Weak perturbation of the $p-$Laplacian

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Criticality vs subcriticality of the Laplace operator

The Laplace operator $-\Delta$ in $L^2(\mathbb{R}^d)$ is unstable for $\mathbf{d = 1, 2}$: given $0 \lneqq V \in C_0(\mathbb{R}^d)$, for **any** $\varepsilon > 0$ there exists $u \in H^1(\mathbb{R}^d)$ such that

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Contrary, if $\mathbf{d \geq 3}$, then there exists $\varepsilon_0 > 0$ such that for **all** $\varepsilon \leq \varepsilon_0$ we have

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In terms of **spectral properties** this mean that

$$\operatorname{spect}_{L^2(\mathbb{R}^d)}(-\Delta - \varepsilon V) \cap (-\infty, 0) \neq \emptyset \qquad \forall\, \varepsilon > 0 \quad d = 1, 2$$

while

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In the language of the criticality theory we say that the Laplace operator in $L^2(\mathbb{R}^d)$ is **critical** if $d = 1, 2$, and **subcritical** if $d \geq 3$.

Equivalent notions

1. **Capacity**: $\mathrm{cap}_2(\Omega, \mathbb{R}^d) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^2,\ u \in C_0^\infty(\mathbb{R}^d),\ u|_\Omega = 1 \right\}$

$$\mathrm{cap}_2(\Omega, \mathbb{R}^d) = 0 \quad \forall\, \Omega \subset \mathbb{R}^d \ \text{ bounded} \qquad d = 1, 2$$

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2. **Heat kernel behavior**: for any $a > 0$ and any $x, y \in \mathbb{R}^d$ we have

$$\int_a^\infty e^{t\Delta}(x,y)\,dt = \infty \quad \text{if } d \leq 2, \qquad \int_a^\infty e^{t\Delta}(x,y)\,dt < \infty \quad \text{if } d \geq 3$$

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3. **Resolvent behavior**: for any $x, y \in \mathbb{R}^d$ we have

$$\lim_{k \to 0} (-\Delta + k^2)^{-1}(x,y) = +\infty \qquad d \le 2,$$

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4. **Existence of an Agmon groundstate**. The equation $\Delta u = 0$ in $\mathbb{R}^d$ admits a positive solution with **minimal growth at infinity** if and only if $d \le 2$. Such a solution satisfies the following condition: $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and there holds

$$\Delta v = 0 \text{ in } \mathbb{R}^d \setminus K \ \wedge\ v > 0 \text{ in } \mathbb{R}^d \setminus K \ \wedge\ u|_{\partial K} \le v|_{\partial K} \ \Rightarrow\ u \le v \ \text{ in } \mathbb{R}^d \setminus K$$

Any constant function is an Agmon groundstate of $-\Delta$ in $\mathbb{R}^d$ with $d = 1, 2$.

Weakly coupled eigenvalues of Schrödinger operators

Let $V \in L^1(\mathbb{R}^d) \cap L^2_{\rm loc}(\mathbb{R}^d)$ be non-negative. If $d = 1, 2$, then the criticality of $-\Delta$ ensures that

$$\lambda(\varepsilon) := \inf \operatorname{spect}(-\Delta - \varepsilon V) < 0 \qquad \forall\, \varepsilon > 0.$$

Obviously, we have

$$\lambda(\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \to 0_+$$

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The problem:

Find the asymptotic expansion of $\lambda(\varepsilon)$ as $\varepsilon \to 0$.

- Note that the perturbation theory à la Kato cannot be applied here.

Weakly coupled eigenvalues of Schrödinger operators

Theorem [Simon 1976]

Let $V \in L^2_{\rm loc}(\mathbb{R}^d), d = 1, 2$, be such that $V(x) = o(|x|^{-3})$ as $|x| \to \infty$.
Assume that $\int_{\mathbb{R}^d} V > 0$. Then, as $\varepsilon \to 0_+$, the operator $-\Delta - \varepsilon V$ has exactly one negative eigenvalue which satisfies

$$\sqrt{-\lambda(\varepsilon)} = \frac{\varepsilon}{2} \int_{\mathbb{R}} V\, dx + o(\varepsilon) \qquad d = 1$$

$$\log(-\lambda(\varepsilon)) = -\frac{2}{\varepsilon} \left(\int_{\mathbb{R}^2} V\, dx \right)^{-1} + o(\varepsilon^{-1}) \qquad d = 2$$

The assumption $\int_{\mathbb{R}^d} V > 0$ guarantees existence of $\lambda(\varepsilon)$ (almost sharp).

