Optimal rate of convergence for a nonlocal-to-local limit in one dimension

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Joint work with
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Anacapri, 7.07.2025

The problem

Target: Quantify the rate of convergence $\rho^{\varepsilon} \rightharpoonup \rho$ where

$$\partial_t \rho^{\varepsilon} - \partial_x(\rho^{\varepsilon}\, \partial_x \rho^{\varepsilon} * \omega_{\varepsilon}) = 0,$$

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Here, ω_ε is a usual mollification kernel: $\omega_\varepsilon = \omega\left(\frac{x}{\varepsilon}\right)$ where $\omega \geq 0$, $\omega(x) = \omega(-x)$, $\int_{\mathbb{R}} \omega(x) \, \mathrm{d}x = 1$.

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The problem is motivated by the interacting particle system: letting

$$X_i'(t) = -\frac{1}{N} \sum_{j \neq i} \partial_x \omega_\varepsilon(X_i(t) - X_j(t)),$$

we have that $\rho^{\varepsilon,N} = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$ solves the nonlocal PDE.

Some literature on this approximation

1. first works by Oelschläger (1990), P.-L. Lions and S. Mas-Galli
2. revisited from the gradient flow point of view: J. A. Carrillo, K. Craig, F. S. Patacchini (2019).
3. convergence for nonlinear diffusion: M. Burger, A. Esposito (2023), J. A. Carrillo, A. Esposito, J. S.-H. Wu (2024), S. Hecht, M. Doumic, B. Perthame, D. Peurichard (2024),
4. tissue growth models: T. Dębiec, B. Perthame, M. Schmidtchen, N. Vauchelet (2024), N. David, T. Dębiec, M. Mandal, M. Schmidtchen (2024), C. Elbar, J. Skrzeczkowski (2025),
5. linear diffusion: K. Craig, M. Jacobs, O. Turanova, (2023), J.A. Carrillo, A. Esposito, J. Skrzeczkowski, J. S.-H. Wu. (2024).
6. rate of convergence: A. Amassad, D. Zhou (2025), J.A. Carrillo, C. Elbar, S. Fronzoni, J. Skrzeczkowski (2025), J.A. Carrillo, P. Gwiazda, J. Skrzeczkowski (2025).

Rate of convergence in one dimension

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The first result given by Amassad and Zhou (2025) gives

$$W_2(\rho^\varepsilon, \rho) \leq C\, \sqrt{\varepsilon},$$

where C depends on the norm of ρ_0 in $W^{1,\infty}$. In their case, ω is any kernel convex for $x > 0$ and $x < 0$.

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In a joint work with J. A. Carrillo, C. Elbar, S. Fronzoni we obtained the same for a particular kernel $\omega(x) = \frac{1}{2} e^{-|x|}$ where C depends only on L^∞ norm of ρ_0. The proof is more elementary, using only the definition of gradient flow.

Strategy of the proof - gradient flow structure

Gradient flow on $\mathbb{R}^d$ is a curve solving

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Similarly, a big class of PDEs of the form

$$\partial_t \rho = \mathrm{div}(\rho \nabla \frac{\delta F}{\delta \rho}[\rho])$$

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$$W_2(\mu, \nu) = \inf_{\gamma} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \mathrm{d}\gamma(x, y) \right\}^{1/2}$$

where $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, $\gamma(A \times \mathbb{R}^d) = \mu(A)$, $\gamma(\mathbb{R}^d \times B) = \nu(B)$.

