Kermack-McKendrick type models for epidemics with nonlocal aggregation terms

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New Perspectives in Nonlocal and Nonlinear PDEs, Anacapri

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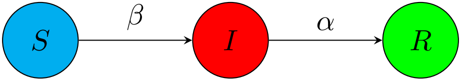
[0] [Di Francesco and Ghaderi Z, 2025]

Summary

Spatially Homogeneous SIR Model

[Kermack and McKendrick, 1927]:

- $S(t)$: The number of susceptible individuals at time t.
- $I(t)$: The number of infected individuals at time t.
- $R(t)$: The number of recovered individuals at time t.
- β : Infectious rate.
- α :Recovery rate.



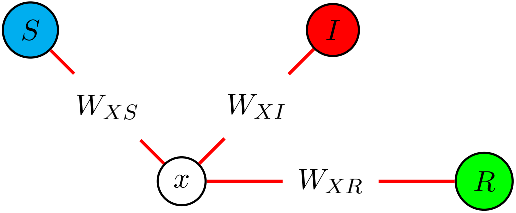
$$\begin{cases} S'(t) = -\beta S(t)I(t) \\ I'(t) = \beta S(t)I(t) - \alpha I(t) \\ R'(t) = \alpha I(t) \end{cases} \tag{1}$$

Specially Nonhomogeneous SIR Model

The spatial movement is mostly described using:

- Diffusion [Langlais and Busenberg, 1997, Kim, 2006]
- Nonlocal movement [Kuniya and Wang, 2018, Yang and Wang, 2023, Bentout et al., 2023, Li, 2023]

Nonlocal Social Interaction: Let x be an individual in compartment X ($X \in \{S, I, R\}$). The movement of x is affected differently by other individuals in each compartment based on their affiliation.



We assume that the kernels satisfy the following conditions.

$$D^\gamma W_{\xi\eta} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$
$$\text{for all multi-indexes } \gamma \text{ with } 1 \le |\gamma| \le 2 \text{ and for all } \xi, \eta \in \{S, I, R\}. \quad (2)$$

$$W_{\xi\eta}(-x) = W_{\xi\eta}(x) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and for all } \xi, \eta \in \{S, I, R\}\,. \quad (3)$$

The operators V_S, V_I, and V_R model nonlocal aggregation of the individuals in each compartment and are defined as:

$$\begin{aligned} V_S[S, I, R] &= W_{SS} * S + W_{SI} * I + W_{SR} * R, \\ V_I[S, I, R] &= W_{IS} * S + W_{II} * I + W_{IR} * R, \\ V_R[S, I, R] &= W_{RS} * S + W_{RI} * I + W_{RR} * R. \end{aligned} \quad (4)$$

Main Model

- $x \in \mathbb{R}^2$, $t \in [0, T]$
- $S(x,t)$: The number of susceptible individuals in x at time t.
- $I(x,t)$: The number of infected individuals in x at time t.
- $R(x,t)$: The number of recovered individuals in x at time t.
- β: Infectious rate.
- α:Recovery rate.

$$\begin{cases} \partial_t S = \operatorname{div}\left(S\nabla V_S[S,I,R]\right) - \beta SI, \\ \partial_t I = \operatorname{div}\left(I\nabla V_I[S,I,R]\right) + \beta SI - \alpha I, \\ \partial_t R = \operatorname{div}\left(R\nabla V_R[S,I,R]\right) + \alpha I. \end{cases} \tag{5}$$

Main Theorem

Theorem 7 [Existence and uniqueness of solutions]

Assume the interaction kernels $W_{\xi\eta}$ satisfy (2) and (3). Assume the initial data S_0, I_0, R_0 belong to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and are nonnegative. Then, for any arbitrary time $T \geq 0$, there exists one and only one

$$(S, I, R) \in L^\infty([0,T]\,;\ (L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)))^3$$

solving (5) in the weak sense and having (S_0, I_0, R_0) as initial datum.

