Phragmèn-Lindelöf type results for a class of parabolic equations on infinite graphs

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New Perspectives in Nonlocal and Nonlinear PDEs

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Statement of the problem

We investigate the *Phragmèn-Lindelöf principle* for parabolic Cauchy problems of the form

$$\begin{cases} \rho(x)u_t - \Delta u = 0 & \text{in } G \times (0,T) \\ u = 0 & \text{in } G \times \{0\}, \end{cases} \tag{1}$$

where

- (G, ω, μ) is an *infinite weighted* graph with *edge-weight* ω and *node (or vertex) measure* μ,
- the *density* ρ is a positive function defined on G,
- Δ denotes the *weighted Laplace operator* on G.

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If time permits, we will also briefly discuss an associated *elliptic* equation.

The results are contained in:

- S. Biagi, G. Meglioli, F.P., *Phragmèn-Lindelöf type theorems for parabolic equations on infinite graphs*, preprint (2025);

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Plan of the Talk

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The problem on $\mathbb{R}^n$

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There is a huge literature on the topic. We recall an important Phragmèn-Lindelöf principle in [Eidelman, Kamin, Porper, 2000].

Suppose that, for some $\rho_0 > 0, q > 0$,

$$\rho(x) \geq \rho_0 (1 + |x|^2)^{\frac{q-2}{2}}.$$

If u is a solution to

$$\begin{cases} \rho(x) u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

and it satisfies, for some $k > 0$,

$$\lim_{|x| \to \infty} \frac{\max_{t \in [0, T]} |u(x,t)|}{e^{k|x|^q}} = 0,$$

then

$$u \equiv 0.$$

The inhomogeneous porous medium equation on $\mathbb{R}^n$

Also problem

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has been largely studied in the literature. In particular, uniqueness of solutions has been addressed.

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Some authors: J.L. Vazquez, G. Reyes, G. Grillo, M. Muratori, S. Kamin, D. Eidus, R. Kersner, A. Tesei, F. P., G. Meglioli, T. W. Petitt.

The problem in the Riemannian setting

In [Grigor'yan, Meglioli, Roncoroni (2025)] uniqueness for problem

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is studied, M being a complete noncompact Riemannian manifold (see also [Grigor'yan], [Ishige, Murata (2001)], [F.P. (2015)] for $\rho \equiv 1$).

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Some *integral* growth conditions on the solution u, depending on the behaviour of ρ, are required. Clearly, the integral condition takes into account also the Riemannian measure of M.

The problem on graphs: motivations

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PDEs on combinatorial graphs provide a powerful framework for modeling diffusion, transport, and dynamic processes on discrete structures. They are crucial in understanding phenomena in networks such as

- social graphs,
- electrical circuits,
- the internet,
- molecular structures,
- neural systems.

Moreover, they are increasingly relevant in

- artificial intelligence,
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For a deeper exploration, see for instance the monographs:

- Keller, Lenz, Wojciechowski, "Graphs and Discrete Dirichlet Spaces";
- L.J. Grady, J. R. Polimeni "Discrete Calculus".

They include several chapters devoted to these topics with many illustrative examples.

The problem on graphs: the mathematical framework

A graph (G, ω, μ) is a triplet, where

- G is a countable set of *vertices* or *nodes*,
- $\omega : G \times G \to [0, +\infty)$ is a function, called *edge weight*,
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where the function ω fulfills

- (*no loops*) $\omega(x, x) = 0 \quad$ for all $x \in G$,
- (*symmetry*) $\omega(x, y) = \omega(y, x) \quad$ for all $(x, y) \in G \times G$,
- (*finite sum*) $\sum_{y \in G} \omega(x, y) < \infty \quad$ for all $x \in G$.

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For any $x, y \in G$, we say that (x, y) is an *edge* of the graph, or that x, y are *neighbors* or *connected* and we will write $x \sim y$ if and only if $\omega(x, y) > 0$.

We specify that we deal with

- *undirected graphs*, that is the edges do not have an orientation;
- it is not possible that an edge connects a vertex to itself, so the graphs do not possess *loops*,
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A collection of vertices $\gamma \equiv \{x_k\}_{k=0}^{n} \subset G$ is called a *path* if $x_k \sim x_{k+1}$ for all $k = 0, \dots, n-1$.

A graph is *connected* if, for any two vertices $x, y \in G$, there exists a path joining x to y.

