Bulk-surface Cahn-Hilliard model for chemically active wetting

Ongoing work with S. Fagioli and J-F. Pietschmann

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What is chemically active wetting

- Classic wetting: Liquid drop on a surface. Governed by equilibrium laws (Young–Dupré law). Droplets adopt spherical cap shapes on passive surfaces.
- Active wetting: Surfaces can actively bind/unbind components using chemical energy. Breaks equilibrium- leads to persistent fluxes and non-standard droplet shapes.

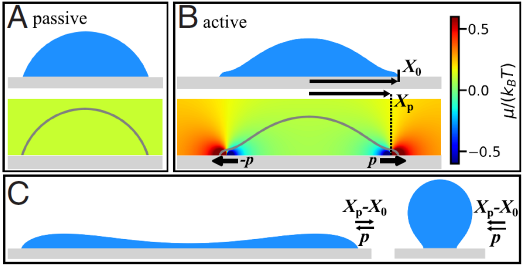

Figure: Passive vs Active Wetting ([Liese et al., 2025]

Modelling active wetting[1]

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- Governing equations include continuum equations for droplet volume fraction ϕ and membrane area fraction ϕ_m :

$$\partial_t \phi = -\nabla \cdot \boldsymbol{j} \quad \partial_t \phi_m = -\nabla_\parallel \cdot \boldsymbol{j}_m - s$$

where $s = s_{off} - s_{on}$ is the net desorption flux.

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- Fluxes:

$$\boldsymbol{j} = -\Lambda(\phi)\nabla\mu \quad \boldsymbol{j}_m = -\Lambda_m(\phi_m)\nabla_{\parallel}\mu_m$$

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$$\boldsymbol{j} = -\Lambda(\phi)\nabla\mu \quad \boldsymbol{j}_m = -\Lambda_m(\phi_m)\nabla_{\parallel}\mu_m$$

- Avoid detail balance: $\frac{s_{on}}{s_{off}} \neq \exp\left[-\frac{\mu_m - \mu}{k_B T}\right]$

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The Bulk-Surface Cahn-Hilliard system

Let V denote the bulk domain and $m \subset \partial V$ denote the membrane domain.

$$\partial_t \phi = -\nabla \cdot \boldsymbol{j} \qquad x \in V, t > 0 \tag{1}$$

$$\partial_t \phi_m = -\nabla_{\|} \cdot \boldsymbol{j}_m - s \qquad x \in m, t > 0 \tag{2}$$

where

$$\begin{array}{ccccccc} \boldsymbol{j} & = & -\Lambda(\phi)\nabla\mu & \quad & \boldsymbol{j}_m & = & -\Lambda_m(\phi)_m \nabla_{\|}\mu_m \\ \Lambda(\phi) & = & D\phi(1-\phi) & & \Lambda_m(\phi_m) & = & D_m\phi_m(1-\phi_m) \\ \mu & = & -\kappa\Delta\phi + f'(\phi) & & \mu_m & = & -\kappa_m\Delta_{\|}\phi_m + f'_m(\phi_m) \end{array}$$

and D, D_m, κ, κ_m are positive constants and s is given by

$$s = k_0(1-\phi_m)(1-\phi_0)(e^{\mu_m} - e^{\mu+\Delta\mu_{act}}), \tag{3}$$

where k_0 is a constant, ϕ_0 is the bulk volume fraction at the memebrane, and $\Delta\mu = \chi_{act}\phi_0$.

