Symmetrization for nonlocal problems

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Workshop on "New Perspectives in Nonlocal and Nonlinear PDEs"

Anacapri

July 7-11, 2025

Symmetrization of a function

Let Ω be an open bounded set of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be a measurable function.

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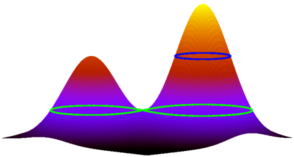
Schwarz symmetrization of f (spherically symmetric decreasing rearrangement), denoted by $f^\sharp$, is such that its level sets are balls of $\mathbb{R}^N$ centered at the origin having the same measure as the corresponding level sets of f.
In other words

$$\{x \in \Omega^\sharp : f^\sharp(x) > t\} = \{x \in \Omega : |f(x)| > t\}^\sharp \qquad t \geq 0.$$



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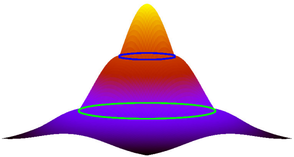

$f^\sharp$

Symmetrization of a function

More precisely, one considers the *distribution function* μ_f of f

$$\mu_f(t) := \left|\{x \in \Omega : |f(x)| > t\}\right|.$$

The function $\mu_f(\cdot)$ is a decreasing function from $\mu_f(0) = |\operatorname{supp}(f)|$ to $\mu_f(+\infty) = 0$ as t increases from 0 to $+\infty$.

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The *one dimensional decreasing rearrangement* of f is

$$f^*(\sigma) = \sup\{t \geq 0 : \mu_f(t) > \sigma\} \qquad \sigma \in [0, +\infty),$$

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The *one dimensional increasing rearrangement* of f is

$$f_*(\sigma) = f^*(|\Omega| - \sigma) \qquad \sigma \in (0, |\Omega|).$$

Symmetrization of a function

The *radially decreasing rearrangement* (or *Schwarz decreasing rearrangement*) of f is (ω_N denotes the measure of the unit ball in $\mathbb{R}^N$)

$$f^\sharp(x) = f^*(\omega_N |x|^N) \qquad x \in \Omega^\sharp;$$

and we call the *radially increasing rearrangement* of f, the function

$$f_\sharp(x) = f_*(\omega_N |x|^N) \qquad x \in \Omega^\sharp.$$

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From the definitions, one deduces that f^*, f_*, $f^\sharp$ and $f_\sharp$ have the same distribution function as f, consequently, rearrangements preserve L^p norms, that is, for all $p \in [1, \infty]$:

$$\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p(0,|\Omega|)} = \|f^\sharp\|_{L^p(\Omega^\sharp)}.$$

Some properties of rearrangements

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$$\int_{\Omega^\sharp} f^\sharp(x) g_\sharp(x)\, \mathrm{d}x \leq \int_\Omega |f\, g|\, \mathrm{d}x \leq \int_{\Omega^\sharp} f^\sharp(x) g^\sharp(x)\, \mathrm{d}x$$

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- $\displaystyle\int_{B_r} f^\sharp(x)\,\mathrm{d}x \le \int_{B_r} g^\sharp(x)\,\mathrm{d}x, \qquad \forall B_r$ (ball of radius r, centered at the origin)

$$\Downarrow$$

$$\|f\|_{L^p(\Omega)} \le \|g\|_{L^p(\Omega)} \qquad 1 \le p \le \infty$$

Symmetrization for local problems

Let us consider the following homogeneous Dirichlet problem in an open bounded set $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -\left(a_{ij}\ u_{x_i}\right)_{x_j} = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the measurable coefficients $a_{ij} = a_{ij}(x)$ satisfy the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega,$$

and the source term $f = f(x)$ is assumed to belong to $L^p(\Omega)$ for suitable $p \geq 1$.

Symmetrization for local problems

A nowadays classical result states that if $u \in H_0^1(\Omega)$ is the weak solution to (1) and $v \in H_0^1(\Omega^\sharp)$ is the weak solution to the "symmetrized problem"

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp, \\ v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

then

$$u^\sharp(x) \leq v(x), \qquad x \in \Omega^\sharp. \tag{2}$$

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An immediate consequence of inequality (2) is, for example, that any norm of u increases under Schwarz symmetrization.

