The concentration comparison for nonlinear diffusion on model manifolds

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The nonlinear diffusion problem

Given a (complete, noncompact) Riemannian model manifold $\mathbb{M}^n$, we consider the following nonlinear diffusion problem:

$$\begin{cases} \partial_t u = \Delta\phi(u) & \text{in } \mathbb{M}^n \times (0, +\infty)\,, \\ u = u_0 \geq 0 & \text{on } \mathbb{M}^n \times \{0\}\,, \end{cases} \tag{CP}$$

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Recall that a model manifold is a Riemannian manifold with a special spherical symmetry, namely its metric g can be written as

$$\mathrm{g} \equiv dr \otimes dr + \psi(r)^2\, \mathrm{g}_{\mathbb{S}^{n-1}}\,,$$

for some smooth $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(0) = 0$ and $\psi'(0) = 1$, where $r \equiv \mathrm{d}(x, o)$ stands for the distance from the pole $o \in \mathbb{M}^n$.

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In the Euclidean space $\mathbb{R}^n$ problem (CP) was studied by several authors such as Vázquez [1982], Eidus [1990], Eidus and Kamin [1994], Carrillo and Vázquez [2007], Grillo, M., and Punzo [2020].

The concept of weak energy solution

Let us set

$$\Phi(u) := \int_0^u \phi(v)\,dv \qquad \forall u \geq 0\,, \tag{1}$$

and denote by $\dot{H}^1(\mathbb{M}^n)$ the closure of $C_c^\infty(\mathbb{M}^n)$ w.r.t. the norm

$$\|f\|^2_{\dot{H}^1} := \|f\|^2_{L^2(B_1(o))} + \|\nabla f\|^2_{L^2(\mathbb{M}^n)}\,.$$

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Definition (Weak energy solutions)

Let $u_0 \geq 0$ with $u_0, \Phi(u_0) \in L^1(\mathbb{M}^n)$. We say that $u \geq 0$ is a weak energy solution of the Cauchy problem (CP) *if*

$$u \in L^1\left(\mathbb{M}^n \times (0,T)\right), \quad \phi(u) \in L^2((0,T); \dot{H}^1(\mathbb{M}^n)) \qquad \forall T > 0\,,$$

and the identity

$$\int_0^{+\infty} \int_{\mathbb{M}^n} u\, \partial_t \xi\, dVdt = \int_0^{+\infty} \int_{\mathbb{M}^n} \langle \nabla \phi(u), \nabla \xi \rangle\, dVdt - \int_{\mathbb{M}^n} u_0(x)\, \xi(x,0)\, dV(x)$$

holds for all $\xi \in C_c^1(\mathbb{M}^n \times [0,+\infty))$, where V is the volume measure of $\mathbb{M}^n$.

Rearrangements and concentration comparison

We denote by $\mathcal{L}_0(\mathbb{M}^n)$ the set of all measurable functions $f : \mathbb{M}^n \to \mathbb{R}$ whose positive upper level sets have finite measure, that is

$$\mu_f(t) := V(x \in \mathbb{M}^n : \ |f(x)| > t) < +\infty \qquad \forall t > 0\,.$$

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Following a well-known procedure (see *e.g.* Bennet and Sharpley [1988]), let us introduce the Schwarz rearrangement $f^\star$ of f:

$$f^\star(x) \equiv f^\star(r) := \int_0^{+\infty} \chi_{\{\mu_f(t) > V(B_r(o))\}} \, dt \, ,$$

i.e. the unique (up to null sets) radially decreasing function about o s.t.

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More in general, owing to the Cavalieri principle, we have

$$\int_{\mathbb{M}^n} F(f) \, dV = \int_{\mathbb{M}^n} F(f^\star) \, dV \tag{2}$$

for all Borel-measurable functions $F : [0, +\infty) \to [0, +\infty)$.

We then consider the following symmetrized Cauchy problem:

$$\begin{cases} \partial_t \overline{u} = \Delta \phi(\overline{u}) & \text{in } \mathbb{M}^n \times (0, +\infty)\,, \\ \overline{u} = u_0^\star \geq 0 & \text{on } \mathbb{M}^n \times \{0\}\,. \end{cases} \tag{CP*}$$

It can be shown that the solution $\overline{u}(\cdot, t)$ is always radially decreasing, hence it coincides at all times with its Schwarz rearrangement.

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We say that (CP) enjoys a concentration comparison if

$$\int_{B_r(o)} u^\star(x,t)\, dV(x) \leq \int_{B_r(o)} \overline{u}(x,t)\, dV(x) \qquad \forall r, t > 0\,,$$

that is, the original solution u is less concentrated than the solution $\overline{u}$ of the symmetrized problem (CP*), and we write $u^\star(\cdot, t) \prec \overline{u}(\cdot, t)$.

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The concentration comparison for (CP) in $\mathbb{R}^n$ was first proved by Vázquez [1982]. Note that a pointwise comparison of the type $u^\star(x,t) \leq \overline{u}(x,t)$ in general fails, differently from the elliptic case. Interestingly, the same phenomenon was recently observed by Ferone and Volzone [2021, 2023] for the fractional Poisson equation.

