Nonlocal approximation of an anisotropic cross-diffusion system

— joint work with T. Dębiec

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New Perspectives in Nonlocal and Nonlinear PDEs
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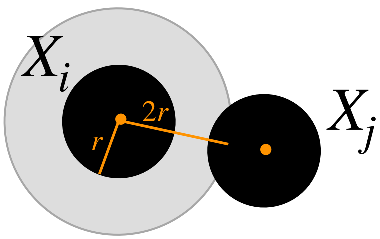
Degenerate Diffusion Equations & Localisation Limits
and where to find them…

Steric Particles & Excluded-Volume Interactions

$$\mathrm{d}\mathbf{X}_i(t) = \sqrt{2D}\,\mathrm{d}\mathbf{W}_{T,i}(t) \quad \text{s.t.} \quad |\mathbf{X}_i - \mathbf{X}_j| \geq 2r$$

$$\partial_t n = D\Delta n + \pi r^2 (N-1)\nabla\cdot(n\nabla n)$$



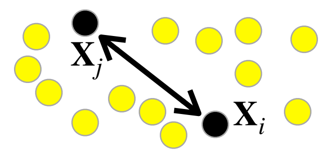
Nonlocal Approximation

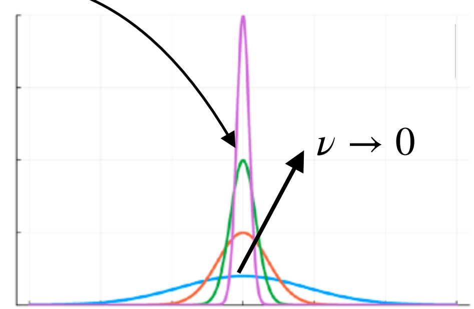
$$\partial_t n = \nabla\cdot(n\nabla K_\nu \star n)$$

$$K_\nu \to \delta_0$$

$$\partial_t n = \nabla\cdot(n\nabla n)$$

localised, repulsive interactions

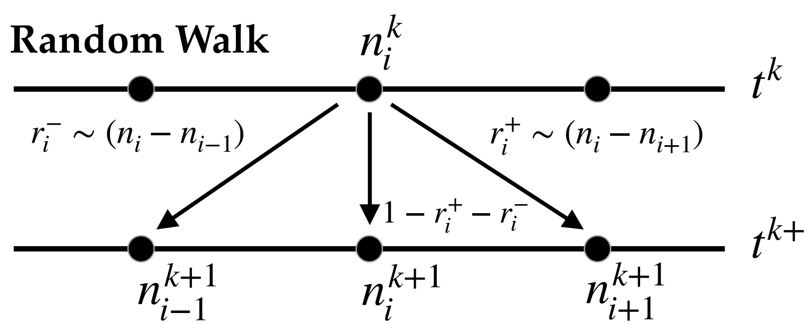




IV Inviscid Limit

$$\partial_t n - \nabla\cdot(n\nabla W) = 0$$
$$-\nu\Delta W + W = n$$
$$W = K_\nu \star n$$

II Biased Random Walk



Localisation Limits
— 'Historical' approach in a nutshell

Strong compactness of n comes from

a bound like $\iint |\nabla V_\nu \star n|^2 \,dxdt \approx \iint |\nabla n|^2 \,dxdt \leq C$

and Aubin-Lions / Riesz-Fréchet-Kolmogorov!

Consider

$$\frac{\partial n}{\partial t} = \nabla \cdot (n \nabla K_\nu \star n)$$

with $K_\nu = V_\nu \star V_\nu$ where $V_\nu \to \delta_0$

as $\nu \to 0$: $$\frac{\partial n}{\partial t} = \nabla \cdot (\underbrace{n}_{\text{strong}} \nabla \underbrace{n}_{\text{weak}})$$

Now consider the system

$$\frac{\partial n^{(i)}}{\partial t} = \nabla \cdot (n^{(i)} \nabla [K_\nu \star (n^{(1)} + n^{(2)})])$$

with

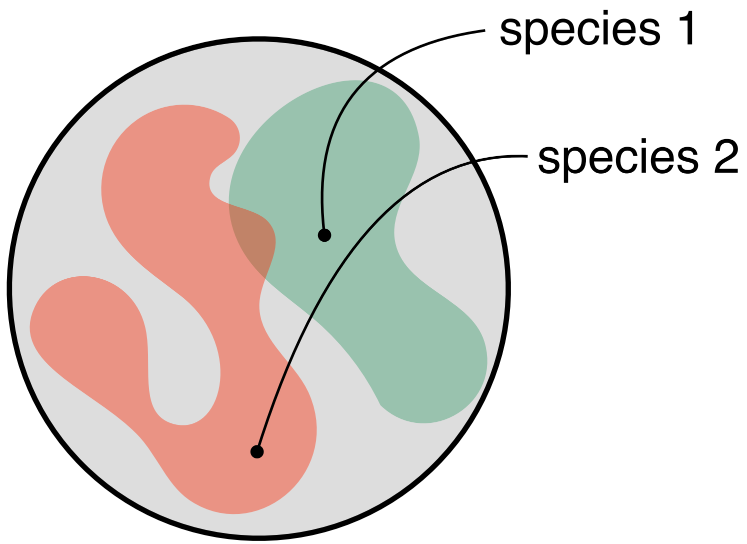
K_ν s.t. $K_\nu \to \delta_0$

as $\nu \to 0$:

