 + m_2)/2 + 1$, $m_2 < m_1 + 2$.

In all dimensions (H3) is a main control on the strength of the anisotropy.

Baroni-Colombo-Mingione 2018 (Variational functionals with double phase)

Main objective of the study

- Identify the ranges of anisotropic exponents where there are self-similar solutions of the fundamental type (source type solutions, Barenblatt solutions), and prove that they explain the asymptotic behaviour of a large class of solutions.

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- Fundamental solutions is the term we use for acceptable solutions that have a Dirac delta as initial trace (i.e, $U(x,t)$ tends to $M\,\delta(x)$ in the sense of measure)
- Fundamental solutions are relevant because they are the asymptotic attractors of all nonnegative solutions of finite mass for a relevant number of Cauchy problems involving PDE's of diffusion type: in particular, for the Heat Equation, the Porous Media Equation (PME) and the Fast Diffusion Equation (FDE) (but this one only in the good range, $m > m_c$).
- These solutions often have a self-similar form. This idea goes back in nonlinear diffusion to the Moscow school, with Zeldovich and Barenblatt, 1950-52.

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- These solutions often have a self-similar form. This idea goes back in nonlinear diffusion to the Moscow school, with Zeldovich and Barenblatt, 1950-52.
- We want to achieve similar results for anisotropic Diffusion (fast, slow or mixed) if the anisotropic exponents allow it. The limitations come from algebra.

Anisotropic diffusion regimes and assumptions

• Fast diffusion regime: (H1F) $m_i < 1 \quad \forall i$. We must require

(H2) $$\overline{m} > m_c \quad (m_c := 1 - \frac{2}{N} \text{ for } N > 2, \quad m_c = 0 \text{ for } N = 2),$$

F. Feo, J. L. Vázquez, B. Volzone - Anisotropic Fast Diffusion Equations - Nonlinear Analysis, 2023 (selfsimilar solutions and asymptotic stabilization)

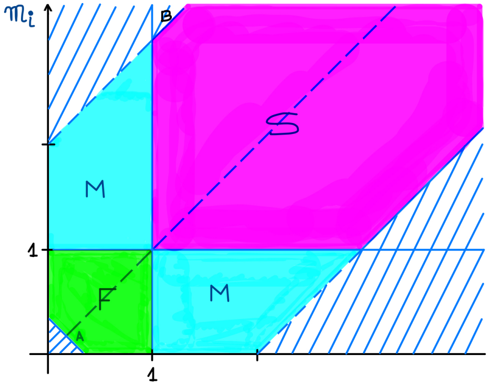
• Slow diffusion regime: (H1S) $m_i > 1 \quad \forall i$. We now require

(H3) $$m_i \le \frac{2}{N} + \overline{m} \qquad \forall i$$

F. Feo, J. L. V., B. Volzone. Asymptotic behaviour of solutions and free boundaries of the anisotropic slow diffusion equation. Submitted. Uploaded to arXiv:2412.12295 [math.AP]

• Mixed regime (H1M): Some exponents are fast, some are slow: $m_i < 1, m_j > 1$. We must require (H2) and (H3). Work is still in progress. We need a stricter (H2') that applies only to the fast exponents

Anisotropic Regions



Anisotropic Regions (only two variables m_i plotted)

Red region: Slow - Green region: Fast - Blue region: Mixed.

Line A: Condition (H2), Lines B: Condition (H3),

Some other references for anisotropic slow and fast diffusion equations

- B. H. Song, 2001 (Anisotropic Diffusions equations: existence and uniqueness)
- B. H. Song, H.Y. Jian, 2005-2006 (Existence and Fundamental Solutions)
- E. Henriques, 2011 (Any locally bounded nonnegative weak solution is locally continuous in Q_T in the two regimes)
- S. Ciani, S. Mosconi, V. Vespri, 2023(Parabolic Harnack estimates for anisotropic slow diffusion)
- Vázquez 2024 The very singular solution for the Anisotropic Fast Diffusion Equation, Nonlinear Anal. 245 (2024),
- Ciani-Henriques, 2025 (Harnack-type estimates in the fast diffution regime and extinction in finite time $\overline{m} < m_c$)

Self-similar solutions I

We are looking for solutions to equation (APME) of the following form

(SSS) $$U(x,t) = t^{-\alpha} F(t^{-\alpha\sigma_1} x_1, .., t^{-\alpha\sigma_N} x_N),$$

with similarity exponents $\alpha > 0, \sigma_1, .., \sigma_N \geq 0$ to be chosen below. It is not clear that this anisotropic form allows for some solution.

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• Writing $y = (y_1, ..., y_N)$ and $y_i = x_i \, t^{-\alpha\sigma_i}$ equation (APME) becomes

$$-t^{-\alpha-1}\left[\alpha F(y) + \sum_{i=1}^{N} \alpha\sigma_i y_i \, F_{y_i}\right] = \sum_{i=1}^{N} t^{-(\alpha m_i + 2\alpha\sigma_i)} (F^{m_i})_{y_i y_i}.$$

We see that time is eliminated as a factor if the algebraic conditions are imposed:

(AC-i) $$\alpha(m_i - 1) + 2\alpha\sigma_i = 1 \quad \text{for all } i = 1, 2, \ldots, N.$$

Note: This works for all types: slow, fast or mixed.

Self-similar solutions II

• We also look for integrable solutions that will enjoy the mass conservation property $\int U(x,t)\,dx =$ constant in time, and this implies that

$$\text{(AC-s)} \qquad \sum_{i=1}^{N} \sigma_i = 1.$$

• Imposing the whole set of algebraic conditions we get a unique choice for the anisotropic exponents

$$\alpha = \frac{N}{N(\overline{m}-1)+2}$$

if $\overline{m} \neq m_c$, and then

$$\sigma_i = \frac{1}{N} + \frac{\overline{m}-m_i}{2} \quad i = 1, \cdots, N.$$

Note that under isotropy $\sigma_i = \frac{1}{N}$.