Approximated problem

We consider, for a fixed $\varepsilon > 0$, the approximated problem

$$\begin{cases} S_t = \operatorname{div}\left(S \nabla V_S[S, I, R]\right) - \beta S I + \varepsilon \Delta S, \\ I_t = \operatorname{div}\left(I \nabla V_I[S, I, R]\right) + \beta S I - \alpha I + \varepsilon \Delta I, \\ R_t = \operatorname{div}\left(R \nabla V_R[S, I, R]\right) + \alpha I + \varepsilon \Delta R. \end{cases} \tag{6}$$

Proposition 8

Assume (2) and let the initial condition satisfy $(S_0, I_0, R_0) \in \left(L^1 \cap L^\infty\left(\mathbb{R}^2\right)\right)^3$. Then, there exists $T > 0$ such that there exists one and only one classical solution $(S, I, R) \in (C^{1,2}_{t,x}(\mathbb{R}^2 \times [0, T]))^3$ to system (6) in $\mathbb{R}^2 \times [0, T]$ with initial condition (S_0, I_0, R_0).

Remark 9

There exists $T > 0$ *such that there exists one and only one classical solution* $u = (u_1, \ldots, u_N) \in (C^{1,2}_{t,x}(\mathbb{R}^d \times [0,T]))^N$ *to the following system in* $\mathbb{R}^d \times [0,T]$.

$$\begin{cases} \partial_t u_i(x,t) = \varepsilon \Delta u_i + \operatorname{div}(u_i \nabla V_i[u_1, \ldots, u_N]) + g_i(u_1, \ldots, u_N), \\ x \in \mathbb{R}^d,\ t > 0, \\ u_i(x,0) = u_i^0(x), \end{cases} \tag{7}$$

where $i = 1, \ldots, N$, *with nonlocal operators of the form*

$$V_i[u_1, \ldots, u_N](x) = \sum_{j=1}^{N} W_{ij} * u_i(x)\,,$$

provided the interaction kernels W_{ij} *satisfy the same assumptions as in* (2), *the functions* $g_i : \mathbb{R}^N \to \mathbb{R}$ *feature* C^1 *regularity, and* $u_i^0 \in L^1 \cap L^\infty$.

Lemma 10 [Non-negativity of solutions]

Under the same assumptions of Proposition 8, assuming further nonnegative initial data, the local solution (S, I, R) provided in Proposition 8 is nonnegative on its maximal existence time interval $[0, T]$.

Proposition 11 [Uniform L^p estimates and global existence]

Let (S, I, R) be the unique local solution to the system (6) found in Proposition 8 and let $p \in [1, +\infty]$. Under the same assumptions of Proposition 8, assuming further $S_0, I_0, R_0 \in \left(L^1 \cap L^\infty\left(\mathbb{R}^2\right)\right)$ are nonnegative functions, then, there exists a constant $C \geq 0$ independent of time and of ε such that

$$\begin{aligned} &\|S(\cdot,t)\|_{L^p(\mathbb{R}^2)} + \|I(\cdot,t)\|_{L^p(\mathbb{R}^2)} + \|R(\cdot,t)\|_{L^p(\mathbb{R}^2)} \\ &\quad \leq \left(1 + \|S_0\|_{L^p(\mathbb{R}^2)} + \|I_0\|_{L^p(\mathbb{R}^2)} + \|R_0\|_{L^p(\mathbb{R}^2)}\right) e^{Ct(1+e^{Ct})}, \end{aligned} \tag{8}$$

for all $t \in [0, T]$. Consequently, the solution (S, I, R) found in Proposition 8 exists globally in time.

Sketch of the proof: the sequence of cut-off functions $\zeta_n : \mathbb{R}^2 \to [0,1]$ such that, for any $n \in \mathbb{N}$, $\zeta_n \in C^2(\mathbb{R})$ and

$$\zeta_n(x) = \begin{cases} 1, & x \in B_n(0) \\ 0, & x \notin B_{n+1}(0) \end{cases} \tag{9}$$

$$|\nabla \zeta_n| + |\Delta \zeta_n| \leq C\, |\zeta_n|\,. \tag{10}$$

- **Step 1 - L^p estimate of S for $p \in [1, +\infty]$.**

$$\frac{d}{dt}\int_{\mathbb{R}^2} S^p \zeta_n dx = p \int_{\mathbb{R}^2} S^{p-1}\left(\varepsilon \Delta S + \operatorname{div}\left(S \nabla V_S[S,I,R]\right) - \beta S I\right)\zeta_n dx$$

$$= -p(p-1)\varepsilon \int_{\mathbb{R}^2} S^{p-2} |\nabla S|^2 \zeta_n dx - p\varepsilon \int_{\mathbb{R}^2} S^{p-1} \nabla S \cdot \nabla \zeta_n dx$$

$$- p(p-1) \int_{\mathbb{R}^2} S^{p-1} \nabla S \cdot \nabla V_S[S,I,R] \zeta_n dx$$

$$- p \int_{\mathbb{R}^2} S^p \nabla V_S[S,I,R] \cdot \nabla \zeta_n dx - \beta p \int_{\mathbb{R}^2} S^p I \zeta_n dx\,.$$