[Weinberger, 1962], [Maz'ya, 1969], [Talenti, 1976]

Symmetrization for local problems

The approach used in most of the papers concerning symmetrization techniques is based on the fact that after the use of a suitable test function allows one can apply Schwarz inequality, Fleming-Rishel formula and isoperimetric inequality in order to obtain a first order differential inequality involving $u^\sharp$ and its radial derivative, namely

$$N\omega_N r^{N-1}\left(-\frac{\mathrm{d}}{\mathrm{d}r}u^\sharp(r)\right) \leq \int_{B_r} f^\sharp(x)\,\mathrm{d}x.$$

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The solution v to the symmetrized problem satisfies the above inequality as an equality and the comparison follows immediately:

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A slightly different approach has been used in [Lions, 1981], where the author observes that one can use the so-called Pólya-Szegő principle which states that, if $u \in H_0^1(\Omega)$, then

$$\int_\Omega |Du|^2 \mathrm{d}x \geq \int_{\Omega^\sharp} |Du^\sharp|^2 \mathrm{d}x. \tag{3}$$

Symmetrization for local problems

The literature about the possible extensions of the comparison result is wide

- elliptic equations with lower order terms
- p-Laplacian type equations
- porous medium equation
- parabolic equations
- anisotropic equations
- ...

Symmetrization for nonlocal problems

Let us consider the following Dirichlet fractional elliptic problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth bounded open set, the source term $f = f(x)$ is assumed to belong to $L^p(\Omega)$ for suitable $p \geq 1$ and $s \in (0, 1)$.

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The fractional Laplacian of a smooth real function u on $\mathbb{R}^N$ can be defined in terms of a hypersingular integral

$$(-\Delta)^s u(x) = \gamma(N,s)\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where the explicit value of the normalization constant $\gamma(N,s)$ is given by

$$\gamma(N,s) = \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

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In those papers a symmetrization result in terms of mass concentration (*i.e.*, an integral comparison, as in the parabolic case) is obtained in a somewhat indirect way.
Indeed, it has been used in an essential way the fact that the fractional problem can be linked to a suitable, local extension problem, whose solution, an extension of u, is defined on an infinite cylinder.
[Caffarelli - Silvestre, 2007], [Stinga - Torrea, 2010], ...

Symmetrization for nonlocal problems

Theorem ([F. - Volzone, ARMA, 2021])

Let $s \in (0,1)$ *and let* $f \in L^p(\Omega)$*, with* $p \geq 2N/(N+2s)$ *when* $N \geq 2$ *and any* $p > 1$ *for* $N = 1$*. If* u *and* v *are the solutions to the following problems*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \qquad \begin{cases} (-\Delta)^s v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\sharp \end{cases}$$

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$$u \prec v \qquad (\textit{comparison of mass concentrations})$$

means that for all $r > 0$ it holds

$$\int_{B_r} u^\sharp(x)\,\mathrm{d}x \leq \int_{B_r} v(x)\,\mathrm{d}x$$

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$$[u]_{H^s} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

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The main novelty is that we give a new proof of the mass concentration comparison which could be of interest because the arguments, based on a suitable Pólya-Szegő, seem to be very flexible with respect to those used in previous papers.

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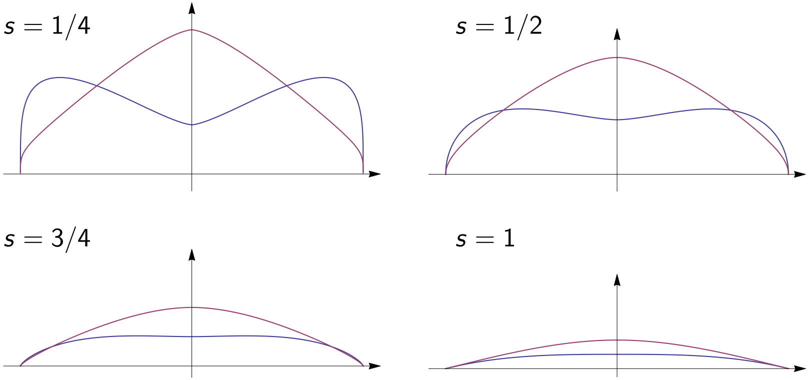

u (blue line) solution corresponding to source term $f = |x|$

v (purple line) solution corresponding to source term $f^\sharp = 1 - |x|$

A general nonlocal operator

Let $K = K(x, y)$ be measurable nonnegative function such that

$$K(x,y) = K(y,x) \quad \forall x, y \in \mathbb{R}^N,$$
$$x \mapsto \int_{\mathbb{R}^N} K(x,y) \min\{|x-y|^2, 1\} dy \in L^1_{loc}(\mathbb{R}^N),$$