Self-similar solutions III. The exponent conditions

• Exponent α: the self-similar solution decays in time in maximum value like a power of time $t^{-\alpha}$. Standard parabolic theory says that the sup norm of the self-similar solution cannot increase in time, hence $\alpha > 0$, hence (H2).

• Exponents σ_i: They control the rate of spatial expansion in each coordinate direction. The self-similar solution does not contract as time passes along any of the coordinate directions if $\sigma_i > 0$.

We easily check that

$$\text{(H2): } (\sum m_i > N-2),\ (\overline{m} > m_c) \iff \alpha > 0$$

$$\text{(H3)} \iff \sigma_i > 0.$$

We call the corresponding set of exponents $\{m_i\}$ with (H2), (H3) the acceptable range of exponents.

Self-similar solutions IV. Profile equation

We are looking for solutions to equation (APME) of the following form

$$U(x,t) = t^{-\alpha} F(t^{-\alpha\sigma_1} x_1, .., t^{-\alpha\sigma_N} x_N),$$

with the above conditions on α and σ_i. We ask that $U \geq 0$.

• It remains to examine the profile function $F(y) = F(y_1, \cdots, y_N)$.
It must satisfy the following nonlinear anisotropic stationary equation in $\mathbb{R}^N$:

(SE) $$\sum_{i=1}^{N} \left[(F^{m_i})_{y_i y_i} + \alpha\sigma_i \, (y_i F)_{y_i} \right] = 0.$$

It is a kind of Anisotropic Fokker-Planck Equation. Conservation of mass must also hold : $\int U(x,t)\, dx = \int F(y)\, dy = M < \infty \ \ \forall t > 0$.

This is a priori a hard equation to solve.

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• We have proved that there exists a suitable solution of this elliptic equation, which is the anisotropic version of the equation of the Barenblatt profiles in the anisotropic equation, we prove it both in fast and slow case.

Barenblatt profile in the isotropic case ($m_i = m$)

• For $m_c < m_i = m < 1$

$$F(y) = \left(C + \frac{\alpha(1-m)}{2mN} |y|^2 \right)^{-1/(1-m)},$$

• for $m > 1$

$$F(y) = \left(C - \frac{\alpha(m-1)}{2mN} |y|^2 \right)_+^{1/(1-m)},$$

$C > 0$ is an arbitrary constant such that can be determined in terms of the initial mass M. Condition $m > m_c$ guaranties that $F \in L^1(\mathbb{R}^N)$ when $m < 1$. In the anisotropic case, we will not get any explicit formula for F.

- The proof is long and complicated, based in many tricks of nonlinear analysis
- we will have existence and uniqueness of self-similar fundamental solutions and
- suitable estimates and in particular decay for fast diffusion regime (all $m_i < 1$)
- properties of the supports for slow diffusion regime (all $m_i > 1$)

Equivalence. Self-similarity and fundamental solution

Equivalence: If there exists a profile F_M of mass M, *i.e.* there exists a solution with mass M to the following equation

$$\text{(SE)} \qquad \sum_{i=1}^{N}\left[(F^{m_i})_{y_i y_i} + \alpha\sigma_i\,(y_i F)_{y_i}\right] = 0.$$

then

$$U_M(x,t) = t^{-\alpha}F_M(t^{-\alpha\sigma_1}x_1,\cdots,t^{-\alpha\sigma_N}x_N)$$

with

$$\alpha = \frac{N}{N(\overline{m}-1)+2} \qquad \sigma_i = \frac{1}{N} + \frac{\overline{m}-m_i}{2}\ \forall i$$

is a fundamental solution of mass M to our equation (*i.e.* the initial datum is a Dirac delta, $M\delta$).

Slow Diffusion regime / Fast diffusion regime

• For lack of time we will focus on the red region

(H1S) $m_i > 1 \quad \forall i$ (Slow Diffusion regime),

Then we assume (H1S) and

(H3) $$m_i \le \frac{2}{N} + \overline{m} \qquad \forall i$$

• Our arguments are adaptable to the fast diffusion case (that was published in 2023) replacing (H3) by (H2). Techniques differ at important calculations, not many. The results are qualitative different.

Key tool. Self-similar variables. Renormalization

- Let us introduce some technical tools. It will be very useful to pass to self-similar parabolic variables, by zooming the original solution according to the self-similar exponents α and σ_i:

$$v(y,\tau)=(t+t_0)^{\alpha}u(x,t), \quad \tau=\log(t+t_0), \quad y_i=x_i(t+t_0)^{-\sigma_i\alpha}$$

with $i=1,..,N$, and α and σ_i as before.

Remark. This change of variables preserves the L^1 norm.

Proposition 1

If $u(x,t)$ is a solution to (APME), then $v(y,\tau)$ is a solution to

$$\text{(ADss)} \qquad v_\tau=\sum_{i=1}^{N}\left[(v^{m_i})_{y_iy_i}+\alpha\sigma_i\,(\,y_i\,v)_{y_i}\right] \quad \mathbb{R}^N\times(\log t_0,+\infty).$$

- This equation is the evolution version of the stationary equation (SE). It does not change with the time-shift t_0. We will prove that v converges to a unique fixed point.

