Large-time behavior for Flocking Hydrodynamic models

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Euler–Alignment system

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$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ &\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) = -\rho \int_{\mathbb{R}^d} \psi(x-y)(u(x) - u(y))\rho(y)\, dy. \end{aligned} \tag{EA}$$

$\psi : \mathbb{R}^d \to \mathbb{R}_+$: communication kernel, radial and decreasing in modulus, decreasing function $\phi : [0, +\infty) \to [0, \infty)$ such that $\psi(x) = \phi(|x|)$.

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Agent-based N-particle model (Cucker–Smale (2007))

$$\begin{aligned} \frac{d}{dt} x_i^N &= v_i^N \\ \frac{d}{dt} v_i^N &= -\frac{1}{N} \sum_j^N \psi(x_i^N - x_j^N)(v_i^N - v_j^N). \end{aligned}$$

math biology, emergent collective dynamics, bird flocks, ...

Flocking phenomenon (Large time behavior)

We consider

$$D_t := \mathrm{Diam}(\mathrm{supp}(\rho(t,\cdot))), \quad V_t := \mathrm{Diam}(u(t,\cdot))$$

Flocking means:

$$D_t \le \overline{D}, \quad V_t \to 0$$

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"Unconditional" flocking occurs if

$$\int_1^\infty \phi(r)\,dr = +\infty.$$

'heavy-tail' condition (long-range interaction). For instance, $\phi(r) \sim r^{-1}$ at the far field. Ha-Tadmor (2008), Ha-Liu(2009).

EA system with interaction potential

Question: What happens if we have interaction as well?

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) = -\rho \nabla W \star \rho - \rho \int_{\mathbb{R}^d} \psi(x-y)(u(x)-u(y))\rho(y)\,dy. \tag{EAI}$$

$W : \mathbb{R}^d \to \mathbb{R}$: interaction potential, $W(x) = W(-x)$ for all $x \in \mathbb{R}^d$

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In particular, we are interested in:

1. Characterisation of limiting profile
2. Conditions for "flocking"
3. Convergence rate

What do we have a priori?

Physical Properties of (EAI):

- Mass, momentum, and the center of mass are conserved.
- Energy dissipation identity

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}|u(x,t)-u(y,t)|^2\rho(x,t)\rho(y,t)\,dxdy\right)\\&\quad+\frac{d}{dt}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}W(x-y)\rho(x,t)\rho(y,t)\,dxdy\right)\\&\quad=-\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(x-y)|u(x,t)-u(y,t)|^2\rho(x,t)\rho(y,t)\,dxdy\le 0\end{aligned}$$

They do not provide any direct clue on the asymptotic behavior of global solutions...

$d = 1$ case with λ-convex potential

First, we consider $d = 1$ case with λ-convex potential.

We assume W to satisfy

$$\lambda(x - y) \leq W'(x) - W'(y) \leq \Lambda(x - y) \tag{1}$$

for $0 < \lambda \leq \Lambda < +\infty$ whenever $x > y$. For instance, $W(x) := \frac{x^2}{2}$ (quadratic attractive potential).

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In the case of $d \geq 1$ with $W(x) = \frac{|x|^2}{2}$, Shu-Tadmor (2020) relaxed the 'heavy-tail' condition to

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In the one-dimensional case, we improve this condition to

$$\phi \ge 0.$$

We crucially employ the structure of $1D$.

$1D$ magic

For the one-dimensional case, by taking ∂_x to the momentum equation in (EAI), we obtain:

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(G) + \partial_x(Gu) = -\partial_x^2 W \star \rho$$

where

$$G := \partial_x u + \psi \star \rho.$$

Ideas used in Carrillo-Choi-Tadmor-Tan (2016) to study the regularity of (EA).