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For every $A \subset G$, $\mu(A) = \sum_{x \in A} \mu(x).$

Let $\mathfrak{F}$ denote the set of all functions $f : G \to \mathbb{R}$. For any $f \in \mathfrak{F}$ and for all $x, y \in G$ let us introduce the *difference operator*

$$\nabla_{xy} f := f(y) - f(x) \,.$$

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The gradient squared of $f \in \mathfrak{F}$ is defined as

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For any $f \in \mathfrak{F}$, the *(weighted) Laplace operator* of (G, ω, μ) is

$$\begin{aligned} \Delta f(x) &:= \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)[f(y) - f(x)] \\ &= \frac{1}{\mu(x)} \sum_{y \sim x} \omega(x, y)(\nabla_{xy} f), \quad x \in G . \end{aligned}$$

Some remarks:

- for any $f, g \in \mathfrak{F}$, the *product rule* holds

$$\nabla_{xy}(fg) = f(x)(\nabla_{xy} g) + (\nabla_{xy} f)g(y) \quad \text{for all } x, y \in G\,;$$

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$$\nabla_{xy}(fg) = f(x)(\nabla_{xy}g) + (\nabla_{xy}f)g(y) \quad \text{for all } x, y \in G\,;$$

- if at least one of the functions $f, g \in \mathfrak{F}$ has *finite* support, then the following *integration by parts formula* holds

$$\begin{aligned}\sum_{x\in G}[\Delta f(x)]g(x)\mu(x) &= -\frac{1}{2}\sum_{x,y\in G}(\nabla_{xy}f)(\nabla_{xy}g)\omega(x,y) \\ &= \sum_{x\in G} f(x)[\Delta g(x)]\mu(x)\,;\end{aligned}$$

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- the operators are somehow *nonlocal* in nature;
- there is no *chain rule*.

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Some uniqueness result for problem (1) with $\rho \equiv 1$ have been established in [Huang (2012)], [X. Huang, M. Keller, M. Schmidt (2020)], assuming that the solution belongs to some weighted ℓ^2 space.

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In these articles, completely different methods from those we will discuss today were used.

The general PL principle

In what follows we always assume that (G, ω, μ) is an infinite, connected, locally finite, undirected, weighted graph.

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Let d be a distance on G (such a d always exists).

Set $S_T := G \times (0, T)$.

Proposition 1

Let $\rho \in \mathfrak{F}$, $\rho > 0$, $x_0 \in G$. Suppose that there exists $Z \in \mathfrak{F}_T$, $Z(x,t) > 0$ in $\overline{S}_T$ such that

$$\rho(x)\,\partial_t Z(x,t) - \Delta Z(x,t) \geq 0 \quad \text{for all } (x,t) \in \overline{S}_T\,. \tag{2}$$

Let u be a solution of problem (1) fulfilling

$$\lim_{d(x,x_0)\to+\infty} \left\{ \max_{t\in[0,T]} \frac{|u(x,t)|}{Z(x,t)} \right\} = 0\,. \tag{3}$$

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This follows from the weak maximum principle applied to finite subgraphs, suitably combined with (3).

A special distance on G

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Let $\Omega \subset G$ be finite subset. By means of $\bar{d}$, we define the combinatorial distance from any $x \in G$ to the subset Ω

$$r(x) := \min_{y \in \Omega} \bar{d}(x, y) \quad \forall x \in G\,.$$

Outer and inner degree (or curvature)

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For any $x \in G$ with $r \equiv r(x) \geq 1$, let

$$\mathfrak{D}_+(x) := \frac{1}{\mu(x)} \sum_{y \in \mathcal{S}_{m+1}(\Omega)} \omega(x,y), \quad \mathfrak{D}_-(x) := \frac{1}{\mu(x)} \sum_{y \in \mathcal{S}_{m-1}(\Omega)} \omega(x,y)\,.$$

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The function $\mathfrak{D}_+ : G \to [0, +\infty)$ is called *outer degree* (or *outer curvature*) w.r.t. Ω, whereas $\mathfrak{D}_- : G \to [0, +\infty)$ is called *inner degree* (or *inner curvature*) w.r.t. Ω.

Given $f \in \mathfrak{F}$, we say that f is *spherically symmetric w.r.t.* Ω if

$$f(x) = f(y) \quad \text{whenever } r(x) = r(y).$$

In this case, with a slight abuse of notation, we write

$$f(x) = f(m) \quad \forall x \in \mathcal{S}_m(\Omega)\,.$$

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Lemma 1

Let $\Omega \subset G$ be a finite set and let $f \in \mathfrak{F}$ be a spherically symmetric function with respect to Ω. Then

$$\Delta f(x) = \mathfrak{D}_+(x)[f(r+1) - f(r)] + \mathfrak{D}_-(x)[f(r-1) - f(r)] \tag{4}$$

for any $x \in G$ with $r \equiv r(x) \geq 1$.