$$\frac{\partial n^{(i)}}{\partial t} = \nabla \cdot \underbrace{(n^{(i)}}_{\text{weak}} \underbrace{\nabla (n^{(1)} + n^{(2)}))}_{\text{weak}}$$

Degond, Mustieles (**1990**), Oelschläger **(1990)**, Lions, Mas-Gallic (**2001**), Carrillo, Craig, Patacchini (**2019**), Burger, Esposito (**2023**), Carrillo, Esposito, Wu (**2023**), Carrillo, Esposito, Skrzeczkowski, Wu (**2024**), Craig, Jacobs, Turanova **(2025)**

Cross-diffusion systems as localisation limits
— What is known…



Burger-Esposito '23:

$$\frac{\partial n^{(i)}}{\partial t} = \nabla \cdot \left(n^{(i)}(a_{i1}\nabla n^{(1)} + a_{i2}\nabla n^{(2)})\right),\ i = 1{,}2$$

Doumic-Hecht-Perthame-Peurichard '24:

$$\frac{\partial n^{(i)}}{\partial t} = \nabla \cdot \left(n^{(i)} \sum_j a_{ij} \nabla n^{(j)}\right), \qquad i = 1,\ldots,N$$

Jüngel-Vetter-Zurek '24:

$$\frac{\partial n^{(i)}}{\partial t} - \epsilon \Delta n^{(i)} = \nabla \cdot \left(n^{(i)} \sum_j a_{ij} \nabla n^{(j)}\right) + n^{(i)}\left(b_{0i} - \sum_j b_{ij} n^{(j)}\right) \qquad i = 1,\ldots,N$$

Do not cover $a_{ij} \equiv 1$.

David-Dębiec-Mandal-S. '24: $a_{ij} \equiv 1$ for $N = 2$

Elbar-Skrzeczkowski '25: $a_{ij} \equiv 1$ for $N = 2$ + nonlinear

Dębiec-Mandal-S. '25: $a_{ij} \equiv 1$ for $N \in \mathbb{N}$ and $N \to \infty$

$$\frac{\partial n^{(i)}}{\partial t} = \nabla \cdot \left(n^{(i)} \nabla \sum_j n^{(j)}\right)$$

Cross-diffusion systems as localisation limits
— Degenerate hyperbolic-parabolic system

two balance laws
(one per species)

$$\frac{\partial n^{(1)}}{\partial t} - \nabla\cdot\left(n^{(1)}\nabla\left(n^{(1)}+n^{(2)}\right)\right) = n^{(1)}G^{(1)}(n)$$

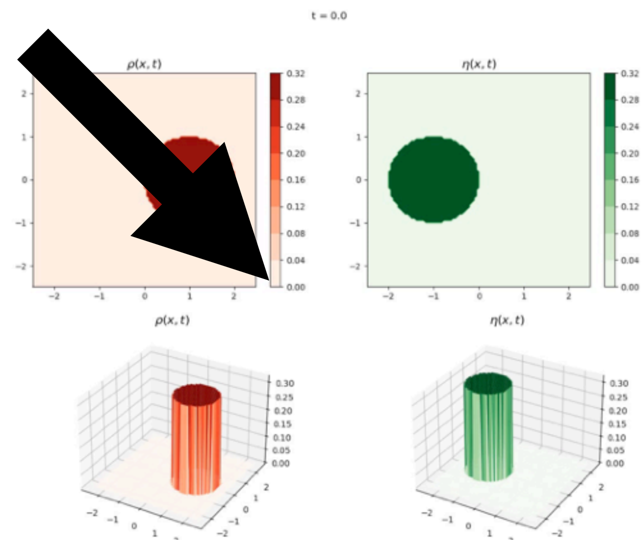
$$\frac{\partial n^{(2)}}{\partial t} - \nabla\cdot\left(n^{(2)}\nabla\left(n^{(1)}+n^{(2)}\right)\right) = n^{(2)}G^{(2)}(n)$$

joint population pressure

$$n := n^{(1)} + n^{(2)}$$

pressure-dependent growth rate

Heinrich, Alert, Wolf, Kosmrli, Cohen 2022



Busenberg/Travis; Gurtin/Pipkin (**1983/84**), Bertsch, Gurtin, Hilhorst, Peletier, Mimura, Izuhara,… (**2014/16**), Carrillo, Huang, S. (**2018**), Carrillo, Fagioli, Santambrogio, S. (**2018**), Gwiazda, Perthame, Świerczewska-Gwiazda (**2019**), Chaplain, Lorenzi, Macfarlane (**2020**), Price & Xu (**2020**), Liu & Xu (**2021**), Jacobs (**2021/22**), David, Debiec, Mandal, S. (**2024**), …

Anisotropic cross-diffusion systems as localisation limits — Motivation.