The Pólya-Szegő inequality

A functional tool that is strictly connected with the above concentration comparison is the Pólya-Szegő inequality, whose $L^2(\mathbb{M}^n)$ version reads

$$\int_{\mathbb{M}^n} |\nabla v^\star|^2 \, dV \le \int_{\mathbb{M}^n} |\nabla v|^2 \, dV \qquad \forall v \in \dot{H}^1(\mathbb{M}^n) \cap \mathcal{L}_0(\mathbb{M}^n) \, , \qquad \text{(PS)}$$

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In $\mathbb{R}^n$ it is well known that (PS) holds, as a consequence of the classical isoperimetric inequality. This was originally established by G. Pólya and G. Szegő in 1951, and later exploited by Talenti [1976] to compute the best constants in the Sobolev inequalities.

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On a general model manifold, there is currently no explicit condition guaranteeing (PS). However, as in the Euclidean space, one can show that it holds provided $\mathbb{M}^n$ supports a centered isoperimetric inequality.

By a centered isoperimetric inequality we mean that

$$\mathrm{Per}(B_r(o)) \leq \mathrm{Per}(\Omega) \qquad \forall \Omega \in \mathscr{B}_b(\mathbb{M}^n)\,, \tag{CII}$$

where $B_r(o)$ is the geodesic ball of radius $r > 0$, centered at the pole, having the same measure V as Ω, the symbol $\mathscr{B}_b(\mathbb{M}^n)$ stands for the class of (bounded) Borel sets in $\mathbb{M}^n$, and we let $\mathrm{Per}(\cdot)$ denote the perimeter functional in the usual dense of De Giorgi.

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Nevertheless, from the remarkable work of Brendle [2013] and a recent preprint of Maggi and Santilli [2023] on constant mean curvature surfaces, a key (necessary) condition seems to be the decreasing monotonicity of the scalar curvature

$$S(x) \equiv S(r) := -(n-1)\left[2\,\frac{\psi''(r)}{\psi(r)} + (n-2)\,\frac{[\psi'(r)]^2 - 1}{\psi(r)^2}\right].$$

Main results

After showing existence and uniqueness of weak energy solutions, we establish the following equivalence result.

Theorem 1 (M., Volzone, 2025)

Let $\mathbb{M}^n$ be a (complete, noncompact) model manifold. Let $\phi \not\equiv 0$ be any continuous nondecreasing function with $\phi(0) = 0$, and Φ its primitive according to (1). *Then the following properties are equivalent:*

(a) $\mathbb{M}^n$ *supports the Pólya-Szegő inequality* (PS)*;*

(b) *For every $u_0 \geq 0$ with $u_0, \Phi(u_0) \in L^1(\mathbb{M}^n)$, it holds*

$$u^\star(\cdot, t) \prec \overline{u}(\cdot, t) \qquad \forall t > 0 \, ,$$

where u is the weak energy solution of the Cauchy problem (CP) *and $\overline{u}$ is the weak energy solution of the symmetrized Cauchy problem* (CP*)*.*

Such a result remains true when restricted to radial functions.

Theorem 2 (M., Volzone, 2025)

Let the assumptions of Theorem 1 hold. Then the following properties are equivalent:

(a) $\mathbb{M}^n$ *supports the Pólya-Szegő inequality* (PS) *restricted to radial functions;*

(b) *For every radial* $u_0 \geq 0$ *with* $u_0, \Phi(u_0) \in L^1(\mathbb{M}^n)$*, it holds*

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$$u^\star(\cdot,t) \prec \overline{u}(\cdot,t) \qquad \forall t > 0\,.$$

In this setting, a clear condition for the validity of (PS) is available.

Proposition (M., Volzone, 2025)

Suppose that the model function ψ *has the following property:*

$$\psi(R)^{n-1} \leq \psi(S)^{n-1} + \psi(T)^{n-1} \quad \text{for every } R, S, T > 0 \text{ s.t.}$$

$$\int_0^R \psi(r)^{n-1}\,dr = \int_0^S \psi(r)^{n-1}\,dr - \int_0^T \psi(r)^{n-1}\,dr\,.$$

Then $\mathbb{M}^n$ *supports the Pólya-Szegő inequality* (PS) *restricted to radial functions. In particular, this is always the case if* ψ *is increasing.*

On the implication (a)⇒(b)

Following an approach originally developed by Vázquez [1982] in $\mathbb{R}^n$, instead of proving the concentration comparison directly on u and $\overline{u}$, it is convenient to work with the following discretized semilinear elliptic Dirichlet problems:

$$\begin{cases} -\Delta\phi(u_{i+1}) + \frac{1}{h}\, u_{i+1} = \frac{1}{h}\, u_i & \text{in } B_R \,, \\ \phi(u_{i+1}) = 0 & \text{on } \partial B_R \,, \end{cases} \tag{3}$$

which correspond to the usual Euler implicit discretization of (CP) in B_R.