The Bulk-Surface Cahn-Hilliard system

Further, f and f_m represent the (respective) free energies given by

$$f(\phi) = \phi \log \phi + (1-\phi)\log(1-\phi) + \chi\phi(1-\phi),$$
$$f_m(\phi_m) = \phi_m \log \phi_m + (1-\phi_m)\log(1-\phi_m) + \chi_m\phi_m(1-\phi_m)$$

The system is subject to the following boundary conditions:

$$\begin{aligned}
\nabla\phi\cdot\boldsymbol{n} &= -\frac{1}{\kappa}\frac{\partial\Omega}{\partial\phi_0}, \quad x\in m\\
\nabla\phi\cdot\boldsymbol{n} &= 0, \quad x\in\partial V\setminus m\\
\nabla_{\parallel}\phi_m\cdot\boldsymbol{t} &= 0, \quad x\in\partial m\\
\boldsymbol{j}\cdot\boldsymbol{n} &= -s, \quad x\in m\\
\boldsymbol{j}\cdot\boldsymbol{n} &= 0, \quad x\in\partial V\setminus m\\
\boldsymbol{j}_m\cdot\boldsymbol{t} &= 0, \quad x\in\partial m.
\end{aligned}$$

where $\Omega(\phi_0)$ is the coupling energy between the bulk and the membrane.

Energy Functional and Strategy

The energy functional related to the system is composed of energies in the bulk and the membrane surface as well as the coupling energy

$$F[\phi, \phi_m] = \int_V f(\phi) + \frac{\kappa}{2}|\nabla\phi|^2 + \int_m f_m(\phi_m) + \frac{\kappa_m}{2}|\nabla_{\parallel}\phi_m|^2 + \Omega(\phi) \tag{4}$$

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Show the existence of weak solutions to the system

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- Prove the existence of the resulting system and use this to approximate solutions to the degenerate problem
- Derive suitable estimates for the approximate solutions that allow to pass to the limit in the approximate equation ([Elliott and Garcke, 1996], but we also fall short!)

Energy estimate and dissipation law

Energy estimates:

$$F[\phi,\phi_m] = \int_V f(\phi) + \frac{\kappa}{2}|\nabla\phi|^2 + \int_m f_m(\phi_m) + \frac{\kappa_m}{2}|\nabla_{\|}\phi_m|^2 + \Omega(\phi)$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F[\phi,\phi_m](t) &= \int_V (f'(\phi) - \kappa\Delta\phi)\partial_t\phi + \int_{\partial V} \kappa\nabla\phi\cdot\boldsymbol{n}\partial_t\phi + \\
&\quad + \int_m (f_m'(\phi_m) - \Delta_{\|}\phi_m)\partial_t\phi_m + \frac{\mathrm{d}}{\mathrm{d}t}\int_m \Omega(\phi) \\
&= \int_V \mu\nabla\cdot(\Lambda(\phi)\nabla\mu) + \int_m \mu_m\left(\nabla_{\|}\cdot(\Lambda_m(\phi_m)\nabla_{\|}\mu_m) - s\right) \\
&= -\int_V \Lambda(\phi)|\nabla\mu|^2 - \int_m \Lambda_m(\phi_m)|\nabla_{\|}\mu_m|^2 + \int_m s(\mu-\mu_m)
\end{aligned}$$

Energy estimate and dissipation law

$$
\begin{aligned}
\int_m s(\mu - \mu_m) =& k_0 \int_m \lambda(\phi)\lambda_m(\phi_m)(\mu_m - (\mu + \Delta\mu_{act}))(\mu - \mu_m) \\
=& - k_0 \int_m \lambda(\phi)\lambda_m(\phi_m)(\mu - \mu_m)^2 - k_0 \int_m \lambda(\phi)\lambda_m(\phi_m)\Delta\mu_{act}(\mu - \mu_m) \\
\leq& k_0 \left(\frac{\alpha}{2} - 1\right) \int_m \lambda(\phi)\lambda_m(\phi_m)(\mu - \mu_m)^2 + \frac{k_0}{2\alpha} \int_m \lambda(\phi)\lambda(\phi_m)\Delta\mu_{act}^2
\end{aligned}
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\end{aligned}
$$

choose $\alpha \in \mathbb{R}$ so that $\beta := k_0 \left(1 - \frac{\alpha}{2}\right) > 0$,

$$
\frac{\mathrm{d}}{\mathrm{dt}} F[\phi, \phi_m](t) + \int_V \lambda(\phi)|\nabla\mu|^2 + \int_m \lambda_m(\phi_m)|\nabla_\parallel \mu_m|^2 + \beta \int_m \lambda(\phi)\lambda_m(\phi_m)(\mu - \mu_m)^2 \leq C
$$