Existence and uniqueness of self-similar fundamental solutions

We assume the Slow Diffusion regime (H1S) and also (H3)

$$m_i \le \frac{2}{N} + \overline{m} \qquad \forall i$$

Theorem 1 (Slow case)

(1) For any mass $M > 0$ there is a unique self-similar fundamental solution $U_M(x,t) \ge 0$ of equation (APME) *with mass M.*

(2) The profile F_M of such a solution is an SSNI (separately symmetric and non-increasing) bounded function.
(3) $F_M(y)$ has compact support in all directions and the positivity set $\Omega(F_M) = \{y : F_M(y) > 0\}$ is open, bounded and star-shaped around the origin.

(4) F_M is C^∞ smooth inside $\Omega(F_M)$ and continuous in $\mathbb{R}^N$.

The asymptotic behaviour of finite mass solutions under assumptions $(H1)$ and $(H3)$

We state the second big theorem.

Theorem 2

Let $u(x,t)$ be the unique solution of the Cauchy problem for equation (APME) *with nonnegative initial datum $u_0 \in L^1(\mathbb{R}^N)$. Let U_M be the unique self-similar fundamental solution with the same mass as u_0. Then,*

$$\lim_{t\to\infty} \|u(\cdot,t) - U_M(\cdot,t)\|_1 = \lim_{\tau\to\infty} \|v(\cdot,\tau) - F_M\|_1 = 0.$$

The convergence also holds in all the L^p norms, $1 < p < \infty$, in the proper scale:

$$\lim_{t\to\infty} t^{\frac{(p-1)\alpha}{p}} \|u(\cdot,t) - U_M(\cdot,t)\|_p = \lim_{\tau\to\infty} \|v(\cdot,\tau) - F_M\|_p = 0,$$

where $\alpha = \frac{N}{N(\overline{m}-1)+2}$.

Support of the solutions. Hausdorff set distance

Our aim: to show how the spatial support of a nontrivial solution $u \ge 0$ with compactly supported and bounded initial data evolves in time under assumptions $(H1S)$ and $(H3)$. A sharp approximation result needs the Hausdorff set distance.

Definition. Let $A, B \subset \mathbb{R}^N$. For any $a \in A$, $b \in B$ we define the distance from a point to a set

$$d(a, B) = \min\{d(a, b) : b \in B\}, \quad d(b, A) = \min\{d(a, b) : a \in A\},$$

and then we define the two directional distances

$$d_1(A, B) = \sup\{d(a, B) : a \in A\},$$

and

$$d_2(A, B) = \sup\{d(b, A) : b \in B\} = d_1(B, A) \quad \text{(we interchange } A \text{ and } B).$$

Then the symmetric distance is defined as

$$d_H(A, B) = \max\{d_1(A, B), d_2(A, B)\}.$$

We will work with bounded sets.

Properties of support of solutions with compactly supported and bounded initial data

This is the third main result,
Let $u(x,t)$ be the unique solution of the Cauchy problem for equation (APME) with nonnegative bounded and compactly supported initial datum u_0 of mass M, when (H1) and (H3) are in force. We examine the long time behaviour. Let $v(y,\tau)$ be the renormalized solution..

Theorem 3

Let F_M be the profile of unique self-similar fundamental solution with the same mass M as u_0. Then we have the set convergence

$$\lim_{\tau\to\infty} d_H(\Omega(v,\tau),\Omega(F_M)) = 0,$$

where v is the renormalized solution. The same formula works for the supports, $d_H(S(v,\tau),S(F_M)) \to 0$.

We formulate the previous asymptotic result in terms of the free boundaries. Let $u(x,t)$ be the unique solution of the Cauchy problem for equation (**??**) with nonnegative bounded and compactly supported initial datum u_0 when (H1) and (H3) are in force. Let $v(y,\tau)$ be the renormalized solution.

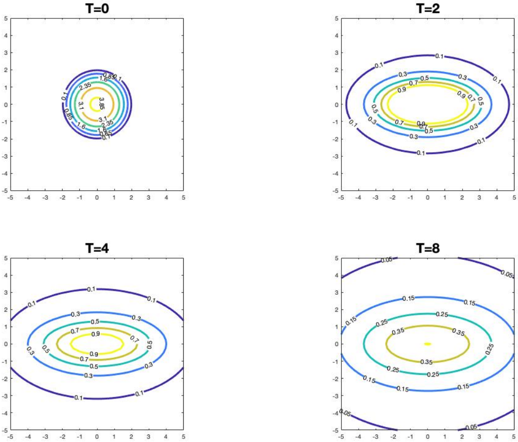
Theorem 4

$$\lim_{\tau\to\infty} d_H(\Gamma(v,\tau),\Gamma(F_M)) = 0,$$

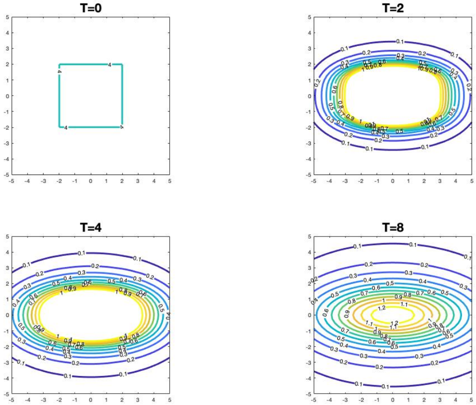
where $\Gamma(v,\tau)$ *is the boundary of the set of positivity of* F_M*,* $\Omega(F_M)$*.*

Some simulations: $N = 2$ and radial data

We have numerical computations with the evolution process that show the appearance of an elongated profile.