One can consider the following general nonlocal operator:

$$\mathcal{L}u(x) = \text{P.V.} \int_{\mathbb{R}^N} K(x,y)(u(x) - u(y)) dy,$$

where the principal value (P.V.) integral is meant in the sense that

$$\mathcal{L}u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} K(x,y)(u(x) - u(y)) dy.$$

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We also suppose that

$$K(x, y) \geq J(x - y) \geq 0$$

and we consider the nonlocal operator associated to $J^\sharp$

$$\mathcal{L}^\sharp v(x) = \text{P.V.} \int_{\mathbb{R}^N} J^\sharp(x - y)(v(x) - v(y))dy.$$

Comments

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– The equations we are treating appear in several contexts, for instance, in Probability theory (generators of stochastic Lévy processes, *i.e.* special stochastic processes with jumps), Fluid mechanics (for example, in the quasi-geostrophic equation) or in Mathematical physics (relativistic Schrödinger operators or the Boltzmann equation), peridynamics theory.
(see, for instance, the survey [Ros-Oton, 2016] and the book [Fernández-Real - Ros-Oton, 2023]).

To deduce an inequality involving $u^\sharp$

Let us consider a solution u to problem

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Using a suitable version of Riesz rearrangement inequality it is possible to prove

$$\int_{B_r} \int_{B_r^c} J^\sharp(x-y)(u^\sharp(x) - u^\sharp(y))\mathrm{d}y\mathrm{d}x \le \int_{B_r} f^\sharp(x)\mathrm{d}x$$

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while for the solution $v = v^\sharp$ to problem

$$\begin{cases} \mathcal{L}^\sharp v = f^\sharp & \text{in } \Omega^\sharp, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\sharp, \end{cases}$$

it holds

$$\int_{B_r} \int_{B_r^c} J^\sharp(x-y)(v(x) - v(y))\mathrm{d}y\mathrm{d}x = \int_{B_r} f^\sharp(x)\mathrm{d}x$$

Theorem (Riesz rearrangement inequality [Almgren - Lieb, 1989])

Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function such that* $F(0,0) = 0$ *and*

$$F(u_2, v_2) + F(u_1, v_1) \geq F(u_2, v_1) + F(u_1, v_2)$$

whenever $u_2 \geq u_1 > 0$ *and* $v_2 \geq v_1 > 0$ $(\partial^2 F/\partial u \partial v \geq 0)$.
Assume that f, g *are nonnegative measurable functions on* $\mathbb{R}^N$*, then we have the inequalities*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(f(x), g(y)) W(ax+by)\, \mathrm{d}x\, \mathrm{d}y \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(f^\sharp(x), g^\sharp(y)) W^\sharp(ax+by)\, \mathrm{d}x\, \mathrm{d}y$$

and

$$\int_{\mathbb{R}^N} F(f(x), g(x))\, \mathrm{d}x \leq \int_{\mathbb{R}^N} F(f^\sharp(x), g^\sharp(x))\, \mathrm{d}x,$$

for any nonnegative function $W \in L^1(\mathbb{R}^N)$ *and any choice of nonzero numbers* a *and* b.

Remark

When $K(x,y) = \dfrac{\gamma(N,s)}{|x-y|^{N+2s}}$, that is $\mathcal{L}u = (-\Delta)^s u$ it turns out that

$$\int_{B_r}\int_{B_r^c} J^\sharp(x-y)\left(u^\sharp(x)-u^\sharp(y)\right)\mathrm{d}y\mathrm{d}x =$$

$$= \gamma(N,s)\int_0^r\left(\int_r^{+\infty}\left(u^\sharp(\tau)-u^\sharp(\rho)\right)\Theta_{N,s}(\tau,\rho)\rho^{N-1}\mathrm{d}\rho\right)\tau^{N-1}\mathrm{d}\tau$$

where

$$\Theta_{N,s}(r,\rho) = \frac{1}{N\omega_N}\int_{|x'|=1}\left(\int_{|y'|=1}\frac{1}{|r\,x'-\rho\,y'|^{N+2s}}\mathrm{d}H^{N-1}(y')\right)\mathrm{d}H^{N-1}(x')$$