A Grönwall argument yields

$$
\|\phi\|_{L^\infty(0,T;H^1(V))} + \|\phi_m\|_{L^\infty(0,T;H^1(m))} + \|\mu\|_{L^2(0,T;H^1(V))} + \|\mu_m\|_{L^2(0,T;H^1(m))} \leq C
$$

Existence for positive mobilities

Galerkin scheme:

$$\begin{aligned}\{\xi_i\}_{i\in\mathbb{N}} \subset & H^1(V): \\ -\Delta\xi_i = & \nu_i\xi_i \quad \text{in} \quad V \\ \nabla\xi_i\cdot\boldsymbol{n} = & 0 \quad \text{on} \quad \partial V\end{aligned} \qquad \begin{aligned}\{\zeta_i\}_{i\in\mathbb{N}} \subset & H^1(m): \\ -\Delta_{\|}\zeta_i = & \theta_i\zeta_i \quad \text{in} \quad m, \\ \nabla_{\|}\zeta_i\cdot\boldsymbol{t} = & 0 \quad \text{on} \quad \partial m\end{aligned}$$

Galerkin ansatz: Let $N \in \mathbb{N}$ and $T > 0$ fixed and for $x \in V, \varsigma \in m, t \in [0,T]$ define

$$\phi^N(t,x) = \sum_{i=1}^N a_i^N(t)\xi_i(x), \qquad \phi_m^N(t,\varsigma) = \sum_{i=1}^N c_i^N(t)\zeta_i(\varsigma)$$

$$\mu^N(t,x) = \sum_{i=1}^N b_i^N(t)\xi_i(x), \qquad \mu_m^N(t,\varsigma) = \sum_{i=1}^N d_i^N(t)\zeta_i(\varsigma),$$

Galerkin approximation for positive mobilities

The scalars a_i, b_i, c_i, d_i are determined such that the following weak formulation holds: for every $\xi \in \mathrm{span}(\xi_1, \ldots, \xi_N)$ and $\zeta \in \mathrm{span}(\zeta_1, \ldots, \zeta_N)$,

$$\int_V \partial_t \phi^N \xi = -\int_V \lambda(\phi^N) \nabla \mu^N \cdot \nabla \xi + \int_m s^N \xi$$

$$\int_V \mu^N \xi = \kappa \int_V \nabla \phi^N \cdot \nabla \xi + \int_V f'(\phi^N) \xi + \int_m \Omega'(\phi^N) \xi$$

$$\int_m \partial_t \phi_m^N \zeta = -\int_V \lambda_m(\phi_m^N) \nabla_\parallel \mu_m^N \cdot \nabla_\parallel \zeta - \int_m s^N \zeta$$

$$\int_m \mu_m^N \zeta = \kappa_m \int_m \nabla_\parallel \phi_m^N \cdot \nabla_\parallel \zeta + \int_m f_m'(\phi_m^N) \zeta$$

$$s^N = k_0 \int_m \lambda(\phi) \lambda_m(\phi_m) \left(\mu_m^N - (\mu^N + \Delta\mu_{act}) \right)$$

together with initial conditions,

$$\phi^N(0) = \sum^N \langle \phi^0, \xi_i \rangle_{L^2(V)} \xi_i \qquad \phi_m^N(0) = \sum^N \langle \phi_m^0, \zeta_i \rangle_{L^2(V)} \zeta_i$$