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} S^p \zeta_n(x)dx \leq (p-1)\int_{\mathbb{R}^2} S^p \Delta V_S[S,I,R]\zeta_n(x)dx \\
+\varepsilon\int_{\mathbb{R}^2} S^p \Delta\zeta_n(x)dx - \int_{\mathbb{R}^2} S^p \nabla V_S[S,I,R]\nabla\zeta_n(x)dx,
\end{aligned}
$$

$$
\begin{aligned}
&\|\nabla V_S[S,I,R]\|_{L^\infty(\mathbb{R}^2)} + \|\Delta V_S[S,I,R]\|_{L^\infty(\mathbb{R}^2)} \\
&\qquad\qquad \leq C\left(\|S\|_{L^1(\mathbb{R}^2)} + \|I\|_{L^1(\mathbb{R}^2)} + \|R\|_{L^1(\mathbb{R}^2)}\right),
\end{aligned}
$$

$$
\|S\|_{L^p(\mathbb{R}^2)} \leq e^{Ct}\|S_0\|_{L^p(\mathbb{R}^2)}.
$$

- **Step 2 - L^p estimate of I for $p \in [1,+\infty]$.**
- **Step 3 - L^p estimate of R for $p \in [1,+\infty]$.**

- **Step 4. Global existence.** Assuming by contradiction that the maximal existence time T in Proposition 8 is finite, then estimate (8) implies we could extend the solution by continuity to time $t = T$ and prove existence and uniqueness of solutions to the Cauchy problem starting at time $t = T$, with local existence for some further interval $[T, T + \tau)$ for some positive $\tau > 0$. This contradicts T being maximal.

  ■

* The result in proposition 11 implies that for all $T \geq 0$ the family of solutions $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ to (6) parametrised by ε is weakly compact in L^p for finite $p > 1$ (resp. weakly-$*$ compact in L^∞) on the space-time domain $\mathbb{R}^2 \times [0, T]$ provided the initial functions S_0, I_0, R_0 belong to $L^p(\mathbb{R}^d)$.

$$\begin{cases} \partial_t S = \operatorname{div}\left(S\nabla V_S[S,I,R]\right) -\beta SI, \\ \partial_t I = \operatorname{div}\left(I\nabla V_I[S,I,R]\right) +\beta SI - \alpha I, \\ \partial_t R = \operatorname{div}\left(R\nabla V_R[S,I,R]\right) + \alpha I. \end{cases}$$

Proposition 15 [Uniform H^1 estimates]

Let (S, I, R) be the unique solution to the system (6). Assume the interaction kernels satisfy (2) and that the initial functions S_0, I_0, R_0 belong to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and are nonnegative. Then, for a given $T \geq 0$, the H^1-norm of $(S(\cdot,t), I(\cdot,t), R(\cdot,t))$ is uniformly bounded on $[0, T]$ with respect to $\varepsilon > 0$.

Proposition [Vanishing viscosity limit]

Let $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ be the unique solution to the system (6) with the diffusion constant $\varepsilon > 0$. Under the assumptions of Theorem 7, there exists a solution

$$(S, I, R) \in L^\infty([0, T]; H^1(\mathbb{R}^2))^3$$

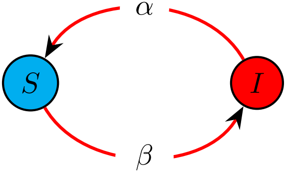
that solves (5) which is the $\varepsilon \searrow 0$ limit of a subsequence of $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ in the $L^2(\mathbb{R}^2 \times [0, T])$-sense for all $T \geq 0$.

Proposition 16 [Uniqueness]

Under the same assumptions of Theorem 7, for a fixed $T \geq 0$, there exists at most one weak solution to (5) with $L^\infty([0, T]\,;\ L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2))$ regularity with a given initial condition (S_0, I_0, R_0) satisfying the assumptions of Theorem 7.

SIS Model

$$\begin{cases} \dot{S} = -\beta SI + \alpha I, \\ \dot{I} = \beta SI - \alpha I\,. \end{cases} \tag{11}$$



Basic Reproduction Number:

$$\mathcal{R}_0 = \frac{\beta N}{\alpha}\,,$$

where $N = S + I$.