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that is,

$$\Theta_{N,s}(r,\rho) = \begin{cases} \dfrac{\alpha_N}{\rho^{N+2s}}\,{}_2F_1\left(\dfrac{N+2s}{2}, s+1; \dfrac{N}{2}; \dfrac{r^2}{\rho^2}\right) & \text{if } 0 \le r < \rho < +\infty \\[2ex] \dfrac{\alpha_N}{r^{N+2s}}\,{}_2F_1\left(\dfrac{N+2s}{2}, s+1; \dfrac{N}{2}; \dfrac{\rho^2}{r^2}\right) & \text{if } 0 \le \rho < r < +\infty \end{cases}$$

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It turns out that

$$\gamma(N, s) \int_0^r \left(\int_r^{+\infty} (u^*(\tau) - u^*(\rho)) \Theta_{N,s}(\tau, \rho)\rho^{N-1} \mathrm{d}\rho \right) \tau^{N-1} \mathrm{d}\tau = r^N (-\Delta)^s_{\mathbb{R}^{N+2}} U(r)$$

A formula for the fractional Laplacian computed on radial function contained in [Ferrari - Verbitsky, 2012] has been used.

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In the case of s-laplacian it is possible to rewrite the above inequality in terms of the spherical mean function.

Let us define the following spherical mean function

$$U(x) = U(|x|) = \frac{1}{|x|^N} \int_0^{|x|} u^*(\rho)\rho^{N-1} \mathrm{d}\rho.$$

It turns out that

$$\gamma(N,s) \int_0^r \left(\int_r^{+\infty} (u^*(\tau) - u^*(\rho)) \Theta_{N,s}(\tau,\rho)\rho^{N-1} \mathrm{d}\rho \right) \tau^{N-1} \mathrm{d}\tau = r^N (-\Delta)^s_{\mathbb{R}^{N+2}} U(r)$$

A formula for the fractional Laplacian computed on radial function contained in [Ferrari - Verbitsky, 2012] has been used.
Then

$$(-\Delta)^s_{\mathbb{R}^{N+2}} U(r) \leq \frac{1}{r^N} \int_0^r f^*(\rho)\rho^{N-1} \mathrm{d}\rho$$

Remark

For the solution v to the symmetrized problem we have

$$(-\Delta)^s_{\mathbb{R}^{N+2}} V(r) = \frac{1}{r^N} \int_0^r f^*(\rho)\rho^{N-1} \mathrm{d}\rho$$

where $V(r)$ is the spherical mean of v,

$$V(x) = V(|x|) = \frac{1}{|x|^N} \int_0^{|x|} v(\rho)\rho^{N-1} \mathrm{d}\rho.$$

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A classical comparison result gives

$$U(r) \leq V(r)$$

that is,

$$u \prec v.$$

Some applications of the method

Fractional $p-$laplacian $(p \geq 2)$

$$\begin{cases} (-\Delta_p)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

[F. -Volzone, DCDS, 2022]

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Nonlocal problem with singularity (elliptic and parabolic)

$$\begin{cases} (-\Delta)^s u = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

[de Bonis, Brandolini, F., Volzone, Asympt. Anal., 2023]
[de Bonis, Brandolini, F., Volzone, Nonlinear Anal. RWA, 2025]

Theorem ([F. - Piscitelli - Volzone, JMPA, 2024])

Let $\mathcal{L}$ and $\mathcal{L}^\sharp$ be defined as above, $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$ such that $c(x) \geq 0$ in Ω. If u and v are the solutions of:

$$\begin{cases} \mathcal{L}u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad \begin{cases} \mathcal{L}^\sharp v + c_\sharp v = f^\sharp & \text{in } \Omega^\sharp, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\sharp, \end{cases}$$

respectively, then

$$u \prec v$$

Moreover we have the following energy estimate:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\,(u(x)-u(y))^2\,dydx + \int_{\mathbb{R}^N} c(x)u^2(x)dx \\ &\quad \leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J^\sharp(x-y)\,(v(x)-v(y))^2\,dydx + \int_{\mathbb{R}^N} c_\sharp(x)v^2(x)dx. \end{aligned}$$

Theorem ([F. - Piscitelli - Volzone, JMPA, 2024])