Galerkin approximation for positive mobilities

Testing with ξ_j, ζ_j correspondingly gives an initial value problem of ODEs for $(a_1, \dots, a_N), (c_1, \dots, c_N)$:

$$a_j'(t) = -\sum_{i=1}^N b_i^N \int_V \lambda \left(\sum_{k=1}^N a_k^N \xi_k \right) \nabla \xi_i \cdot \nabla \xi_j + \int_m s^N \xi_j$$

$$b_j(t) = \kappa \nu_j a_j^N + \int_V f' \left(\sum_{i=1}^N a_i^N \xi_i \right) \xi_j - \omega \int_m \xi_j$$

$$c_j'(t) = -\sum_{i=1}^N d_i^N \int_m \lambda_m \left(\sum_{k=1}^N c_k^N \zeta_k \right) \nabla_\parallel \zeta_i \cdot \nabla_\parallel \zeta_j - \int_m s^N \zeta_j$$

$$d_j(t) = \kappa_m \theta_j c_j^N + \int_m f_m' \left(\sum_{i=1}^N c_i^N \zeta_i \right) \zeta_j$$

$$a_j^N(0) = \langle \phi^0, \xi_j \rangle_{L^2(V)} \quad c_j^N(0) = \langle \phi_m^0, \zeta_j \rangle_{L^2(m)}$$

Compactness and limit passage

We obtain $\phi^N, \mu^N \in C([0,T]; H^1(V), \phi_m^N, \mu_m^N \in C([0,T]; H^1(m)$ satisfing the weak formulation.
From the energy estimate after a Grönwall argument we obtain

- $\|\phi^N\|_{L^\infty(0,T;H^1(V))}, \|\phi_m^N\|_{L^\infty(0,T;H^1(m))} \leq C$
- $\|\mu^N\|_{L^2(0,T;H^1(V))}, \|\mu_m^N\|_{L^2(0,T;H^1(m))} \leq C$
- $\|\mu^N\|_{L^\infty(0,T;(H^1(V))')}, \|\mu_m^N\|_{L^\infty(0,T;(H^1(m))')} \leq C$
- $\|\partial_t\phi^N\|_{L^2(0,T;(H^1(V))')}, \|\partial_t\phi^N\|_{L^2(0,T;(H^1(V))')} \leq C$

Compactness and limit passage

Sending $N \to \infty$ and applying Aubi-Lions compactness lemma, there exist ϕ, ϕ_m, μ, μ_m such that

$$
\begin{aligned}
\phi^N &\to \phi \quad \text{strongly in} \quad C([0,T]; L^2(V)) \\
\phi_m^N &\to \phi_m \quad \text{strongly in} \quad C([0,T]; L^2(m)) \\
\phi^N &\to \phi \quad \text{weak}-* \text{ in} \quad L^\infty([0,T]; H^1(V)) \\
\phi_m^N &\to \phi_m \quad \text{weak}-* \text{ in} \quad L^\infty([0,T]; H^1(m)) \\
\mu^N &\to \mu \quad \text{weakly in} \quad L^2([0,T]; H^1(V)) \\
\mu_m^N &\to \mu_m \quad \text{weakly in} \quad L^2([0,T]; H^1(m)) \\
\partial_t \phi^N &\to \partial_t \phi \quad \text{weakly in} \quad L^2([0,T]; (H^1(V))') \\
\partial_t \phi_m^N &\to \partial_t \phi_m \quad \text{weakly in} \quad L^2([0,T]; (H^1(m))')
\end{aligned}
$$

Nondegenrate limit

ϕ, ϕ_m, μ, μ_m satisfy the following weak formulation: for all $\xi \in L^2(0,T;H^1(V)), \xi_m \in L^2(0,T;H^1(m)), \zeta \in H^1(V), \zeta_m \in H^1(m)$,