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Inspired by this, we introduce

$$\omega(x,t) := u + \Psi \star \rho, \quad \Psi(r) := \int_0^r \phi(s)ds \in L^\infty_{loc}(\mathbb{R}_+)$$

evaluated at $(\eta(x,t),t)$, along the "forward characterstic" of fluid,

$$\partial_t \eta(\cdot,t) = u(\eta(\cdot,t),t), \quad \eta(x,0) = x.$$

η-ω coordinates

We can obtain

$$\partial_t \eta(x,t) = \omega(x,t) - \int_{\mathbb{R}} \Psi(\eta(x,t) - \eta(z,t))\rho_0(z)\, dz,$$
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For fixed $x, y \in \mathrm{supp}(\rho_0)$ $(x > y)$, we have

$$\begin{aligned}\partial_t(\eta(x,t) - \eta(y,t)) &= \omega(x,t) - \omega(y,t)\\ &\quad - \left(\int_{\mathbb{R}} \left(\Psi(\eta(x,t) - \eta(z,t)) - \Psi(\eta(y,t) - \eta(z,t))\right)\rho_0(z)\,dz\right)\end{aligned}$$

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$$\partial_t(\omega(x,t) - \omega(y,t)) \le -\lambda(\eta(x,t) - \eta(y,t)).$$

In particular, we have

$$\partial_t \left(\lambda(\eta(x,t) - \eta(y,t))^2 + (\omega(x,t) - \omega(y,t))^2 \right) \le 0,$$

from which we derive

$$\lambda D_t^2 \le \sup_{x,y \in \mathrm{supp}(\rho_0)} \left(\lambda (D_0)^2 + (\omega(x,0) - \omega(y,0))^2 \right).$$

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Suppose that we assume $\phi(r) > 0$ for all $r \geq 0$. Then, we observe that

$$\phi(\eta(x,t) - \eta(y,t)) \geq \phi(D_t) \geq \phi(\overline{D}) =: \phi_m > 0.$$

In this case, can we chracterise the asymptotic profile and obtain the convergence rate as well?

Main Result 1

Theorem

Let $d = 1$ and (ρ, u) be a global-in-time classical solution of the problem (EAI) *and W satisfy* (1) *with compactly supported ρ_0. We denote*

$$\eta_c(t) := \int_{\mathbb{R}^n} x\rho_0(x)\,dx + tu_\infty, \quad u_\infty := \int_{\mathbb{R}^n} u_0(x)\rho_0(x)\,dx.$$

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Then, we have stability as:

- *If the communication kernel $\psi : \mathbb{R} \to (0, \infty)$ is bounded above, then we have*

$$W_\infty(\rho(t), \delta_{\eta_c(t)}) + \|u(t,\cdot) - u_\infty\|_{L^\infty(supp(\rho(t)))} \to 0$$

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exponentially fast as $t \to \infty$.
- *If the communication kernel $\psi : \mathbb{R} \to (0, \infty)$ is weakly singular as*
$$\psi(x) = |x|^{-\alpha} \quad with \quad \alpha \in (0, 1) \quad for\ all \quad 0 \le r \le R,$$
for some $R > 0$, then we have
$$W_\infty(\rho(t), \delta_{\eta_c(t)}) = \Theta\left(t^{-\frac{1}{\alpha}}\right), \quad \|u(t, \cdot) - u_\infty\|_{L^\infty(supp(\rho(t)))} = \mathcal{O}\left(t^{-\frac{1-\alpha}{\alpha}}\right)$$
as $t \to \infty$.

Why polynomial decay?

Why do we have polynomial decay for the weakly singular kernel?

The particles conglomerate into a single point as time goes to infinity, and the local singularity of the communication kernel kicks in to slow down the process significantly, compared to the bounded kernel case.

Strategy of proof

Denoting

$$X(t) := \eta(x,t) - \eta(y,t), \quad Y(t) := \omega(x,t) - \omega(y,t)$$

- ϕ bounded above: we get

$$\begin{aligned} \dot{X} &\sim Y - cX \\ \dot{Y} &\sim -X. \end{aligned}$$

  where $\phi_m \leq c \leq \phi(0)$. We conduct "hypocoercivity estimates" to obtain exponential decay.