The media in which tissues grow are characterised by their anisotropic nature, for instance, generated by the extracellular matrix.

Consider the following system

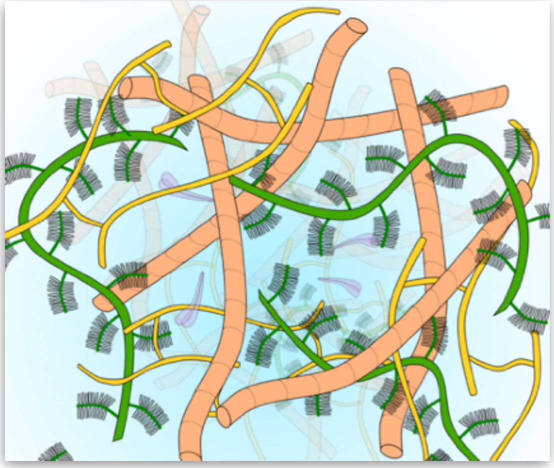
$$\frac{\partial n_\nu^{(1)}}{\partial t} = \nabla \cdot (n_\nu^{(1)} A \nabla m_\nu) + n_\nu^{(1)} G^{(1)}(n_\nu)$$
$$\frac{\partial n_\nu^{(2)}}{\partial t} = \nabla \cdot (n_\nu^{(2)} A \nabla m_\nu) + n_\nu^{(2)} G^{(2)}(n_\nu)$$

coupled through the anisotropic Brinkman law

$$-\nu \nabla \cdot (A \nabla m_\nu) + m_\nu = n_\nu$$

where $A = A(x,t) \in \mathbb{R}^{d\times d}$ is a given anisotropy tensor.

Extra**C**ellular **M**atrix



$A = A(x,t)$

Incorporating an anisotropy tensor in the model comes at the price of several analytical challenges…

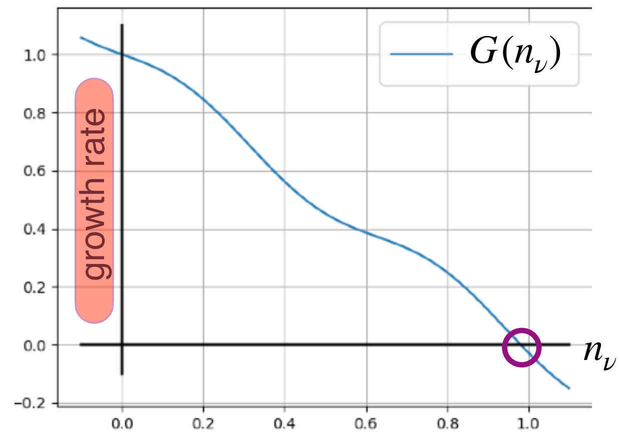
Anisotropic cross-diffusion systems as localisation limits
— Assumptions on data, growth, and anisotropy.

The anisotropy tensor:

- $A \in W^{1,\infty}(0,T; L^\infty(\mathbb{R}^d)) \cap L^{\frac{2q}{q-d}}(0,T; \dot{W}^{1,q}(\mathbb{R}^d))$, for some $q \geq d$
- $A(x,t)$ is symmetric for a.e. $(x,t) \in \mathbb{R}^d \times (0,T)$
- $A(x,t)$ is $\lambda-$uniformly elliptic for $\xi \cdot A(x,t)\xi \geq \lambda |\xi|^2, \forall \xi \in \mathbb{R}^d$, and a.e. (x,t)

The growth rates:

- $G^{(i)} : \mathbb{R} \to \mathbb{R}$ are C^1
- $(G^{(i)})' \leq -\alpha < 0$, for some $\alpha > 0$
- call $\bar{n}^{(i)} > 0$ s.t. $G^{(i)}(\bar{n}^{(i)}) = 0$, and set $\bar{n} := \bar{n}^{(1)} + \bar{n}^{(2)}$



The initial data:

- $0 \leq n^{(i),\text{in}} \leq \bar{n}^{(i)}$, for $i = 1,2$
- $n^{(i),\text{in}} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$
- $|x|^2(n^{(1),\text{in}} + n^{(2),\text{in}}) \in L^1(\mathbb{R}^d)$

„bounded and tight densities“

Main result — localisation limit for anisotropic system
— Statement

Under those assumptions, we prove the vanishing viscosity / localisation limit, $\nu \to 0$. Formally, the system becomes:

$$n_0 = n_0^{(1)} + n_0^{(2)}$$

$$\frac{\partial n_0^{(i)}}{\partial t} = \nabla \cdot (n_0^{(i)} A \nabla n_0) + n_0^{(i)} G^{(i)}(n_0), \qquad i = 1,2$$

a degenerate parabolic cross-diffusion system constituting the anisotropic analogue of the multi-species porous medium system associated to Darcy's law $v = -A\nabla(n_0^{(1)} + n_0^{(2)})$.