Some simulations: $N = 2$ a square initial configuration



Computations by F. del Teso, UAM

p-Laplacian

Idea of the Anisotropic p-Laplacian

Anisotropic Fast p-Laplacian. Our assumptions

Let us turn to the anisotropic p-Laplacian case and see what can be done. We consider the APL equation in any dimension $N \geq 2$:

$$\text{(APL)} \qquad u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} \quad \text{in} \quad Q := \mathbb{R}^N \times (0, +\infty),$$

with the assumptions

$$\text{(H1)} \qquad 1 < p_i < 2 \qquad \text{(Fast Diffusion regime)},$$

and also

$$\text{(H2)} \qquad \bar{p} > p_c := \frac{2N}{N+1}.$$

Here $\overline{p}$ is the harmonic mean:

$$\frac{1}{\overline{p}} := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

This is shown to be good anisotropic fast diffusion range in the sense coined for the isotropic case.

Anisotropic p-Laplacian: other assumptions

People studying anisotropy in the Calculus of Variations are concerned about the amount of separation among the anisotropic exponents as a strong difficulty in the theory (see reference). In that sense we have

$$(H1) + (H2)$$

$$\Downarrow$$

(H3)
$$p_i \le \frac{N+1}{N}\bar{p} \qquad \forall i$$

Remarks: For $N = 2$ we have $p_1/2 < p_2 < 2p_1$. In all dimensions (H3) is a control on the strength of the anisotropy.

(H3) does not appear in the isotropic case and we would have to take it into account here only when $p_i > 2$!

Baroni-Colombo-Mingione 2018 (Variational functionals with double phase)

Pairs p_1, p_2 satisfying conditions (H2)-(H3)

We show a general diagram when $p_1, p_2 \le 2$ or $p_1, p_2 \ge 2$.

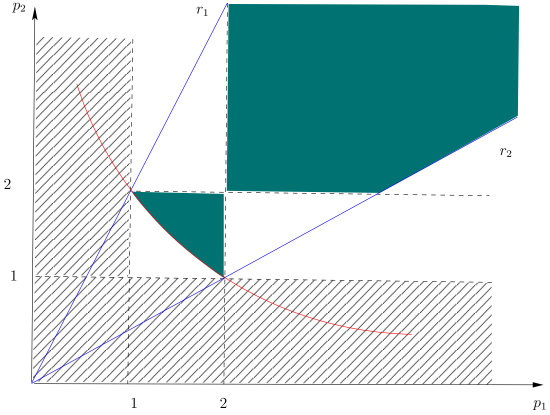


Figure: Red curve: $(p_1 - 2/3)(p_2 - 2/3) > 4/3$, region between blue lines: $\frac{1}{2} < \frac{p_1}{p_2} < 2$

Tools and proofs

Tools: the scaling transformations

- The equation is invariant under the following rescaling

$$\widehat{u}(x,t) = \mathcal{T}_k(u) = k^{\alpha} u(k^{\alpha\sigma_1} x_1, \cdots, k^{\alpha\sigma_N} x_N)$$

with scaling parameter $k > 0$. It preserves the mass

- There are other rescalings which change the mass. Thus, we obtain the profile F_M of mass M starting from the profile F_1 of mass 1 by the rescaling

$$\mathcal{T}_\kappa[F(y)] = \kappa\, F(\kappa^{\frac{1-m_1}{2}} y_1, \cdots, \kappa^{\frac{1-m_N}{2}} y_N)$$

with scaling parameter $\kappa > 0$. It changes the mass

$$\int_{\mathbb{R}^N} \mathcal{T}_\kappa[F(y)]\, dy = \kappa^{\beta} \int_{\mathbb{R}^N} F(y)\, dy$$

with $\beta = \frac{N}{2}(\overline{m} - m_c)$ ($\beta > 0$ since $\overline{m} > m_c$).

New tool: L^1-L^∞ smoothing effect

This works under under assumptions $(H2)$ and $(H3)$

Theorem 5

For any nonnegative $u_0 \in L^1(\mathbb{R}^N)$, the solution $u(x,t)$ is a uniformly bounded function for each $t > 0$:

$$\|u(\cdot,t)\|_\infty \leq C\, t^{-\alpha} \|u_0\|_1^{2\alpha/N}, \quad C = C(m_1, \cdots, m_N, N)$$

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- classical parabolic Moser iteration
- starting point

Proposition 2 (Troisi; di Blasio, F., Zecca '24)

Let $\lambda_i > 0$ and $1 \leq \widetilde{r} < N$. Then for every nonnegative function $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$\left\| u^{\overline{\lambda}} \right\|_{L^{\widetilde{r}^*}} \leq C_S \left\| \left(\prod_{i=1}^N \left| \partial_{x_i} u^{\lambda_i} \right| \right)^{1/N} \right\|_{L^{\widetilde{r}}} \leq C_S \prod_{i=1}^N \left\| \partial_{x_i} u^{\lambda_i} \right\|_{L^{r_i}}^{1/N}, \tag{2}$$

where $\frac{1}{\widetilde{r}} = \sum_{i=1}^N \frac{1}{r_i}$, $\widetilde{r}^* = \frac{N\widetilde{r}}{N-\widetilde{r}}$, $\overline{\lambda} = \frac{1}{N}\sum_{i=1}^N \lambda_i$ and C_S is a positive constant depending on N and $\widetilde{r}$.

New tool: Qualitative properties of the solutions

Let u be a nonnegative solution of the Cauchy problem for (APME) with nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$. As a consequence of the Aleksandrov's reflection principle we have:

Theorem 6

If u_0 is a symmetric function in each variable x_i, and also a nonincreasing function in $|x_i|$ for all i a.e., then $u(x,t)$ is also symmetric and a nonincreasing function in $|x_i|$ for all i for all $t > 0$.