Spherically symmetric graphs

The weighted graph (G, μ, ω), endowed with the combinatorial distance r, is said to be *weakly spherically symmetric* with respect to a finite subset $\Omega \subset G$, if the outer and inner degrees $\mathfrak{D}_{\pm}$ are spherically symmetric with respect to Ω.

Therefore, on a weakly symmetric graph,

$$\mathfrak{D}_{\pm}(x) = \mathfrak{D}_{\pm}(m) \quad \forall x \in S_m(\Omega) .$$

Trees

Let (G, ω, μ) be a weakly symmetric graph w.r.t. $\Omega = \{o\}$, for some fixed point $o \in G$ (which is usually referred to as the *root of* G). Suppose that

- $\omega : G \times G \to \{0, 1\}$;

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- $\omega : G \times G \to \{0, 1\}$;
- $\omega|_{S_m(\Omega)\times S_m(\Omega)} = 0$;
- $\mu(x) = 1$ for every $x \in G$;
- there exists $b : \mathbb{N} \to \mathbb{N}$, which is called the *branching function*, such that

$$\mathfrak{D}_+(x) = b(m), \quad \mathfrak{D}_-(x) = 1 \quad \text{for every } x \in S_m(\Omega) \text{ and } m \in \mathbb{N}.$$

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$$\mathfrak{D}_+(x) = b(m), \quad \mathfrak{D}_-(x) = 1 \quad \text{for every } x \in S_m(\Omega) \text{ and } m \in \mathbb{N}.$$

In this case, G is a *tree* with branching function $b(m)$.

General PL principle with combinatorial distance

Theorem 1

Let $\Omega \subset G$ be a finite subset. Suppose that, for some $\rho_0 > 0$,

$$\rho \in \mathfrak{F}, \quad \rho(x) \geq \rho_0 \frac{\mathfrak{D}_+(x)}{r+1} \quad \text{for all } \; x \in G, \tag{5}$$

Let u be a solution of problem (1) fulfilling

$$\lim_{r \to +\infty} \frac{1}{\tilde{Z}(x)} \left\{ \max_{t \in [0,T]} |u(x,t)| \right\} = 0\,, \tag{6}$$

$$\tilde{Z}(x) := e^{B(r+1)}\,, \quad \text{for all } \; x \in G \quad (B > 0). \tag{7}$$

Then

$$u \equiv 0 \quad \text{in } \; S_T.$$

Proof. We define the function

$$Z(x,t) := e^{A(1+Qt)(r+1)} \qquad (A > 0, Q > 0).$$

By means of Lemma 1, we show that Z fulfills the assumptions of Proposition 1 with $d = r$.

Theorem 2

Let $\Omega \subset G$ be a finite subset. Suppose that

$$\rho \in \mathfrak{F}, \quad \rho(x) \geq \frac{\mathfrak{D}_+(x)}{r+1}\, e^{\rho_0 \log^\beta (r+2)} \quad \text{for all } \; x \in G, \tag{8}$$

for some $\beta \in (0, 1]$ and $\rho_0 > 0$. Let u be a solution of problem (1) fulfilling

$$\lim_{r \to +\infty} \frac{1}{\hat{Z}(x)} \left\{ \max_{t \in [0,T]} |u(x,t)| \right\} = 0\,, \tag{9}$$

where, for some $B > 0$,

$$\hat{Z}(x) := e^{B(r+1) \log^\beta (r+2)}\,, \quad \text{for all } \; x \in G. \tag{10}$$

Then

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Proof. We define the function

$$Z(x,t) := e^{A(1+Qt)(r+1)\log^{\beta}(r+1)} \qquad (A>0, Q>0, 0<\beta<1).$$

By means of Lemma 1, we show that Z fulfills the assumptions of Proposition 1, with $d=r$.