Starting point: weak solutions $n_\nu^{(i)} \in L^\infty(0,T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ nonnegative, s.t.:

$$-\int_0^T \int_{\mathbb{R}^d} n_\nu^{(i)} \partial_t \varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} n_\nu^{(i)} (A \nabla m_\nu) \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \varphi n_\nu^{(i)} G^{(i)}(n_\nu) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} \varphi(x,0) n^{(i),\mathrm{in}}(x) \mathrm{d}x$$

for each $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$, as well as

$$-\nu \nabla \cdot (A \nabla m_\nu) + m_\nu = n_\nu$$

almost everywhere.

Main result — localisation limit for anisotropic system — Statement

Under those assumptions, we prove the vanishing viscosity / localisation limit, $\nu \to 0$.
Formally, the system becomes:

$$n_0 = n_0^{(1)} + n_0^{(2)}$$

$$\frac{\partial n_0^{(i)}}{\partial t} = \nabla \cdot (n_0^{(i)} A \nabla n_0) + n_0^{(i)} G^{(i)}(n_0), \qquad i = 1,2$$

a degenerate parabolic cross-diffusion system constituting the anisotropic analogue of the multi-species porous medium system associated to Darcy's law $v = -A\nabla(n_0^{(1)} + n_0^{(2)})$.

For $\nu > 0$, let $(n_\nu^{(1)}, n_\nu^{(2)}, m_\nu)_{\nu>0}$ be a family of weak solutions to the nonlocal anisotropic model with initial data $(n^{(1),\mathrm{in}}, n^{(2),\mathrm{in}})$. Then, there exists a subsequence such that

- $n_\nu^{(i)} \rightharpoonup n_0^{(i)}$, weakly-* in $L^\infty(0,T; L^1 \cap L^\infty(\mathbb{R}^d))$, $i = 1,2$
- $n_\nu \to n_0$, strongly in $L^2(0,T; L^2(\mathbb{R}^d))$
- $\nabla m_\nu \to \nabla n_0$, strongly in $L^2(0,T; L^2(\mathbb{R}^d))$
- $m_\nu \to n_0$, strongly in $L^2(0,T; L^2(\mathbb{R}^d))$

where $n_0 = n_0^{(1)} + n_0^{(2)}$ and $(n_0^{(1)}, n_0^{(2)})$ is a weak solution to local system with initial data $(n^{(1),\mathrm{in}}, n^{(2),\mathrm{in}})$.

Proof. (localisation limit for anisotropic system)
— A priori estimates

L^∞**–bound** for n_ν.

Let (x^*, t^*) be a maximum point for n_ν, and assume that $\bar{n} < n(x^*, t^*)$. In such a point we have

$$\frac{\partial n_\nu}{\partial t} = 0, \quad \nabla n_\nu = 0, \quad G^{(i)}(n_\nu) < 0.$$

Therefore, at point (x^*, t^*) there holds

$$0 = \frac{\partial n_\nu}{\partial t} = \underbrace{\nabla n_\nu \cdot A \nabla m_\nu}_{=0} + n_\nu \nabla \cdot (A \nabla m_\nu) + \underbrace{n_\nu^{(1)} G^{(1)}(n_\nu) + n_\nu^{(2)} G^{(2)}(n_\nu)}_{<0}$$
$$< \frac{1}{\nu}\left(m_\nu - n_\nu\right) \leq 0$$

Proof of the last inequality. Set $C := n(x^*, t^*)$ and test the anisotropic Brinkman equation by $|C - m_\nu|_- := \max(m_\nu - C, 0)$ to get

$$-\nu |C - m_\nu|_- \nabla \cdot (A \nabla (m_\nu - C)) + |C - m_\nu|_- (m_\nu - C) = |C - m_\nu|_- (n_\nu - C) \leq 0$$

where the last inequality holds because $n_\nu \leq n_\nu(x^*, t^*) = C$.

Proof. (localisation limit for anisotropic system)
— A priori estimates

Integrating

$$-\nu|C - m_\nu|_- \nabla \cdot (A\nabla(m_\nu - C)) + |C - m_\nu|_-(m_\nu - C) = |C - m_\nu|_-(n_\nu - C) \le 0$$

in space and time yields

$$\nu \int_0^T \int_{\mathbb{R}^d} \nabla |C - m_\nu|_- \cdot A\nabla |C - m_\nu|_- \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} |C - m_\nu|_-^2 \,\mathrm{d}x\mathrm{d}t \le 0$$

By ellipticity of the anisotropy tensor A, both terms are nonnegative, and we conclude

$$|C - m_\nu|_- = 0, \qquad \text{i.e.,} \qquad m_\nu \le C$$

Hence, $m_\nu \le n_\nu$ in a maximum point. However, then $0 = \partial_t n_\nu < 0$, which is a contradiction, and therefore $n_\nu \le \bar{n}$.