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For short we call this property: SSNI (separately symmetric and nonincreasing)

Proposition 3

Any non-negative self-similar fundamental solution with finite mass is SSNI

Idea of the proof:

- SSNI is an asymptotic property
- self-similar solutions necessary verify asymptotic properties for all time

The constructed solution with L^1 data

For any nonnegative $u_0 \in L^1(\mathbb{R}^N)$ there is a unique function $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ such that $u, u^{m_i} \in L^1_{loc}(Q)$ for all $i = 1, ..., N$, and equation (APME) holds in the distributional sense in $Q = \mathbb{R}^N \times (0, +\infty)$. Moreover

1) $u(x,t)$ is a uniformly bounded function for each $t > 0$ and $L^1 - L^\infty$ estiamte holds.
2) Let $Q_\tau = \mathbb{R}^n \times (\tau, \infty)$. We have $\partial_i u^{m_i} \in L^2(Q_\tau)$ for every i and suitable energy estimates are satisfied. Equation (APME) holds in the weak sense in Q_τ for every $\tau > 0$.
3) The maps $S_t : u_0 \mapsto u(\cdot, t)$ generate a semigroup of L^1 ordered contractions in $L^1_+(\mathbb{R}^n)$. The L^1-contraction estimates are satisfied.
4) for all $t > 0$ we have $\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx$.
5) If $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then item 2) holde with $\tau = 0$ and $u(x,t)$ is uniformly bounded and continuous in space and time.

Difficulties and strategies

- In the isotropic case the self-similar fundamental solutions are radially symmetric with respect to the space variables, so the existence calculations become one-dimensional and the uniqueness is guaranteed.
- In the anisotropic framework we have to prove the existence and uniqueness of self-similar fundamental solution with finite mass.
- The properties of the profile F_M are different in the two ranges.
- The uniqueness proof uses the same strategy for all $m_i > 1$ or all $m_i < 1$ (few differences).
- The existence proof uses a fixed point argument but there is a substantial difference in the two ranges:
  - When all $m_i > 1$ we need to prove that "the flow does not leave the assigned box" after a certain time. We compare the solution with a suitable explicit 1D travelling wave.
  - When all $m_i < 1$ we use a crucial barrier function (a supersolution) to have a control of a generic solution and then a control of the L^1 profile F_M;

Fast Anisotropic upper barrier construction

The construction of an upper barrier in an outer domain will play a key role in the proof of existence of the self-similar fundamental solution.

Proposition 4

Let α, a_i, σ_i be as defined before. Take $\delta > 0$ and $\theta_i \geq 2$ such that

$$\frac{1}{\sigma_i} < \delta\theta_i < \frac{2}{1-m_i}\,. \tag{3}$$

Let $\Omega_r = \{y \in \mathbb{R}^N : \sum_{i=1}^N |y_i|^{\theta_i} \geq r\}$ be an anisotropic outer domain, where $r > 0$ depending of these parameters Then the function

$$\overline{F}(y) = \left(\sum_{i=1}^N |y_i|^{\theta_i}\right)^{-\delta} \tag{4}$$

is a supersolution to equation (SE) in the domain Ω_r and $\overline{F} \in L^1(\Omega_r)$.

Remarks. 1) We first observe that our hypotheses (H1), (H2) and the value of σ_i guarantee that

(5) $$\frac{1}{\sigma_i} < \frac{2}{1-m_i}.$$

so the choice of $\theta_i\delta$ is possible.

2) In choosing the exponents for the supersolution we can take the spatial decay $\theta_i\delta$ as close as we want to the dimensional exponent $2/(1-m_i)$.

3) It is proved in

Vázquez 2024 The very singular solution for the Anisotropic Fast Diffusion Equation,

that the decay of $F(y)$ in the i direction $(0, ...r_i, ...)$ is really $F(y) \sim C\, r_i^{-2/(1-m_i)}$.

Uniqueness of a self-similar fundamental solution: sketch of the proof

- Any non-negative self-similar fundamental solution $U_M(x,t)$ with mass $M>0$ is SSNI.
- U_1 and U_2 are two different self-similar fundamental solutions with the same mass $M>0$ and profiles $F_1,\ F_2$.
- the functional $J[U_1,U_2](t)=\int_{\mathbb{R}^N}(U_1(x,t)-U_2(x,t))_+\,dx$ is nonnegative and nonincreasing in time
- $J[U_1,U_2](t)=\int_{\mathbb{R}^N}(F_1(x)-F_2(x))_+\,dx=:c_0\geq 0$
- we construct solutions U_*, U^* such that $U_*(x,1)=\min\{F_1(x),F_2(x)\}$, $U^*(x,1)=\max\{F_1(x),F_2(x)\}$
- we show that

$$U_*=\min\{U_1,U_2\}\,,\quad U^*=\max\{U_1,U_2\}$$

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- we show that
$$U_*=\min\{U_1,U_2\}\,,\quad U^*=\max\{U_1,U_2\}$$
- for every $t>1$ U_1 and U_2 are positive at $x=0$ because they are SSNI with positive mass and then by continuity they are positive in $I(0)$ for every time $t\geq 1$, t close to 1. Then the equation is locally not degenerate.
- by a Strong Maximum Principle arguments, we show that this is possible only if $U_1(x,t_1)=U_2(x,t_1)$ for all x with t_1, then $c_0=0$, so $U_1\leq U_2$ for all x and t. Recalling they have the same mass we conclude $U_1\equiv U_2$.

Existence of a self-similar fundamental solution: sketch of the proof

Let S_τ be the semigroup map associated to the rescaled flow, *i.e.* the v flow, $S_\tau v_0 = v(y,\tau)$. We take $t_0 = 1$, then $\tau_0 = 0$.
Let us call $\mathcal{K}$ the set (constants $M, L, R > 0$ to be chosen):

A1) $\int_{\mathbb{R}^N} \varphi(x)\,dx = M$.