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Powerful abstract tools such as the Crandall-Liggett theorem guarantee that $\{u_i\}$, suitably interpolated, converges locally uniformly as $h \to 0^+$ in $L^1(B_R)$ to the solution u_R of the localized version of (CP) in B_R with homogeneous Dirichlet boundary conditions. A further passage to the limit as $R \to +\infty$ yields (monotone) convergence to u. The same can be done for $\overline{u}$.

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A key ingredient to prove the concentration comparison for (3) is estimating

$$\frac{d}{dt} \int_{\{\phi(u_i)>t\}} |\nabla\phi(u_i)|^2 \, dV\,,$$

which, for the parabolic problem, is always harder. On the contrary, it is easier to handle for (3) upon adapting classical arguments by Talenti [1976, 1979].

Sketch of proof of the implication (b)⇒(a)

Our proof is inspired by an argument that was successfully exploited for the heat equation (see *e.g.* Lieb and Loss [2001]). For simplicity, we discuss the case when ϕ is bijective only.

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First of all, given any nonnegative $v \in L^\infty(\mathbb{M}^n) \cap H^1_c(\mathbb{M}^n)$ and any $h > 0$, from the weak formulations of u and $\overline{u}$ one can deduce the following identity:

$$-\int_{\mathbb{M}^n} u(x,h)\, v(x)\, dV(x) = \int_{\mathbb{M}^n} \left\langle \nabla v, \int_0^h \nabla\phi(u)\, dt \right\rangle dV - \int_{\mathbb{M}^n} u_0\, v\, dV\,. \quad (4)$$

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Similarly, if with repeat the same computations with u replaced by $\overline{u}$ and v replaced by $v^\star$, we obtain

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Since $u^\star(\cdot,h) \prec \overline{u}(\cdot,h)$ by assumption and $\overline{u}(\cdot,h)$ is radially decreasing, thanks to the Hardy-Littlewood inequality we have:

$$\int_{\mathbb{M}^n} u(x,h)\, v(x)\, dV(x) \le \int_{\mathbb{M}^n} u^\star(x,h)\, v^\star(x)\, dV(x) \le \int_{\mathbb{M}^n} \overline{u}(x,h)\, v^\star(x)\, dV(x)\,.$$

It is then feasible to pick $u_0 = \phi^{-1}(v)$, which belongs to $L^1(\mathbb{M}^n)$ since v is uniformly bounded with compact support and $\phi^{-1}(0) = 0$. For the same reason, we also have $\Phi(u_0) \in L^1(\mathbb{M}^n)$. Moreover, we notice that

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By combining (4), (5), the Hardy-Littlewood inequality, and (7), we end up with

$$\int_{\mathbb{M}^n} \left\langle \nabla v^\star, \frac{1}{h}\int_0^h \nabla\phi(\overline{u})\,dt \right\rangle dV \le \int_{\mathbb{M}^n} \left\langle \nabla v, \frac{1}{h}\int_0^h \nabla\phi(u)\,dt \right\rangle dV \qquad \forall h > 0\,,$$

so that by letting $h \to 0^+$ and using (6) we finally obtain

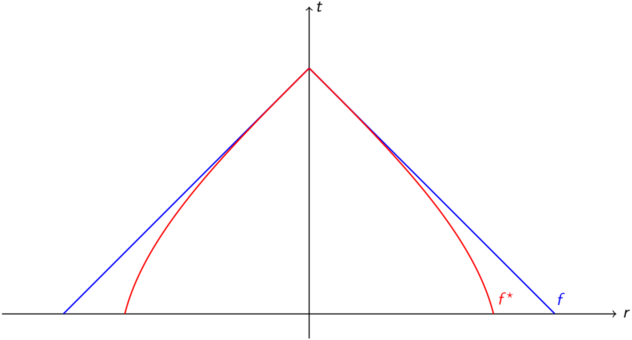
$$\int_{\mathbb{M}^n} |\nabla v^\star|^2\,dV \le \int_{\mathbb{M}^n} |\nabla v|^2\,dV\,.$$

On the failure of the Pólya-Szegő inequality

Assume that $\mathbb{M}^n$ is a model manifold such that $S(\hat{o}) > S(o)$ for some $\hat{o} \in \mathbb{M}^n$. Consider the function

$$f(x) = (1 - \mathrm{d}(x, \hat{o}))^+$$

and its Schwarz rearrangement $f^\star$. The following figure depicts the radial (w.r.t. to o and $\hat{o}$, respectively) approximate profiles of f and $f^\star$:



Surprisingly, $f^\star$ looks steeper than f...

By exploiting the well-known asymptotic formula

$$V(B_r(x)) = \frac{\omega_n}{n}\, r^n \left(1 - \frac{S(x)}{6(n+2)}\, r^2 + \mathcal{O}\left(r^3\right)\right),$$

we can turn the above intuition into a rigorous result.

Theorem 3 (M., Volzone, 2025)

Let $\mathbb{M}^n$ be any model manifold such that there exists some $\hat{o} \in \mathbb{M}^n$ for which $S(\hat{o}) > S(o)$. Then $\mathbb{M}^n$ does not support the Pólya-Szegő inequality (PS).

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THANK YOU FOR YOUR ATTENTION!

GRAZIE PER L'ATTENZIONE!