Proof. (localisation limit for anisotropic system)
— A priori estimates

L^∞**–bound.** $n(x,t) \leq \bar{n}$.

L^1**– bound.** Integrate both equations to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n_\nu \mathrm{d}x \leq \max_{i=1,2} G^{(i)}(0)\int_{\mathbb{R}^d} n_\nu \mathrm{d}x$$

Regularity of m_ν. From testing the Brinkman equation

$$-\nu\nabla\cdot(A\,\nabla m_\nu) + m_\nu = n_\nu$$

by m_ν and using ellipticity, we deduce

$$\|\nu^{1/2}\nabla m_\nu\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \leq C$$

for some $C > 0$ independent of ν.

Proof. (localisation limit for anisotropic system)
— Regularity of the velocity field

Using the Brinkman equation $-\nu\nabla\cdot(A\nabla m_\nu)+m_\nu=n_\nu$ and properties of the anisotropy tensor A

$$A\in L^\infty(0,T;L^\infty(\mathbb{R}^d)),\qquad \nabla A\in L^{\frac{2q}{q-d}}(0,T;L^q(\mathbb{R}^d))$$

we can derive the following regularity:

- $\nabla\cdot(A\nabla m_\nu)\in L^\infty(0,T;L^\infty(\mathbb{R}^d))$, directly from Brinkman
- $m_\nu\in L^\infty(0,T;H^1(\mathbb{R}^d))$, testing by m_ν + ellipticity
- $D^2m_\nu\in L^2(0,T;L^2(\mathbb{R}^d))$, testing by $-\Delta m_\nu$ + ellipticity + interpolation
- $A\nabla m_\nu\in L^2(0,T;H^1(\mathbb{R}^d))$

$$\begin{aligned}
&\|\nabla(A\nabla m_\nu)\|^2_{L^2(0,T;L^2(\mathbb{R}^d))}\\
&\quad\le \|A\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\|D^2m_\nu\|^2_{L^2(0,T;L^2(\mathbb{R}^d))}+\int_0^T\int_{\mathbb{R}^d}|\nabla A|^2|\nabla m_\nu|^2\,\mathrm{d}x\mathrm{d}t\,.\\
&\qquad\qquad +2\|A\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_0^T\int_{\mathbb{R}^d}|\nabla m_\nu||\nabla A||D^2m_\nu|\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Proof. (localisation limit for anisotropic system)
— Towards dissipation control

Claim. There holds, uniformly in ν:

$$\int_0^T \int_{\mathbb{R}^d} n_\nu |\nabla m_\nu|^2 \mathrm{d}x\mathrm{d}t \leq C$$

Proof. Using the symmetry of A, we observe

$$\begin{aligned}\int_{\mathbb{R}^d} n_\nu \partial_t m_\nu \mathrm{d}x &= \int_{\mathbb{R}^d} \left(-\nu \nabla \cdot (A \nabla m_\nu) + m_\nu\right) \partial_t m_\nu \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \partial_t \left(-\nu \nabla \cdot (A \nabla m_\nu) + m_\nu\right) m_\nu \mathrm{d}x - \nu \int_{\mathbb{R}^d} \nabla m_\nu \cdot \partial_t A \nabla m_\nu \mathrm{d}x\end{aligned}$$

Hence, using the equation for n_ν and the ellipticity of A,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n_\nu m_\nu \mathrm{d}x &= \int_{\mathbb{R}^d} \partial_t n_\nu \, m_\nu \mathrm{d}x - \frac{\nu}{2}\int_{\mathbb{R}^d} \nabla m_\nu \cdot \partial_t A \nabla m_\nu \mathrm{d}x \\ &\leq -\int_{\mathbb{R}^d} n_\nu \nabla m_\nu \cdot A \nabla m_\nu \mathrm{d}x + C\int_{\mathbb{R}^d} n_\nu m_\nu \mathrm{d}x - \frac{\nu}{2}\int_{\mathbb{R}^d} \nabla m_\nu \cdot \partial_t A \nabla m_\nu \mathrm{d}x \\ &\leq -\lambda \int_{\mathbb{R}^d} n_\nu |\nabla m_\nu|^2 \mathrm{d}x + C\int_{\mathbb{R}^d} n_\nu m_\nu \mathrm{d}x + \frac{\nu}{2}\|\partial_t A\|_{L^\infty(\mathbb{R}^d)} \|\nabla m_\nu\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