Optimality of ρ on trees

Theorem 3

Let (G, ω, μ) be a tree as above, with *constant branching function* $b(r) = b_0 \geq 2$. Assume that $\rho \in \mathfrak{F}$, $\rho > 0$ on G fulfills

$$\rho(x) \leq c_0(1+r)^{-\alpha} \quad \text{for any } x \in G,$$

for some $c_0 > 0, \alpha > 1$. Then for every fixed $\gamma \in \mathbb{R}$ and every $u_0 \in \mathfrak{F}$ satisfying

$$u_0 \geq \gamma \text{ on } G \quad \text{and} \quad u_0 \equiv \gamma \text{ out of } B_{\hat{R}}(o), \tag{11}$$

there exists a solution u to problem (1) such that

$$u(x, t_0) \to \gamma \text{ as } r(x) \to +\infty \quad \text{for every } t_0 > 0. \tag{12}$$

The previous theorem is a consequence of the following general criterium.

Theorem 4

Let $\rho \in \mathfrak{F}$, $\rho > 0$. We assume that there exist a function $h \in \mathfrak{F}$ and a ball $B_{\hat{R}}(o) \subseteq G$ such that

$$\begin{aligned} \text{i) } & \Delta h \leq -\rho \quad \text{in } \; G \setminus B_{\hat{R}}(o), \\ \text{ii) } & h > 0 \quad \text{in } \; G, \\ \text{iii) } & h(x) \to 0 \text{ as } r(x) \to +\infty. \end{aligned} \tag{13}$$

Then there exist infinitely many bounded solutions u of problem (1). In particular, for every fixed $\gamma \in \mathbb{R}$ and every $u_0 \in \mathfrak{F}$ satisfying (11) there exists a solution u to problem (1) such that (12) holds.

In order to obtain the existence of *infinitely many bounded* solutions on a tree, we use the previous criterium, after having shown that

$$h(x) = \frac{K}{[1 + r(x)]^{\beta}}, \quad x \in G,$$

satisfies (13), for suitable $K > 0, \beta > 0$.

Further results on $\mathbb{Z}^n$

We now consider the $n-$dimensional *integer lattice graph*, i.e. $G = \mathbb{Z}^n$. We recall that, $x \sim y$ if and only if there exists $k \in \{1, \ldots, n\}$ such that $x_k = y_k \pm 1$ and $x_i = y_i$ for $i \neq k$.

We define the edge weight and the node measure as

$$\omega : \mathbb{Z}^n \times \mathbb{Z}^n \to [0, +\infty); \qquad \omega(x, y) = \begin{cases} 1 & \text{if } y \sim x \\ 0 & \text{if } y \not\sim x\,, \end{cases}$$

$$\mu(x) = \sum_{y \in \mathbb{Z}^n} \omega(x, y) = 2n\,.$$

We equip the graph $(\mathbb{Z}^n, \omega, \mu)$ with the Euclidean distance

$$|x - y| = \left(\sum_{k=1}^{n} |x_k - y_k|^2 \right)^{\frac{1}{2}} \quad (x, y \in \mathbb{Z}^n) . \tag{14}$$

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This is not true on $\mathbb{Z}^n$.

PL principle on $\mathbb{Z}^n$, $n \geq 3$

Theorem 5

Let $G = \mathbb{Z}^n, n \geq 3$. Let u be a solution of equation (1) and ρ such that, for some $\alpha \geq 2$,

$$\rho \in \mathfrak{F}, \quad \rho(x) \geq c_0 \, (1 + |x|)^{-\alpha} \quad \text{for all} \;\; x \in G.$$

Furthermore, assume that u fulfills

$$\lim_{|x| \to \infty} \frac{\max_{t \in [0,T]} |u(x,t)|}{\overline{Z}(x)} = 0,$$

where

$$\overline{Z}(x) := \begin{cases} e^{B|x|^{\min\{1, 2-\alpha\}}} & \text{if } \alpha \in [0,2) \\ e^{B \log^2(2+|x|^2)} & \text{if } \alpha = 2 \end{cases}. \tag{15}$$

Then

$$u(x) \equiv 0 \quad \forall x \in G.$$

Proof. We define

$$Z(x) := \begin{cases} e^{(A+Qt)(1+|x|^2)^{\min\{1,2-\alpha\}}} & \text{if } \alpha \in [0,2), \\ e^{(A+Qt)\log^2(2+|x|^2)} & \text{if } \alpha = 2\,. \end{cases}$$

We show that Z satisfies the hypotheses of Proposition 1, by direct computation and by using the following