$$\begin{aligned}
\int_0^T \langle \xi(t), \partial_t \phi(t) \rangle_{H^1(V),(H^1(V))'} &= -\int_V \lambda(\phi)\nabla\mu \cdot \nabla\xi + \int_m s\xi \\
\int_V \mu\zeta &= \kappa \int_V \nabla\phi \cdot \nabla\zeta + \int_V f'(\phi)\zeta + \int_m \Omega'(\phi)\zeta \\
\int_0^T \langle \xi_m(t), \partial_t \phi_m(t) \rangle_{H^1(m),(H^1(m))'} &= -\int_m \lambda(\phi_m)\nabla_{\|}\mu_m \cdot \nabla_{\|}\xi_m + \int_m s\xi_m \\
\int_m \mu_m\zeta_m &= \kappa_m \int_m \nabla_{\|}\phi_m \cdot \nabla\zeta_m + \int_V f_m'(\phi_m)\zeta_m
\end{aligned}$$

Strategy to approximate the degenerate limit

In the degenerate case, we approximate them by solutions of nondegenerate equations with positive (small) mobility. We also modify the bulk and membrane free-energies f, f_m so that it is defined (not singular) on all $\mathbb{R}$. We introduce for each $\varepsilon > 0$,

$$\Lambda^\varepsilon(u) := \begin{cases} \Lambda(\varepsilon) & \text{for} \quad u \le \varepsilon \\ \Lambda(u) & \text{for} \quad \varepsilon < u < 1-\varepsilon \\ \Lambda(1-\varepsilon) & \text{for} \quad u \ge 1-\varepsilon \end{cases} \qquad \Lambda_m^\varepsilon(u) := \begin{cases} \Lambda_m(\varepsilon) & \text{for} \quad u \le \varepsilon \\ \Lambda_m(u) & \text{for} \quad \varepsilon < u < 1-\varepsilon \\ \Lambda_m(1-\varepsilon) & \text{for} \quad u \ge 1-\varepsilon \end{cases}$$

Let $\phi^\varepsilon, \phi_m^\varepsilon, \mu^\varepsilon, \mu_m^\varepsilon$ denote the weak solutions to

$$\begin{aligned} \partial_t\phi &= \nabla\cdot\Lambda^\varepsilon(\phi)\nabla\mu && \text{in} \quad V_T & \partial_t\phi_m &= \nabla_\parallel\cdot\Lambda_m^\varepsilon(\phi_m)\nabla_\parallel\mu_m - s && \text{in} \quad m_T \\ \mu &= -\kappa\Delta\phi + (f^\varepsilon)'(\phi) && \text{in} \quad V_T & \mu_m &= -\kappa_m\Delta_\parallel\phi_m + (f_m^\varepsilon)'(\phi_m) && \text{in} \quad m_T \\ \kappa\nabla\phi\cdot\boldsymbol{n} &= -\Omega'(\phi) && \text{on} \quad m_T & \nabla\phi\cdot\boldsymbol{t} &= 0 && \text{on} \quad \partial m_T \end{aligned}$$

Degenerate limit - goals and requirements

Goal: to pass to the $\varepsilon \to 0$ limit in the weak formulation: for any $\xi \in L^2(0,T;H^1(V)), \xi_m \in L^2(0,T;H^1(m))$

$$\int_0^T \langle \xi, \partial_t \phi^\varepsilon \rangle_{H^1,(H^1)'} = -\int_{V_T} \Lambda^\varepsilon(\phi^\varepsilon)\nabla(-\kappa\Delta\phi^\varepsilon + (f^\varepsilon)'(\phi^\varepsilon)) \cdot \nabla\xi + \int_{m_T} s^\varepsilon \xi$$

$$\int_0^T \langle \xi_m, \partial_t \phi_m^\varepsilon \rangle_{H^1,(H^1)'} = -\int_{m_T} \Lambda_m^\varepsilon(\phi_m^\varepsilon)\nabla_\parallel(-\kappa_m\Delta_\parallel\phi_m^\varepsilon + (f_m^\varepsilon)'(\phi_m^\varepsilon)) \cdot \nabla_\parallel\xi_m - \int_{m_T} s^\varepsilon \xi_m$$

$$s^\varepsilon = k_0\Lambda^\varepsilon(\phi^\varepsilon)\Lambda_m(\phi_m^\varepsilon)\left(-\kappa\Delta\phi^\varepsilon + \kappa_m\Delta_\parallel\phi_m^\varepsilon + (f_m^\varepsilon)'(\phi_m^\varepsilon) - (f^\varepsilon)'(\phi^\varepsilon) - \Delta\mu_{act}\right)$$