Proof. (localisation limit for anisotropic system)
— Towards dissipation control

Moment & entropy control. With those properties we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|x|^2 n_\nu \mathrm{d}x \leq C\int_{\mathbb{R}^d}|x|^2 n_\nu \mathrm{d}x + C\int_{\mathbb{R}^d} n|\nabla m_\nu|^2 \mathrm{d}x$$

and therefore

$$\sup_{\nu>0}\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}|x|^2 n_\nu \mathrm{d}x \leq C$$

as well as

$$\sup_{\nu>0}\sup_{0\leq t\leq T}\int_{\mathbb{R}^d} n_\nu|\log n_\nu|\mathrm{d}x \leq C$$

Testing with $\xi \in L^2(0,T;H^1(\mathbb{R}^d))$ yields

$$\int_{\mathbb{R}^d}\xi\partial_t n_\nu \mathrm{d}x \leq C\|\sqrt{n_\nu}\,\nabla m_\nu\|_{L^2(\mathbb{R}^d)}\|\nabla\xi\|_{L^2(\mathbb{R}^d)} + C\|\xi\|_{L^2(\mathbb{R}^d)}$$

and we get

$$\|n_\nu\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \leq C$$

Proof. (localisation limit for anisotropic system)
— Dissipation control

Then, we derive the entropy identity

$$\mathscr{H}[n_\nu(T)] - \mathscr{H}[n^{\mathrm{in}}] - \int_0^T \int_{\mathbb{R}^d} n_\nu \nabla \cdot (A \nabla m_\nu) \mathrm{d}x \mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{R}^d} \log n_\nu (n_\nu^{(1)} G^{(1)}(n_\nu) + n_\nu^{(2)} G^{(2)}(n_\nu)) \mathrm{d}x \mathrm{d}t$$

by writing an equation for $n_\epsilon := n \star_{x,t} \eta_\epsilon$, testing with $\log(\delta + n_\epsilon)$ and integrating.

Letting $\epsilon \to 0$ and then $\delta \to 0$, the identity follows.

As a consequence, we obtain

$$-\int_0^T \int_{\mathbb{R}^d} n_\nu \nabla \cdot (A \nabla m_\nu) \mathrm{d}x \mathrm{d}t \le C, \qquad \approx \int_0^T \int_{\mathbb{R}^d} \nabla n \cdot (A \nabla n) \mathrm{d}x \mathrm{d}t \le C,$$

with $C > 0$ independent of ν.

Proof. (localisation limit for anisotropic system) — Strong convergence of m_ν

Now, we proceed with the following steps

- $n_\nu^{(i)} \rightharpoonup n_0^{(i)}$, $n_\nu \rightharpoonup n_0 := n_0^{(1)} + n_0^{(2)}$, weakly in $L^2(0,T;L^2(\mathbb{R}^d))$
- $m_\nu \rightharpoonup n_0$, weakly in $L^2(0;T;L^2(\mathbb{R}^d))$, by passing to the limit in the Brinkman equation

$$-\nu \iint \nabla \cdot (A\nabla\phi) m_\nu \mathrm{d}x\mathrm{d}t + \iint \phi m_\nu \mathrm{d}x\mathrm{d}t = \iint \phi n_\nu \mathrm{d}x\mathrm{d}t$$

Moreover: $m_\nu \to n_0$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ and $\nabla m_\nu \rightharpoonup \nabla n_0$ weakly in $L^2(0,T;L^2(\mathbb{R}^d))$

$$\lambda \iint |\nabla m_\nu|^2 \mathrm{d}x\mathrm{d}t \le \iint \nabla m_\nu \cdot A\nabla m_\nu \mathrm{d}x\mathrm{d}t \le -\iint m_\nu \nabla\cdot(A\nabla m_\nu)\mathrm{d}x\mathrm{d}t$$

$$\le -\iint n_\nu \nabla\cdot(A\nabla m_\nu)\mathrm{d}x\mathrm{d}t - \iint (m_\nu - n_\nu)\nabla\cdot(A\nabla m_\nu)\mathrm{d}x\mathrm{d}t \le C$$

which yields $\nabla m_\nu \rightharpoonup \nabla n_0$ weakly in $L^2(0,T;L^2(\mathbb{R}^d))$ and $n_0 \in L^2(0,T;H^1(\mathbb{R}^d))$.

For strong compactness, we need some control on $\partial_t m_\nu$ ...