A2) $0 \le \varphi(x) \le L$

A3) $\varphi(x) = 0$ if $x \notin Q(R) = \{x : |x_i| \le R \quad \forall i = 1,\ldots,N\}$.

A4) φ is SSNI.

$\mathcal{K}$ is a closed and convex subset of $L^1(\mathbb{R}^N)$ and is not empty if $M \le 2^N L\, R^N$.

Proposition 5

For every $\tau_1 > 0$ and for every $M > 0$, there is a choice of L and R such that, under the above conditions A1)-A4), the flow map S_{τ_1} satisfies

$$S_{\tau_1}(\mathcal{K}) \subset \mathcal{K}.$$

Existence of a self-similar fundamental solution: sketch of the proof

Proof of $v(x,\tau_1)=0$ if $x\notin Q(R)$(choice of R,M and L are fixed). DIBUJO

- we fix direction e_1 and the domain $D_1=\{x: x_1\ge R\}$
- we compare u in D_1 for times $0<t<t_1$ with a one dimensional super-solution of the PME in the direction e_1.
- $\overline{u}(x,t)$ is the explicit 1D travelling wave with speed $A<1$ of the form

  $$\overline{u}^{m_1-1}(x,t)=cA(At+K-x_1)_+,$$

  where $c\le\frac{m_1-1}{m_1}$,A is fixed and $K>0$ will be chosen later.
- We check that we can apply the parabolic comparison with a convenient choice of K.
- we conclude that $u(x,t)\le\overline{u}(x,t)\quad$ in $D_1\ge R,\ t>0$.
- we translate the comparison in v variables obtaining $v(x,\tau_1)=0$ if $x\notin Q(R)$

Existence of a self-similar fundamental solution: sketch 2

- it is possible to prove that the image set $Y = S_{\tau_1}(\mathcal{K}(L_1))$ is relatively compact in L^1;
- by Schauder Fixed Point Theorem, for any $\tau_1 > 0$ there exists at least a fixed point $\phi_{\tau_1} \in \mathcal{K}$, *i. e.*, $S_{\tau_1}(\phi_{\tau_1}) = \phi_{\tau_1}$.
- we get periodicity for the orbit $V_{\tau_1}(y,\tau)$ starting at $\tau = 0$ from $V_{\tau_1}(y,0) = \phi_{\tau_1}(y)$:

$$V_{\tau_1}(y,\tau + k\tau_1) = V_{\tau_1}(y,\tau) \quad \forall \tau > 0,\ k \in \mathbb{N}.$$

- By an argument similar to the one used in the uniqueness, we prove that any periodic solution V_{τ_1} of our renormalized problem must be stationary in time: $V_{\tau_1}(y,\tau) = F(y)$.

Properties of the support of F_M under assumptions $(H1)$ and $(H3)$

F_1 is the profile of self-similar fundamental solution U_1 of mass $M = 1$.

$$\Omega(F_1) = \{y : F_1(y) > 0\} \quad \text{is the set of positivity of } F_1$$

$\Omega(F_1) = \{y : F_1(y) > 0\}$ is open, bounded and star-shaped around the origin.

The closure of $\Omega(F_1)$ is called the support of F_1.

Proposition 6

There are some positive constants $c_1, \cdots, c_N$ depending only on the parameters of the problem such that $\Omega(F_1)$ is contained in the box

$$\Omega(F_1) \subset [-c_1, c_1] \times \cdots \times [-c_N, c_N].$$

In the previous proposition the optimal constants are attained on the coordinate axes by the monotonicity properties of F_1.

The asymptotic behaviour of finite mass solutions under assumptions $(H1)$ and $(H3)$

We restate Theorem 2.

Theorem 7

Let $u(x,t)$ be the unique solution of the Cauchy problem for equation (APME) *with nonnegative initial datum $u_0 \in L^1(\mathbb{R}^N)$. Let U_M be the unique self-similar fundamental solution with the same mass as u_0. Then,*

$$\lim_{t\to\infty} \|u(\cdot,t) - U_M(\cdot,t)\|_1 = \lim_{\tau\to\infty} \|v(\cdot,\tau) - F_M\|_1 = 0.$$

The convergence also holds in all the L^p norms, $1 < p < \infty$, in the proper scale:

$$\lim_{t\to\infty} t^{\frac{(p-1)\alpha}{p}} \|u(\cdot,t) - U_M(\cdot,t)\|_p = \lim_{\tau\to\infty} \|v(\cdot,\tau) - F_M\|_p = 0,$$

where $\alpha = \frac{N}{N(\overline{m}-1)+2}$.

- L^p-version of Theorem is a consequence of L^1-version + $\|U_M(\cdot,t)\|_\infty \le Ct^{-\alpha}$ + $\|u(\cdot,t\|_\infty \le Ct^{-\alpha}$

The asymptotic behaviour of finite mass solutions

- More difficulties with respect to the isotropic case and some differences in the two ranges.
- Proof in two steps:
  - proof for bounded and compactly supported data;
  - generalization to L^1 data (we use the monotonicity of F_M with respect to M).
- IDEA: "four-step method" by Kamin-Vázquez 88
  - u is a solution with mass M
  - family of rescaled solutions with mass M: $u_\lambda(x,t) = \lambda^\alpha u(\lambda^{\frac{\alpha}{N}} x, \lambda t)$
  - passing to the limit (up a subsequence): $u_\lambda(x,t) \to U(x,t)$ for $\lambda \to +\infty$ (using some estimates)
  - identification: $U = U_M$
    - U is a fundamental self-similar solution with mass M
    - uniqueness of self-similar fundamental solution

L^∞-convergence under some additional conditions on the data

We have a stronger asymptotic convergence result under our initial conditions of bounded and compactly supported data.