Let $\mathcal{L}$ and $\mathcal{L}^\sharp$ be defined as above, $c \in L^\infty(\Omega \times (0,T))$ is nonnegative, $f \in L^2(\Omega \times (0,T))$. If u and v are the solutions of:

$$\begin{cases} u_t + \mathcal{L}u + cu = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

and

$$\begin{cases} v_t + \mathcal{L}^\sharp v + c_\sharp v = f^\sharp & \text{in } \Omega^\sharp, \\ v = 0 & \text{in } (\mathbb{R}^N \setminus \Omega^\sharp) \times (0,T), \\ v(x,0) = v_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where $u_0 \in L^2(\Omega)$ and $v_0 = v_0^\sharp \in L^2(\Omega^\sharp)$ are such that $\displaystyle\int_{B_r} u_0^\sharp(x)dx \le \int_{B_r} v_0(x)dx$ $\forall r > 0$, then

$$\int_{B_r} u^\sharp(x,t)\,dx \le \int_{B_r} v(x,t)\,dx \quad \forall r > 0 \;\; \forall t \in [0,T].$$

Comparison principle

As discussed above we get

$$\int_{B_r}\int_{B_r^c} J^\sharp(x-y)\left(u^\sharp(x)-u^\sharp(y)\right)\mathrm{d}y\mathrm{d}x+\int_{B_r} c_\sharp(x)u^\sharp(x)\mathrm{d}x \le \int_{B_r} f^\sharp(x)\mathrm{d}x,$$

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while, as regards the weak solution v to the symmetrized problem we have

$$\int_{B_r}\int_{B_r^c} J^\sharp(x-y)\left(v(x)-v(y)\right)\mathrm{d}y\mathrm{d}x+\int_{B_r} c_\sharp(x)v(x)\mathrm{d}x=\int_{B_r} f^\sharp(x)\mathrm{d}x.$$

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Subtracting, we have:

$$\int_{B_r}\int_{B_r^c} J^\sharp(x-y)\left(u^\sharp(x)-u^\sharp(y)\right)dydx-\int_{B_r}\int_{B_r^c} J^\sharp(x-y)\left(v(x)-v(y)\right)\mathrm{d}y\mathrm{d}x$$
$$+\int_{B_r} c_\sharp(x)(u^\sharp(x)-v(x))\mathrm{d}x \le 0.$$

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It remains to prove that

$$\int_{B_r} u^\sharp(x)dx \le \int_{B_r} v(x)dx \qquad \forall r\ge 0.$$

Proposition

Let u, v be two nonnegative, radial and summable functions on B_R. Let us assume that the function

$$r \in [0, R] \mapsto \int_{B_r} (u(x) - v(x))dx$$

admits a positive maximum point at $\bar{r} > 0$, that is,

$$0 < \int_{B_{\bar{r}}} (u(x) - v(x))dx = \max_{r \in [0,R]} \left(\int_{B_r} (u(x) - v(x))dx \right).$$

Then, if $\mathfrak{h}$ is a positive radially increasing function such that $(u(x) - v(x))\mathfrak{h}(x)$ is summable on $B_{\bar{r}}$, we have

$$\int_{B_{\bar{r}}} (u(x) - v(x))\mathfrak{h}(x)dx > 0.$$

An analogous result holds true if the following function admits a negative minimum point

$$r \in [0, R] \mapsto \int_{B_R \setminus B_r} (u(x) - v(x))dx$$

Comparison principle

In order to prove the desired comparison result we suppose by contradiction that

$$r \in [0, R] \mapsto \int_{B_r} (u^\sharp(x) - v(x))dx$$

admits a positive maximum point at $\bar{r} > 0$ and, using the above Proposition we show that inequality

$$\int_{B_r} \int_{B_r^c} J^\sharp(x-y)\left(u^\sharp(x) - u^\sharp(y)\right) dydx - \int_{B_r} \int_{B_r^c} J^\sharp(x-y)\left(v(x) - v(y)\right) \mathrm{d}y\mathrm{d}x \\ + \int_{B_r} c_\sharp(x)(u^\sharp(x) - v(x))\mathrm{d}x \leq 0.$$

gives a contradiction at $r = \bar{r}$.

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gives a contradiction at $r = \bar{r}$.
In order to apply the Proposition, we use the fact that

- $\Phi_1(x) = \int_{B_r^c} J^\sharp(x-y)dy$ is radially increasing for any $|x| < r$;
- $\Phi_2(y) = \int_{B_r} J^\sharp(x-y)dx$ is radially decreasing for any $|y| > r$.