Proof. (localisation limit for anisotropic system)
— Strong convergence of m_ν

Time control. Let $\varphi \in H^1(\mathbb{R}^d)$ and compute $\int \varphi \partial_t m_\nu \mathrm{d}x$. But first solve: $-\nu \nabla \cdot (A \nabla \xi) + \xi = \varphi$.
Testing by ξ gives

$$\nu \iint \nabla \xi \cdot (A \nabla \xi) \mathrm{d}x\mathrm{d}t + \iint |\xi|^2 \mathrm{d}x\mathrm{d}t = \iint \xi \varphi \mathrm{d}x\mathrm{d}t \leq \frac{1}{2} \iint |\xi|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2} \iint |\phi|^2 \mathrm{d}x\mathrm{d}t$$

and thus, independently of ν

$$\iint |\xi|^2 \mathrm{d}x\mathrm{d}t \leq \iint |\phi|^2 \mathrm{d}x\mathrm{d}t$$

Similarly, testing by $-\nabla \cdot (A \nabla \xi)$ and using ellipticity gives, again, independently of ν:

$$\iint |\nabla \xi|^2 \mathrm{d}x\mathrm{d}t \leq \lambda^{-1} \|A\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \iint |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi \partial_t m_\nu \mathrm{d}x &= \int (-\nu \nabla \cdot (A \nabla \xi) + \xi) \partial_t m_\nu \mathrm{d}x = \int \nu \nabla \partial_t m_\nu \cdot (A \nabla \xi) + \xi \partial_t m_\nu \mathrm{d}x \\
&= \int (-\nu \nabla \cdot (A \nabla \partial_t m_\nu) + \partial_t m_\nu) \xi \mathrm{d}x = \int \partial_t (-\nu \nabla \cdot (A \nabla m_\nu) + m_\nu) \xi \mathrm{d}x - \nu \int \nabla \xi \cdot (\partial_t A \nabla m_\nu) \mathrm{d}x \\
&= \int \partial_t n_\nu \xi \mathrm{d}x - \nu \int \nabla \xi \cdot (\partial_t A \nabla m_\nu) \mathrm{d}x \leq C(\|\partial_t n_\nu\|_{H^{-1}}, \|\nabla m_\nu\|_{L^2}) \|\xi\|_{H^1(\mathbb{R}^d)} \\
&\leq C \|\varphi\|_{H^1(\mathbb{R}^d)}
\end{aligned}$$

Proof. (localisation limit for anisotropic system)
— Strong convergence of n_ν

Now, we proceed with the following steps

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- $m_\nu \rightharpoonup n_0$, weakly in $L^2(0;T;L^2(\mathbb{R}^d))$
- $m_\nu \to n_0$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ (Aubin-Lions)
- $\nabla m_\nu \rightharpoonup \nabla n_0$ weakly in $L^2(0,T;L^2(\mathbb{R}^d))$ (Banach-Alaoglu)
- next: $n_\nu \to n_0$, strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ — ellipticity + entropy-identity because…

$$\|n_\nu - n_0\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \lesssim \|m_\nu - n_0\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \|m_\nu - n_\nu\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \to 0$$

$$\int_0^T \int_{\mathbb{R}^d} |m_\nu - n_\nu|^2 \mathrm{d}x\mathrm{d}t = \nu \int_0^T \int_{\mathbb{R}^d} (m_\nu - n_\nu)\nabla \cdot (A\nabla m_\nu)\mathrm{d}x\mathrm{d}t$$

$$= -\nu \int_0^T \int_{\mathbb{R}^d} n_\nu \nabla \cdot (A\nabla m_\nu) + \nabla m_\nu \cdot A\nabla m_\nu \mathrm{d}x\mathrm{d}t \leq C\nu$$

Proof. (localisation limit for anisotropic system)
— Strong convergence of n_ν

Now, we proceed with the following steps

- $n_\nu^{(i)} \rightharpoonup n_0^{(i)}$, $n_\nu \rightharpoonup n_0 := n_0^{(1)} + n_0^{(2)}$, weakly in $L^2(0,T;L^2(\mathbb{R}^d))$
- $m_\nu \rightharpoonup n_0$, weakly in $L^2(0;T;L^2(\mathbb{R}^d))$
- $m_\nu \to n_0$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ (Aubin-Lions)
- $\nabla m_\nu \rightharpoonup \nabla n_0$ weakly in $L^2(0,T;L^2(\mathbb{R}^d))$ (Banach-Alaoglu)
- next: $n_\nu \to n_0$, strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ — ellipticity + entropy-identity because…
- deduce that n_0 is a weak solution to

$$\frac{\partial n_0}{\partial t} - \nabla \cdot (n_0 A \nabla n_0) = n_0^{(1)} G^{(1)}(n_0) + n_0^{(2)} G^{(2)}(n_0)$$

with initial data $n^{(1),\mathrm{in}} + n^{(2),\mathrm{in}}$.

We **do not** know if $n_0^{(1)}, n_0^{(2)}$ satisfy the corresponding equations!