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For our PDEs in one dimension:

$$\partial_t \rho^\varepsilon - \partial_x(\rho^\varepsilon \, \partial_x \rho^\varepsilon * \omega_\varepsilon) = 0 \implies F_\varepsilon[\rho] = \frac{1}{2}\int_{\mathbb{R}} \rho * \omega_\varepsilon \, \rho$$

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Strategy of the proof - gradient flow structure (3)

In one dimension, both $F_\varepsilon[\rho] = \frac{1}{2}\int_{\mathbb{R}} \rho * \omega_\varepsilon\, \rho$ and $F[\rho] = \frac{1}{2}\int_{\mathbb{R}} \rho^2$ are convex. Hence,

$$F_\varepsilon(\rho_s) \geq F_\varepsilon(\rho_t^\varepsilon) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\rho_t^\varepsilon, \rho_s),$$

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Therefore, by chain rule and $F_\varepsilon[\rho] \leq F[\rho]$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\rho_t^\varepsilon, \rho_t) \leq F_\varepsilon(\rho_t) - F(\rho_t) + F(\rho_t^\varepsilon) - F_\varepsilon(\rho_t^\varepsilon) \leq F(\rho_t^\varepsilon) - F_\varepsilon(\rho_t^\varepsilon).$$

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Simple a priori estimates for the nonlocal problem shows that $\{\rho_t^\varepsilon\}$ and $\{\varepsilon \, \partial_x^2 \rho_t^\varepsilon * \omega_\varepsilon\}$ are bounded in L^2.

The rate $\sqrt{\varepsilon}$ did not seem to be optimal

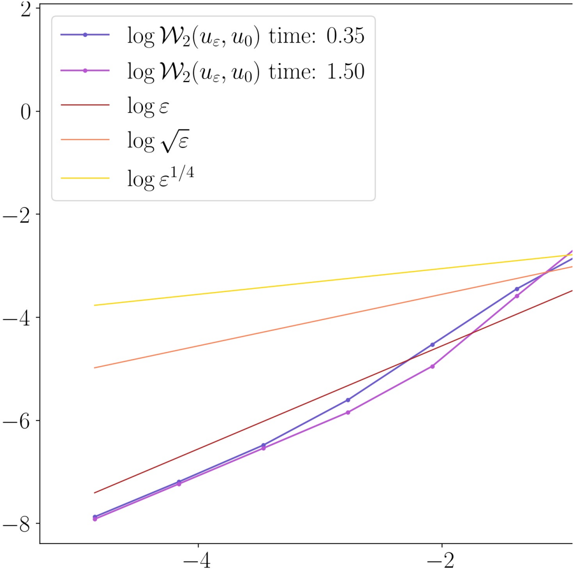

Figure: $\log \varepsilon$ on the x-axis

New formula for W_2 along two gradient flows

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$$\partial_t \rho = \mathrm{div}\,(\rho\, v_1[\rho])$$

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and both equations have smooth velocity fields, we can prove (joint work with J.A. Carrillo, P. Gwiazda)

$$W_2(\rho_t, \mu_t) \leq L \int_0^t \left(\int_\Omega |v_1[\rho_s] - v_2[\rho_s]|^2 \, \rho_s \, \mathrm{d}x \right)^{1/2} \mathrm{d}s.$$

Advantages of the new formula

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Applications to stability analysis of PDEs with respect to parameter, for example for a general aggregation-diffusion equation

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}(\rho \nabla W * \rho)$$

we get Lipschitz continuity with respect to m, V and W.

Stability estimates for aggregation-diffusion

$$\partial_t \mu = \Delta \mu^m + \operatorname{div}(\mu \nabla V_1) + \operatorname{div}(\mu \nabla K_1 * \mu)$$

$$\text{velocity field: } v_1[\mu] = \frac{m}{m-1} \nabla \mu^{m-1} + \nabla V_1 + \nabla K_1 * \mu$$

where

$$L \approx \exp\left(\left(\|\nabla^2 V_2\|_{L^\infty} + \|\nabla^2 K_2\|_{L^\infty} \right) t \right).$$

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Then, we get with $m < n$

$$W_2(\mu_t, \nu_t) \le L \int_0^t \Big(\int_{\mathbb{R}^d} \Big[\Big| \frac{m}{m-1} \nabla {\mu_s}^{m-1} - \frac{n}{n-1} \nabla {\mu_s}^{n-1} \Big|^2 + |\nabla (V_1 - V_2)|^2 + |\nabla (K_1 - K_2) * \mu_s|^2 \Big] \mu_s \Big)^{1/2} \mathrm{d}s$$