Parabolic case

We consider the problem where the time derivative u_t is replaced by a difference quotient

$$\begin{cases} \mathcal{L}u_{n+1} + c_n u_{n+1} + \dfrac{u_{n+1}}{\Delta t} = f_n + \dfrac{u_n}{\Delta t} & \text{in } \Omega \\ u_{n+1} = 0, & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $n = 0, ..., k-1$ and $u_0 = u_0(x)$ is given in the initial conditions.

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where $n = 0, ..., k-1$ and $u_0 = u_0(x)$ is given in the initial conditions.
The corresponding symmetrized problems are

$$\begin{cases} \mathcal{L}^\sharp v_{n+1} + (c_n)_\sharp v_{n+1} + \dfrac{v_{n+1}}{\Delta t} = (f_n)^\sharp + \dfrac{v_n}{\Delta t} & \text{in } \Omega^\sharp \\ v_{n+1} = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\sharp, \end{cases}$$

with $v_0 = v_0(x)$.

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with $v_0 = v_0(x)$.
From the comparison result

$$\int_{B_r} u_{n+1}^\sharp(x)\mathrm{d}x \leq \int_{B_r} v_{n+1}(x)\mathrm{d}x \qquad \forall r > 0.$$

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From the comparison result

$$\int_{B_r} u^\sharp_{n+1}(x)\mathrm{d}x \le \int_{B_r} v_{n+1}(x)\mathrm{d}x \qquad \forall r > 0.$$

it is possible to deduce

$$\int_{B_r} u^\sharp(x,t)\mathrm{d}x \le \int_{B_r} v(x,t)\mathrm{d}x \qquad \text{for all } r > 0, t \in (0,T).$$

An eigenvalue problem

Let us consider the following nonlocal eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda\, u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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We are interested in the case where $\lambda = \lambda_1(\Omega)$ is the first eigenvalue. We denote by $B_R \subset \mathbb{R}^N$ the ball (centered at the origin) having the same first eigenvalue as Ω, that is $\lambda_1(B_R) = \lambda_1(\Omega)$.

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Theorem ([Brandolini - de Bonis - F. - Volzone, in preparation])

Let u be a positive eigenfunction for corresponding to $\lambda = \lambda_1(\Omega)$ and let v the positive solution to the following eigenvalue problem

$$\begin{cases} (-\Delta)^s v = \lambda_1(\Omega)\, v & \text{in } B_R, \\ v = 0 & \text{on } \mathbb{R}^N \setminus B_R, \end{cases}$$

satisfying

$$||v||_{L^1(B_R)} = ||u||_{L^1(\Omega)}.$$

Then:

$$\int_{B_r} u^\sharp(x)\,\mathrm{d}x \le \int_{B_r} v(x)\,\mathrm{d}x, \tag{5}$$

where B_r is the ball (centered at the origin) having radius r.

Remarks

- The proof of the Theorem uses the fact that the spherical mean function $U(x)$ satisfies an inequality involving the s-laplacian in dimension $(N+2)$ and some relations between the eigenvalues of balls in dimensions N and $(N+2)$ proved in [Dyda, 2012].

Remarks

- The proof of the Theorem uses the fact that the spherical mean function $U(x)$ satisfies an inequality involving the s-laplacian in dimension $(N+2)$ and some relations between the eigenvalues of balls in dimensions N and $(N+2)$ proved in [Dyda, 2012].
- Estimate (5) implies, for $p > 1$,

$$||u||_{L^p(\Omega)} \leq ||v||_{L^p(B_R)} = \frac{||v||_{L^p(B_R)}}{||v||_{L^1(B_R)}} ||u||_{L^1(\Omega)}.$$

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The above inequality has been proved for the first eigenvalue of the (local) laplacian; in such a case, for $p = 2$ and $N = 2$, it reads as

$$\int_\Omega u^2 \mathrm{d}x \leq \frac{\lambda}{4\pi} \left(\int_\Omega |u| \mathrm{d}x \right)^2 \qquad \text{(Payne-Rayner inequality)}$$

[Payne - Rayner, 1972], [Kohler-Jobin, 1977], [Chiti, 1982], [Alvino - F. - Trombetti, 1998]

Remarks

- The paper by Chiti is based on some comparison results, contained in a different paper, which state a comparison result in terms of mass concentration in the same spirit as our theorem.