- $\phi(r) = r^{-\alpha}$ near $r \sim 0$ case: we end up with

$$\begin{aligned} \dot{X} &\sim Y - X^{-\alpha} X \\ \dot{Y} &\sim -X. \end{aligned}$$

  We carefully study the dynamics of (X, Y) in the phase plane.

$d = 1$ with $W(x) = -|x| + x^2/2$ case

$W(x) = -|x| + x^2/2$: (repulsive) Poisson + (attractive) quadratic potential.

cf. Carrillo-Choi-Zatorska (2016): $\psi =$ constant.

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For general bounded or weakly singular ψ, we obtain a uniform bound $D_t \le \overline{D}$ under

$$\phi \geq 0.$$

Major difficulty is "oscillatory behavior".

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Lemma
Suppose that $(X(t), Y(t)) \in C^1(\mathbb{R}_+; \mathbb{R}^2)$ *satisfies*

$$\dot{X} = Y - A$$

$$\dot{Y} = -(X - c)$$

for some $c > 0$ *and* $X(t), A(t) \ge 0$ *for all* $t \ge 0$. *We further assume that there exists* $B > 0$ *such that for any* $t \ge 0$ *with* $X(t) \le c$ *then* $A(t) \le B$. *Then we have*

$$X(t) \le c + \max\left\{\sqrt{(X(0) - c)^2 + Y(0)^2},\ \max\{Y(0), B + 1\} + \frac{c^2}{2}\right\}$$

for all $t \ge 0$.

Main Results 2: $d = 1$ with $W(x) = -|x| + x^2/2$ case

Theorem

Let $d = 1$ and (ρ, u) be a global-in-time classical solution of the problem (EAI) *where $W(x) = -|x| + x^2/2$. For*

$$\rho_\infty(x,t) := \frac{1}{2}1_{\{\eta_c(t)-1 \le x \le \eta_c(t)+1\}},$$

we have the following stability:

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$$W_\infty(\rho(t), \rho_\infty(t)) + \|u(t,\cdot) - u_\infty\|_{L^\infty(supp(\rho(t)))} \to 0$$
exponentially fast as $t \to \infty$.
- *If the communication kernel $\psi : \mathbb{R} \to (0,\infty)$ is locally integrable at the origin, then we have*
$$W_2(\rho(t), \rho_\infty(t)) + \|u(t,\cdot) - u_\infty\|_{L^2(\mathbb{R}, d\rho(t))} \to 0$$
as $t \to \infty$.

Remark. We expect exponential decay for weakly singular case.

Main Results 3: multi-D with λ-convex potential

Theorem

Let $d \geq 1$ and (ρ, u) be a global-in-time classical solution of the problem (EAI) *with interaction potential $W(x)$ satisfying*

$$\lambda|x-y|^2 \leq \langle \nabla W(x) - \nabla W(y), x-y \rangle \leq \Lambda |x-y|^2.$$

We assume the communication kernel ψ verifies $\psi(x) \geq \psi_m$ for all $x \in \mathbb{R}^d$ for some $\psi_m > 0$.

- *If the communication weight function ψ is bounded from above, then we have*

$$W_2(\rho(t), \delta_{\eta_c(t)}) + \|u(t,\cdot) - u_\infty\|_{L^2(\mathbb{R}^d, d\rho(t))} \to 0$$

  as $t \to \infty$ exponentially fast.

- *If the communication weigh $\psi(s)$ is given as*

$$\psi(x) = |x|^{-\alpha}$$

  near $|x| \sim 0$ for some $\alpha \in (0,2)$, then we have

$$W_2(\rho(t), \delta_{\eta_c(t)}) + \|u(t,\cdot) - u_\infty\|_{L^2(\mathbb{R}^d, d\rho(t))} \lesssim t^{\frac{2-\alpha}{2\alpha}}$$

  as $t \to \infty$.

Grazie Mille !