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Idea behind the proof of the new formula

The starting point is the Bressan formula: if S is an autonomous semigroup corresponding to the solution μ_t and ρ_t is any map

$$W_2(\rho_t, \mu_t) \le \int_0^t \liminf_{h \to 0} \frac{W_2(\mathcal{S}_h \rho_t, \rho_{t+h})}{h} \, \mathrm{d}s$$

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We consider a W_2 geodesic hopefully connecting $\mathcal{S}_h\rho_t$ and ρ_{t+h}

$$\partial_\tau \gamma_\tau + \mathrm{div} \left(\gamma_\tau \ (v_1[\rho_{t+\tau}] - v_2[\mathcal{S}_\tau \rho_t])\right) = 0,$$
$$\rho_0^3 = \mathcal{S}_h \rho_t.$$

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By the Benamou-Brenier formula

$$\frac{W_2(\mathcal{S}_h \rho_t, \rho_{t+h})}{h} \approx \left[\frac{1}{h} \int_0^h \left|v_1[\rho_{t+\tau}] - v_2[\mathcal{S}_h \rho_t]\right|^2 \gamma_\tau \, \mathrm{d}\tau\right]^{\frac{1}{2}} + \text{error},$$

where the error comes from the fact that $\rho_{t+h} \neq \gamma_h$.

Application of the new formula to the limit problem

In one dimension, for any kernel ω_ε which is convex for $x > 0$ and $x < 0$, the nonlocal equation

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generates Lipschitz semigroup with constant $L = 1$.

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$$W_2(\rho^\varepsilon_t, \rho_t) \leq L \int_0^t \left(\int_{\mathbb{R}} |\partial_x \rho_s - \partial_x \rho_s * \omega_\varepsilon|^2 \, \rho_s \, \mathrm{d}x \right)^{1/2} \mathrm{d}s.$$

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Note that here ρ_s solves $\partial_t \rho_s = \frac{1}{2} \partial_x^2 \rho_s^2$ which enjoys quite a lot of regularity, even in the presence of vacuum.

Regularity for the quadratic PME in 1D

$$\partial_t \rho_s = \frac{1}{2}\partial_x^2 \rho_s^2 = \rho_s\, \partial_x^2 \rho_s + |\partial_x \rho_s|^2$$

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under assumptions on initial conditions:

$$\rho_0, \partial_x \rho_0 \in L^\infty(\mathbb{R}), \qquad |\partial_x^2 \rho_0|_- \in \mathcal{M}(\mathbb{R}).$$

Application of the new formula to the limit problem (2)

$$\begin{aligned} W_2^2(\rho_t^\varepsilon, \rho_t) &\leq t \int_0^t \int_{\mathbb{R}} \left|\partial_x \rho_s - \partial_x \rho_s * \omega_\varepsilon\right|^2 \rho_s \, \mathrm{d}x \, \mathrm{d}s. \\ &\leq t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\partial_x \rho_s(x-y) - \partial_x \rho_s(x)\right|^2 \rho_s(x) \, \omega_\varepsilon(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \end{aligned}$$

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Target: control the (RHS) by $C\,\varepsilon^2$.

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Difficulty: in $(*)$, $|\partial_x^2 \rho_s|^2$ will not be evaluated at x!