Steady States:

- *disease free equilibrium* $(S, I) = (N, 0)$ in case $\mathcal{R}_0 < 1$
- *endemic equilibrium* $(S, I) = \left(\frac{\alpha}{\beta}, N - \frac{\alpha}{\beta}\right)$ in case $\mathcal{R}_0 > 1$

A specific repulsive-attractive potential

We restrict for simplicity to the one-space dimensional case and we set

$$W(x) = x^2 - \gamma |x| \tag{12}$$

for some constant $\gamma > 0$.

$$\begin{aligned} \partial_t S &= \partial_x (S W' * (S + I)) - \beta S I + \alpha I \,, \\ \partial_t I &= \partial_x (I W' * (S + I)) + \beta S I - \alpha I \,. \end{aligned}$$

Basic Reproduction Number:

$$\mathcal{R}_0 = \frac{M\beta}{\gamma\alpha}$$

where $N = S + I$ and

$$M = \int_{\mathbb{R}} N(x) dx > 0 \,.$$

.

Theorem 19 [Steady states]

The stationary system of the model (13) with W given by (12) with $\gamma > 0$ has the one-parameter family of solutions

$$S(x) = \frac{M}{\gamma}\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)\,, \qquad I(x) = 0$$

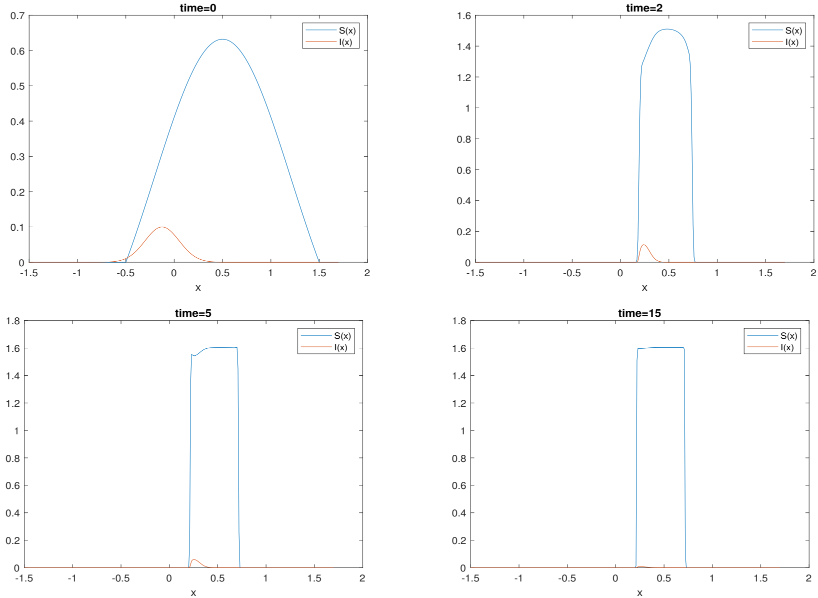
for all $M \geq 0$. Moreover, it has the additional one-parameter family of solutions

$$S(x) = \frac{\alpha}{\beta}\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)\,, \qquad I(x) = \left(\frac{M}{\gamma} - \frac{\alpha}{\beta}\right)\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)$$

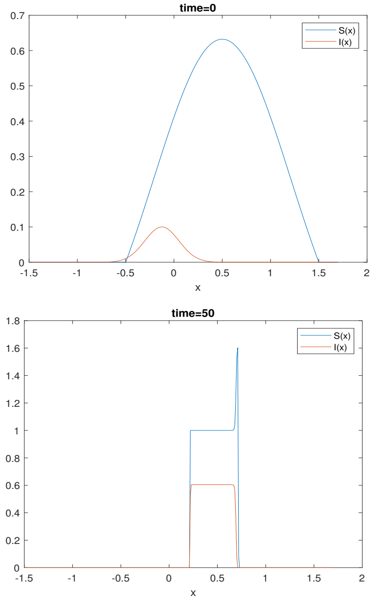
for all

$$M \in \left(\frac{\alpha\gamma}{\beta}, +\infty\right)\,.$$

Numerical Simulations - Disease Free Equilibrium



Numerical Simulations - Endemic Equilibrium



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THANK YOU FOR YOUR ATTENTION