Theorem 8

Under the extra conditions that u_0 is bounded and compactly supported, then

$$\lim_{t\to\infty} t^{\alpha}\|u(\cdot,t)-U_M(\cdot,t)\|_\infty = \lim_{\tau\to\infty}\|v(\cdot,\tau)-F_M\|_\infty = 0,$$

where $\alpha = \frac{N}{N(\overline{m}-1)+2}$.

Sketch of the proof of the L^∞-asymptotic behaviour

Assumptions:$(H1)$ and $(H3)$ and u_0 bounded and compactly supported.

1) Uniform convergence in $B_\varepsilon(0)$ with $\varepsilon > 0$
   - quantitative positive lemma in $B_\varepsilon(0)$
   - $v(y,\tau)$ is a uniformly non-degenerate solution of (APME) in $B_\varepsilon(0) \times (\tau_1, +\infty)$ with $\tau_1 > 0$
   - Hölder continuity and Ascoli-Arzelà Theorem
2) Uniform convergence in $\{y : |y| \geq \varepsilon', F(y) \geq \varepsilon'\}$ for $0 < \varepsilon' < \varepsilon$
   - by contradiction
   - by Aleksandrov reflection principle we get

Proposition 7

Let u be a nonnegative solution of the Cauchy problem for (APME) with bounded initial data supported in the box $Q(\vec{a}) = [-a_1, a_1] \times \cdots \times [-a_N, a_N]$. Let $x_0 = (x_{01}, \cdots, x_{0N})$ be a point in $\mathbb{R}^N$ with all coordinates $x_{0i} \geq a_i$ and $K(x_0)$ be the conical region $K(x_0) = \{x : x_i > x_{0i}\ \forall i\}$. Then $u(x,t)$ is monotone nonincreasing along every straight line that starts at x_0 and enters $K(x_0)$.

3) Uniform convergence in $\{y : |y| \geq \varepsilon', F(y) \leq \varepsilon'\}$ for $0 < \varepsilon' < \varepsilon$

Support of the solutions. Hausdorff set distance

Our aim: to show how the spatial support of a nontrivial solution $u \geq 0$ with compactly supported and bounded initial data evolves in time under assumptions $(H1S)$ and $(H3)$. A sharp approximation result needs the Hausdorff set distance.

Definition. Let $A, B \subset \mathbb{R}^N$. For any $a \in A$, $b \in B$ we define the distance from a point to a set

$$d(a, B) = \min\{d(a, b) : b \in B\}, \quad d(b, A) = \min\{d(a, b) : a \in A\},$$

and then we define the two directional distances

$$d_1(A, B) = \sup\{d(a, B) : a \in A\},$$

and

$$d_2(A, B) = \sup\{d(b, A) : b \in B\} = d_1(B, A) \quad \text{(we interchange } A \text{ and } B\text{).}$$

Then the symmetric distance is defined as

$$d_H(A, B) = \max\{d_1(A, B), d_2(A, B)\}.$$

We will work with bounded sets.

Properties of support of solutions with compactly supported and bounded initial data

This is the third main result,
Let $u(x, t)$ be the unique solution of the Cauchy problem for equation (APME) with nonnegative bounded and compactly supported initial datum u_0 of mass M, when (H1) and (H3) are in force. We examine the long time behaviour. Let $v(y, \tau)$ be the renormalized solution..

Theorem 9

Let F_M be the profile of unique self-similar fundamental solution with the same mass M as u_0. Then we have the set convergence

$$\lim_{\tau \to \infty} d_H(\Omega(v, \tau), \Omega(F_M)) = 0,$$

where v is the renormalized solution. The same formula works for the supports, $d_H(S(v, \tau), S(F_M)) \to 0$.

Idea of the proof

- mass rescaling : $M = 1$
- Step 1: approximation of family $\Omega(v,\tau)$ to $\Omega(F_1)$ from inside, *i.e.*

$$d_1(\Omega(F_1), \Omega(v,\tau)) \to 0 \quad \text{as } \tau \to +\infty \tag{6}$$

  - for every $\varepsilon > 0$ we set $G_\varepsilon = \{y : F_1(y) \geq \varepsilon\}$. By uniform convergence result, we have $G_\varepsilon \subset \Omega(v,\tau)$ for large τ, then $d_1(G_\varepsilon, \Omega(v,\tau)) = 0$ for large τ;
  - $d_1(\Omega(F_1), G_\varepsilon) \to 0$ as $\varepsilon \to 0$;
  - then by triangular inequality we get (6).

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- Step 1: approximation of family $\Omega(v, \tau)$ to $\Omega(F_1)$ from inside, *i.e.*

(6) $$d_1(\Omega(F_1), \Omega(v, \tau)) \to 0 \quad \text{as } \tau \to +\infty$$

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  - $d_1(\Omega(F_1), G_\varepsilon) \to 0$ as $\varepsilon \to 0$;
  - then by triangular inequality we get (6).
- Step 2: approximation of family $\Omega(v, \tau)$ to $\Omega(F_1)$ from outside, *i.e.* $d_1(\Omega(v, \tau), \Omega(F_1)) \to 0$ as $\tau \to +\infty$
  This step is more delicate.

Step 2: $d_1(\Omega(v,\tau),\Omega(F_1)) \to 0 \quad$ as $\tau \to +\infty$

PROBLEM: A family of positivity sets of a converging family of functions may fail to approximate the limit when the convergence of supports is examined, because of the presence of "thin positivity tails" of evanescent intensity that must be however counted in the sense of positive sets even if their mass is small.