Application of the new formula to the limit problem (3)

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x \rho_s(x-y) - \partial_x \rho_s(x)|^2 \, \rho_s(x) \, \omega_\varepsilon(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s$$

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We split for two sets:

$$A := \{(s,x,y) : \rho_s(x) \leq 2 \, \|\partial_x \rho_0\|_{L^\infty} \, |y|\},$$
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On A we estimate (x, y, s integrals not written)

$$2\|\partial_x \rho_0\|_{L^\infty} \, \left(\int_0^1 |\partial_x^2 \rho_s(x - \theta y)| \, |y| \, \mathrm{d}\theta \right) \, 2 \, \|\partial_x \rho_0\|_{L^\infty} \, |y| \, \omega_\varepsilon(y)$$

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After Fubini

$$4t \, \|\partial_x \rho_0\|_{L^\infty}^2 \, \|\partial_x^2 \rho_s\|_{L^1} \int_{\mathbb{R}} |y|^2 \omega_\varepsilon(y) \, \mathrm{d}y \leq$$
$$\leq 8t \, \|\partial_x \rho_0\|_{L^\infty}^2 \, \| |\partial_x^2 \rho_0|^- \|_{L^1} \, \varepsilon^2 \int_{\mathbb{R}} |y|^2 \omega(y) \, \mathrm{d}y.$$

Application of the new formula to the limit problem (4)

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x \rho_s(x-y) - \partial_x \rho_s(x)|^2 \, \rho_s(x) \, \omega_\varepsilon(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s$$

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$$\frac{\rho_s(x)}{\rho_s(x-\theta y)} = \frac{\rho_s(x) - \rho_s(x-\theta y)}{\rho_s(x-\theta y)} + 1 \leq \frac{\theta \, |y| \, \|\partial_x \rho_0\|_{L^\infty}}{\rho_s(x-\theta y)} + 1$$

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$$\rho_s(x-\theta y) \geq \rho_s(x) - \theta \, |y| \, \|\partial_x \rho_0\|_{L^\infty} \geq |y| \, \|\partial_x \rho_0\|_{L^\infty}.$$

so we obtain that $\rho_s(x)$ and $\rho_s(x-\theta y)$ are the same, up to a numerical constant.

Application of the new formula to the limit problem (5)

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x \rho_s(x-y) - \partial_x \rho_s(x)|^2 \, \rho_s(x) \, \omega_\varepsilon(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s$$

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Changing $\rho_s(x) \to \rho_s(x - \theta y)$ and estimating the difference of derivatives

$$\int_0^1 |\partial_x^2 \rho_s(x-\theta y)|^2 \, |y|^2 \rho_s(x-\theta y) \, \omega_\varepsilon(y)$$

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Integrating

$$\|\sqrt{\rho_s}\partial_x^2 \rho_s\|_{L^2_{t,x}} \int_{\mathbb{R}} |y|^2 \, \omega_\varepsilon(y) \, \mathrm{d}y \leq \varepsilon^2 \|\sqrt{\rho_s}\partial_x^2 \rho_s\|_{L^2_{t,x}} \int_{\mathbb{R}} |y|^2 \, \omega(y) \, \mathrm{d}y$$

concludes the proof.

Difficulty when applying the formula

We stated

$$W_2(\rho_t^\varepsilon, \rho_t) \leq L \int_0^t \left(\int_{\mathbb{R}} \left|\partial_x \rho_s - \partial_x \rho_s * \omega_\varepsilon\right|^2 \rho_s \, \mathrm{d}x \right)^{1/2} \mathrm{d}s.$$

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In fact, it is not possible to apply our general formula in this form:

- to regularize velocity field for the PME, we need to work on a bounded domain (we cannot add a constant),
- nonlocal problem does not like the bounded domain (accumulation at the boundary) so we need to work on the whole space.

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In fact, we prove using our formula

$$W_2(\rho_t^\varepsilon, \rho_t^\delta) \leq \int_0^t \left(\int_{\mathbb{R}} \left| \partial_x \rho_s^\delta * \omega_\delta - \partial_x \rho_s^\delta * \omega_\varepsilon \right|^2 \rho_s^\delta \, \mathrm{d}x \right)^{1/2} \mathrm{d}s.$$

and we pass to the limit $\delta \to 0$.

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THANK YOU!