Proof. (localisation limit for anisotropic system)
— Strong Compactness of ∇m_ν

The limit, n_0, satisfies the entropy identity:

$$\begin{aligned}\mathscr{H}[n_0(T)] - \mathscr{H}[n^{\mathrm{in}}] - \int_0^T \int_{\mathbb{R}^d} n_0 \nabla \cdot (A \nabla m_0) \mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_{\mathbb{R}^d} \log n_0 (n_0^{(1)} G^{(1)}(n_0) + n_0^{(2)} G^{(2)}(n_0)) \mathrm{d}x\mathrm{d}t\end{aligned}$$

Comparing it with that for n_ν, we find

$$\limsup_{\nu\to 0} \int_0^T \int_{\mathbb{R}^d} \nabla m_\nu \cdot A \nabla m_\nu \mathrm{d}x\mathrm{d}t \leq \int_0^T \int_{\mathbb{R}^d} \nabla n_0 \cdot A \nabla n_0 \mathrm{d}x\mathrm{d}t\,.$$

But since A is symmetric and positive definite, and $m_\nu \rightharpoonup n_0$:

$$\liminf_{\nu\to 0} \int_0^T \int_{\mathbb{R}^d} \nabla m_\nu \cdot A \nabla m_\nu \mathrm{d}x\mathrm{d}t \geq \int_0^T \int_{\mathbb{R}^d} \nabla n_0 \cdot A \nabla n_0 \mathrm{d}x\mathrm{d}t\,.$$

By ellipticity of A

$$\lambda \|\nabla m_\nu - \nabla n_0\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \leq \int_0^T \int_{\mathbb{R}^d} (\nabla m_\nu - \nabla n_0) \cdot A(\nabla m_\nu - \nabla n_0) \mathrm{d}x\mathrm{d}t \to 0,$$

Proof. (localisation limit for anisotropic system)
— Passing to the limit

To summarise, we have

- $n_\nu^{(i)} \rightharpoonup n_0^{(i)}$, $n_\nu \rightharpoonup n_0 := n_0^{(1)} + n_0^{(2)}$, weakly in $L^2(0,T; L^2(\mathbb{R}^d))$
- $m_\nu, n_\nu \to n_0$, strongly in $L^2(0;T; L^2(\mathbb{R}^d))$
- $\nabla m_\nu \to \nabla n_0$, strongly in $L^2(0;T; L^2(\mathbb{R}^d))$

We can pass to the limit in

$$-\int_0^T \int_{\mathbb{R}^d} n_\nu^{(i)} \partial_t \varphi \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} n_\nu^{(i)} (A \nabla m_\nu) \cdot \nabla \varphi \, \mathrm{d}x \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \varphi n_\nu^{(i)} G^{(i)}(n_\nu) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^d} \varphi(x,0) n^{(i),\mathrm{in}}(x) \mathrm{d}x$$

and obtain

$$-\int_0^T \int_{\mathbb{R}^d} n_0^{(i)} \partial_t \varphi \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} n_0^{(i)} (A \nabla n_0) \cdot \nabla \varphi \, \mathrm{d}x \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \varphi n_0^{(i)} G^{(i)}(n_0) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^d} \varphi(x,0) n^{(i),\mathrm{in}}(x) \mathrm{d}x$$

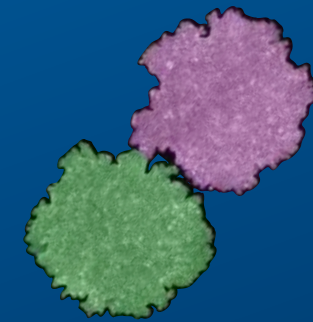
Summary of the strategy.

- use entropy (in)equality to deduce new uniform bounds for the velocity field (weak convergence)
- compare with the entropy structure of limit equation (upgrade to strong convergence)

Thank you for your attention!

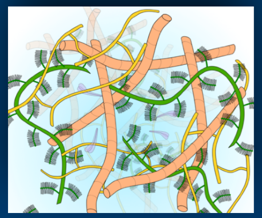
2024: David, Dębiec, Mandal, **S.**: *A Degenerate Cross-Diffusion System as the Inviscid Limit of a Nonlocal Tissue Growth Model, SIMA 2024.*

$$\partial_t n^{(i)} = \nabla \cdot \left(n^{(i)} \nabla (n^{(1)} + n^{(2)}) \right) + n^{(i)} G^{(i)}(n)$$



2025: Dębiec, **S.**: *Nonlocal approximation of an anisotropic cross-diffusion system,* in *Nonlin. Anal. 2025.*

$$\partial_t n^{(i)} = \nabla \cdot \left(n^{(i)} A(x,t) \nabla (n^{(1)} + n^{(2)}) \right) + n^{(i)} G^{(i)}(n)$$



2025: Dębiec, Mandal, **S.**: *From Finite to Continuous Phenotypes in (Visco-) Elastic Tissue Growth Models,* in *JDE 2025.*

$$\partial_t n(x,t;a) = \nabla_x \cdot \left(n(x,t;a) \nabla_x \int_0^1 n(x,t;a) \mathrm{d}a \right) + n(x,t;a) G(n;a)$$

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