STRATEGY: We combine the uniform convergence of $v(y,\tau)$ in the positivity set of F_1 with a new comparison in the exterior of $\Omega(F_1)$ with a suitable upper barrier.
We know that there exists a mass $M_2 > 1$ such that $v(y,\tau) \le F_{M_2}(y)$ and then $\Omega(v,\tau)$ is contained in $\Omega(F_{M_2})$ for τ big enough. This means that for all large τ the solution vanishes outside a large set, more precisely

$$v(y,\tau) = 0 \quad \text{for} \;\; y \notin \Omega(F_{M_2}).$$

To conclude Step 2 we only have to reduce the role of the mass M_2 to a smaller size of mass M_3 near $M = 1$. The reduction part is delicate.

A comparison argument

$$\mathcal{S}_t^{(2)}(\Omega(F_1)) = \Omega(U_1, t) \quad \mathcal{S}_k^{(1)}(\Omega(F_1)) = \Omega(F_M)$$

These two different expansions have produced different kinds of “distorted balls” but they are mutually comparable.

Proposition 8

Let $\varepsilon > 0$ be close to 0. There exist positive constants c_1 and c_2 such that

$$E_{1+c_1\varepsilon}(\Omega(F_1)) \subset \mathcal{S}_{1+\varepsilon}^{(1)}(\Omega(F_1)) \subset E_{1+c_2\varepsilon}(\Omega(F_1)).$$

There also exist positive constants c_3 and c_4 such that

$$E_{1+c_3\varepsilon}(\Omega(F_1)) \subset \mathcal{S}_{1+\varepsilon}^{(2)}(\Omega(F_1)) \subset E_{1+c_4\varepsilon}(\Omega(F_1)).$$

$E_\lambda(\Omega(F_1)) = \{z = \lambda y,\ y \in \Omega(F_1)\}$ linear expansion of $\Omega(F_1)$ with parameter $\lambda > 0$

Some remarks

- Our result on the large-time behaviour of supports is deduced in the isotropic case from a simplified argument. Indeed
  - the support of the profile is a ball of radius R (depending on the total mass M);
  - the support of the self-similar fundamental solution (Barenblatt solution) is a ball of radius $R(t)$;
  - the supports of $v(\cdot, \tau)$ with compact initial support are proved to be approximate balls (introducing minimal and maximal radius), then evolve in time to fill a ball of radius $R(t)$ given by the closest Barenblatt solution;
  - see Theorem 18.8 in

    The Porous Medium Equation: Mathematical Theory by Vázquez;
- this known isotropic result extends in the anisotropic case to the result of our theorems: in the anisotropic case the support evolves in time to fill an "anisotropic" ball which is given by the support of the closest Self-similar Fundamental solution.

Some remarks

- Our result on the large-time behaviour of supports is deduced in the isotropic case from a simplified argument. Indeed
  - the support of the profile is a ball of radius R (depending on the total mass M);
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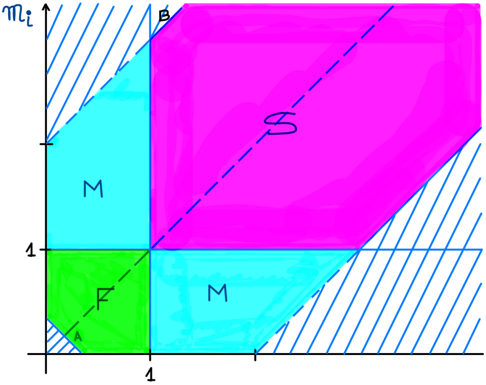
- this known isotropic result extends in the anisotropic case to the result of our theorems: in the anisotropic case the support evolves in time to fill an "anisotropic" ball which is given by the support of the closest Self-similar Fundamental solution.

Some works in progress and open problems

- Analysis of the mixed slow-fast case:
  $\exists m_i < 1$, $\exists m_j \geq 1$:
  (2025) We prove the existence of a fundamental solution with compact support only in the slow directions and with an L^1-profile in the other directions.

- Investigation in the range $\overline{m} \leq m_c$;
- Investigation in the range $m_i \geq \frac{2}{N} + \overline{m}$
- Study of problems in bounded domains with suitable boundary conditions;
- To adapt our strategy to anisotropic $p-$laplacian;
- The anisotropic p-Laplacian when all $p_i < 2$ is partially studied in our first paper

Anisotropic Regions



Anisotropic Regions (only two variables m_i plotted)

Red region: Slow - Green region: Fast - Blue region: Mixed.

Line A: Condition (H2), Lines B: Condition (H3),

Our work since 2021

- Filomena Feo, J. L. Vázquez, Bruno Volzone. Anisotropic p-Laplacian Evolution of Fast Diffusion type - Adv. Nonlinear Stud. 21 (2021), no. 3, 523-555.
- F. Feo, J. L. V., B. Volzone - Anisotropic Fast Diffusion Equations - Nonlinear Analysis, 233 (2023), Paper No. 113298
- J. L. V. The very singular solution for the Anisotropic Fast Diffusion Equation and its consequences - Nonlinear Anal. 245 (2024), Paper No. 113556.
- F. Feo, J. L. V., B. Volzone. Asymptotic behaviour of solutions and free boundaries of the anisotropic slow diffusion equation. Submitted. Uploaded to arXiv:2412.12295 [math.AP]
- F. Feo, J. L. V., B. Volzone. Anisotropic diffusion equations in the mixed case. april 2025, in progress.

Thank you for your attention

Grazie

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