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$$s^\varepsilon = k_0\Lambda^\varepsilon(\phi^\varepsilon)\Lambda_m(\phi_m^\varepsilon)\left(-\kappa\Delta\phi^\varepsilon + \kappa_m\Delta_\parallel\phi_m^\varepsilon + (f_m^\varepsilon)'(\phi_m^\varepsilon) - (f^\varepsilon)'(\phi^\varepsilon) - \Delta\mu_{act}\right)$$

We need:

- Uniform bounds for $\phi^\varepsilon, \phi_m^\varepsilon$ in $L^2(0,T;H^1)$ and for $\nabla\Delta\phi^\varepsilon$ in $L^2(0,T,L^2)$.
- Strong convergence for $\phi^\varepsilon, \nabla\phi^\varepsilon, \phi_m^\varepsilon, \nabla\phi_m^\varepsilon$ in $L^2(0,T,L^2)$
- Weak convergence for $\Delta\phi^\varepsilon, \Delta\phi_m^\varepsilon$ in $L^2(0,T,L^2)$

Limitation(s)

Since $\nabla\mu^\varepsilon \in L^2(V_T), \nabla_\parallel\mu_m^\varepsilon \in L^2(m_T)$ and $\nabla(f^\varepsilon)'(\phi^\varepsilon) = (f^\varepsilon)''(\phi^\varepsilon)\nabla\phi^\varepsilon \in L^2(V_T)$, we have $\nabla\Delta\phi^\varepsilon \in L^2(V_T), \nabla_\parallel\Delta_\parallel(\phi_m^\varepsilon) \in L^2(m_T)$.

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But, we do not yet have a uniform bound of the sort

$$\int_{V_T} |\nabla\phi^\varepsilon|^2 + \int_{V_T} |\nabla\Delta\phi^\varepsilon|^2 \leq C(T)$$

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$$\int_{V_T} |\nabla\phi^\varepsilon|^2 + \int_{V_T} |\nabla\Delta\phi^\varepsilon|^2 \leq C(T)$$

$\vdots$

Dealing with the degenerate mobilities ([Elliott and Garcke, 1996])

Tackling the degenerate mobility: Define $\Psi^\varepsilon, \Psi^\varepsilon_m$ as follows

$$(\Psi^\varepsilon)''(u) = \frac{1}{\Lambda^\varepsilon(u)}, \quad (\Psi^\varepsilon)'(0) = \Psi^\varepsilon(0) = 0 \quad (\Psi^\varepsilon_m)''(u) = \frac{1}{\Lambda^\varepsilon_m(u)}, \quad (\Psi^\varepsilon_m)'(0) = \Psi^\varepsilon_m(0) = 0$$

Since $(\Psi^\varepsilon)'', (\Psi^\varepsilon_m)''$ are bounded, we have $(\Psi^\varepsilon)'(\phi^\varepsilon) \in L^2(0,T;H^1(V))$ and $(\Psi^\varepsilon_m)'(\phi^\varepsilon_m) \in L^2(0,T;H^1(m))$ are admissible test functions.