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Remark 1

Let $x \in \mathbb{Z}^n$ and consider some $y \in \mathbb{Z}^n$, $y \sim x$. Then

$$|y|^2 - |x|^2 = (|x|^2 \pm 2x_k + 1) - |x|^2 = \pm 2x_k + 1,$$

$$(|y|^2 - |x|^2)^2 = 4x_k^2 + 1 \pm 2x_k\,.$$

Thus, by summing over all the $y \sim x$ we get

$$\sum_{y\sim x}\left(|y|^2 - |x|^2\right) = 2n, \quad \text{and} \quad \sum_{y\sim x}(|y|^2 - |x|^2)^2 = 8|x|^2 + 2n\,. \qquad (16)$$

A comment about $\rho \equiv 1$

Let $\rho \equiv 1$. In this case, $u \equiv 0$ is the unique solution, provided

$$\lim_{|x|\to\infty} \frac{\max_{t\in[0,T]} |u(x,t)|}{e^{B|x|}} = 0\,.$$

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This is in agreement with a result in [Huang (2012)], where it is shown that we have nonuniqueness of solutions u satisfying

$$\max_{t\in[0,T]} u(x,t) \sim e^{c|x|\log|x|} \quad \text{as } |x| \to \infty\,.$$

for a suitable $c > 0$.

Optimality of decay condition of ρ on $\mathbb{Z}^n$, $n \geq 3$

Theorem 6

Let $G = \mathbb{Z}^n, n \geq 3$. Assume that

$$\rho \in \mathfrak{F}, \quad 0 < \rho(x) \leq c_0 \, (1 + |x|)^{-\alpha} \quad \text{for all } \; x \in G,$$

for some $\alpha > 2$. Then for every fixed $\gamma \in \mathbb{R}$ and every $u_0 \in \mathfrak{F}$ satisfying (11) there exists a solution u to problem (1) sastisfying (12).

Special cases: $\mathbb{Z}^2$ and the antitree

Now we see that on $\mathbb{Z}^2$ and anti-trees, problem (1) admits a unique solution satisfying an appropriate growth condition at infinity, for every $\rho \in \mathfrak{F}, \rho > 0$ in G.

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Theorem 7

Let $\rho \in \mathfrak{F}, \rho > 0$ in $\mathbb{Z}^2$. Let u be a solution of problem (1) fulfilling

$$\lim_{|x| \to +\infty} \frac{1}{\log(\log |x|^2))} \left\{ \max_{t \in [0,T]} |u(x,t)| \right\} = 0 \, .$$

Then

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Proof

$$Z(x,t) := K e^{\gamma t} \log(\log(|x|^2 + 4))$$

satisfies the hypotheses of Proposition 1 for suitable $K > 0, \gamma > 0$.

Anti-tree

Let $\Omega = \{o\}$ for some point $o \in G$. Let $s : \mathbb{N} \to \mathbb{N}$ be given by

$$s(m) = \mathrm{card}[S_m(o)] \quad \text{for all } m \in \mathbb{N}.$$

We then say that G is an *anti-tree* with sphere size s if

$$\mathfrak{D}_{\pm}(x) = s(m) \qquad \text{for all } x \in S_{m\pm1}(o), m \in \mathbb{N}, m \geq 1.$$

Theorem 8

Let G be an anti-tree with size s. Let $\rho \in \mathfrak{F}, \rho > 0$ in G. Let u be a solution of problem (1) fulfilling

$$\lim_{r \to +\infty} \frac{1}{r} \left\{ \max_{t \in [0,T]} |u(x,t)| \right\} = 0 \,,$$

Then

$$u \equiv 0 \quad \text{in} \;\; S_T \,.$$

Theorem 8

Let G be an anti-tree with size s. Let $\rho \in \mathfrak{F}, \rho > 0$ in G. Let u be a solution of problem (1) fulfilling

$$\lim_{r \to +\infty} \frac{1}{r} \left\{ \max_{t \in [0,T]} |u(x,t)| \right\} = 0\,,$$

Then

$$u \equiv 0 \quad \text{in} \;\; S_T\,.$$

Proof. We define

$$Z(x,t) := e^{\gamma t}(Kr + 1).$$

By Lemma 1 we show that Z fulfills the hypotheses of Proposition 1.

Elliptic equations

Similar results hold also for the elliptic Schrödinger equation

$$\Delta u - V(x)u = 0 \quad \text{in } G\,,$$

where $V > 0$, V can tend to 0 at infinity.

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The threshold effects for the decay condition of V (in dependence of the graph) are similar.

Future perspectives

- Uniqueness in ℓ^∞ of solutions of the porous medium equation

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- Uniqueness of possible *unbounded* solutions.

Thank you!