$$\begin{aligned}\int_0^T \langle (\Psi^\varepsilon)'(\phi^\varepsilon), \partial_t \phi^\varepsilon \rangle_{H^1,(H^1)'} &= -\int_{V_T} \Lambda^\varepsilon(\phi^\varepsilon)\nabla(-\kappa\Delta\phi^\varepsilon + (f^\varepsilon)'(\phi^\varepsilon)) \cdot (\Psi^\varepsilon)''(\phi^\varepsilon)\nabla\phi^\varepsilon + \\ &\quad + \int_{m_T} s^\varepsilon (\Psi^\varepsilon)'(\phi^\varepsilon)\end{aligned}$$

$$\begin{aligned}\int_0^T \langle (\Psi^\varepsilon_m)'(\phi^\varepsilon_m), \partial_t \phi^\varepsilon_m \rangle_{H^1,(H^1)'} &= -\int_{m_T} \Lambda^\varepsilon_m(\phi^\varepsilon_m)\nabla(-\kappa_m\Delta\phi^\varepsilon_m + (f^\varepsilon_m)'(\phi^\varepsilon_m)) \cdot (\Psi^\varepsilon_m)''(\phi^\varepsilon_m)\nabla\phi^\varepsilon_m \\ &\quad - \int_{m_T} s^\varepsilon (\Psi^\varepsilon_m)'(\phi^\varepsilon_m)\end{aligned}$$

Dealing with the degenerate mobilites

It follows that

$$\int_V \Psi^\varepsilon(\phi^\varepsilon(T)) + \kappa \int_{V_T} |\Delta\phi^\varepsilon|^2 + (f^\varepsilon)''(\phi^\varepsilon)|\nabla\phi^\varepsilon|^2 \leq \int_V \Psi^\varepsilon(\phi^0) + \int_{m_T} s^\varepsilon(\Psi^\varepsilon)'(\phi^\varepsilon) + \\ - \int_{m_T} \Delta\phi^\varepsilon\Omega'(\phi^\varepsilon)$$

$$\int_m \Psi^\varepsilon_m(\phi^\varepsilon_m(T)) + \kappa_m \int_{m_T} |\Delta\phi^\varepsilon_m|^2 + (f^\varepsilon_m)''(\phi^\varepsilon_m)|\nabla\phi^\varepsilon_m|^2 \leq \int_m \Psi^\varepsilon_m(\phi^0_m) - \int_{m_T} s^\varepsilon(\Psi^\varepsilon_m)'(\phi^\varepsilon_m)$$

Dealing with the degenerate mobilites

It follows that

$$\int_V \Psi^\varepsilon(\phi^\varepsilon(T)) + \kappa \int_{V_T} |\Delta\phi^\varepsilon|^2 + (f^\varepsilon)''(\phi^\varepsilon)|\nabla\phi^\varepsilon|^2 \leq \int_V \Psi^\varepsilon(\phi^0) + \int_{m_T} s^\varepsilon(\Psi^\varepsilon)'(\phi^\varepsilon) + \\ - \int_{m_T} \Delta\phi^\varepsilon\Omega'(\phi^\varepsilon)$$

$$\int_m \Psi_m^\varepsilon(\phi_m^\varepsilon(T)) + \kappa_m \int_{m_T} |\Delta\phi_m^\varepsilon|^2 + (f_m^\varepsilon)''(\phi_m^\varepsilon)|\nabla\phi_m^\varepsilon|^2 \leq \int_m \Psi_m^\varepsilon(\phi_m^0) - \int_{m_T} s^\varepsilon(\Psi_m^\varepsilon)'(\phi_m^\varepsilon)$$

Further, for any $z > 1$ and $z < 0$, by making an expansion of $\Psi^\varepsilon(z)$ around $1-\varepsilon$ and $-\varepsilon$ respectively. we obtain that

$$\int_V (\phi^\varepsilon - 1)_+^2 + \int_V (\phi_-^\varepsilon)^2 \leq C\varepsilon \int_V \Psi^\varepsilon(\phi^\varepsilon)$$

Missing links and future ideas

1. In order to have the right compactness to pass to the limit $\varepsilon \to 0$, we need a uniform bound $\|\phi^\varepsilon\|_{L^2(0,T;H^2(V))} + \|\phi^\varepsilon\|_{L^2(0,T;H^2(V))} \leq C$, which is missing for now.
2. New source term and binding energy
3. Particle approximation
4. Stationary states

References

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