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- The leading order terms in the above equations were derived already in

Landau-Lifschitz: *Course of Theoretical Physics, Volume III,* **Edition 1949**

as a (solved) **exercise** . . . ☹

Note added in proof. The lowest order perturbation terms for weak coupling have been obtained in Landau and Lifshitz, "Quantum Mechanics," pp. 156–157. I should like to thank J. Klauder for bringing this to my attention.

The sub-leading term

A test function argument shows that $-\Delta - \varepsilon V$ has a negative eigenvalue for any $\varepsilon > 0$ even if

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$$\sqrt{-\lambda(\varepsilon)} = \frac{\varepsilon}{2} \int_{\mathbb{R}} V(x)\, dx - \frac{\varepsilon^2}{4} \int_{\mathbb{R}^2} V(x)|x-y|\, V(y)\, dx dy + o(\varepsilon^2)$$

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- What happens if $\int_{\mathbb{R}} V(x) dx = 0$? Simon shows that in this case

$$\int_{\mathbb{R}^2} V(x)|x-y|\, V(y)\, dx dy < 0 \qquad ☺$$

Main ingredients of Simon's proof

The Birman-Schwinger principle: suppose for simplicity that $V \geq 0$. The BS-principle then says that $-\Delta - \varepsilon V$ has eigenvalue $\lambda(\varepsilon) < 0$ **if and only if** the operator

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Notice that the operator $\sqrt{V} \big(-\Delta - \lambda(\varepsilon) \big)^{-1} \sqrt{V}$ is

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- Similar analysis is performed for $d = 2$ (much harder)

Extensions and generalizations

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- Dirac and Pauli operators **[Baur 2025]**

- All the above operators are linear.

The non-linear problem

We consider the weak coupling problem for the perturbed $p-$Laplacian which acts as

$$(-\Delta_p - V)(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) - V|u|^{p-2}\, u, \qquad p > 1$$

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- The operator $-\Delta_p$ is **critical** in $\mathbb{R}^d$ if and only if $d \leq p$.

Therefore we further generalize the weak coupling problem by considering

$$\begin{aligned} &-\Delta_p - \varepsilon V && \text{if} \quad d \leq p && \text{(TE \& RF)} \\ &-\Delta_p - W - \varepsilon V && \text{if} \quad d > p && \text{(UD \& YP)} \end{aligned}$$

Here W is a **background potential** which makes $-\Delta_p - W$ **critical** in $\mathbb{R}^d$.

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- The sharp weight in the Hardy inequality;

$$W(x) = \left(\frac{d-p}{p}\right)^p |x|^{-p}$$

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- **The goal** is to study the asymptotic behavior of

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- $\lambda : [0, \infty) \to \mathbb{R}$ is a continuous concave function of ε.

- The results depend heavily on the relation between p and d.

Main results: $d < p, \ W = 0$

Theorem [Ekholm-Frank-K 2016]

Let $p > d \geq 1$. Let $V \in L^1(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} V\, dx > 0$. Then, as $\varepsilon \to 0_+$

$$\lambda(\varepsilon) = \frac{d-p}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} \varepsilon^{\frac{p}{p-d}} \left(S_{d,p} \int_{\mathbb{R}^d} V(x)\, dx\right)^{\frac{p}{p-d}} + o(\varepsilon^{\frac{p}{p-d}})$$

where $S_{d,p}$ is the sharp constant in the Gagliardo-Nirenberg inequality

$$\|u\|_\infty^p \leq S_{d,p} \, \|\nabla u\|_p^d \, \|u\|_p^{p-d}, \qquad u \in W^{1,p}(\mathbb{R}^d), \quad d < p.$$

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$$\lambda(\varepsilon) = \frac{d-p}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} \varepsilon^{\frac{p}{p-d}} \left(S_{d,p} \int_{\mathbb{R}^d} V(x)\, dx\right)^{\frac{p}{p-d}} + o(\varepsilon^{\frac{p}{p-d}})$$

where $S_{d,p}$ is the sharp constant in the Gagliardo-Nirenberg inequality

$$\|u\|_\infty^p \leq S_{d,p} \, \|\nabla u\|_p^d \, \|u\|_p^{p-d}, \qquad u \in W^{1,p}(\mathbb{R}^d), \quad d < p.$$

- The hypothesis $V \in L^1(\mathbb{R}^d)$ is almost optimal: if $V_+ \notin L^1(\mathbb{R}^d)$ and $V_- \in L^1(\mathbb{R}^d)$. Then, as $\varepsilon \to 0_+$

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-\frac{p}{p-d}} \, \lambda(\varepsilon) = -\infty.$$

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- The value of $S_{d,p}$ is known only for $d = 1$: we have $S_{1,p} = \frac{p}{2}$.
- Hence for $p = 2$ we recover the leading term in the Simon's result.

Main results: $d = p, \ W = 0$

Theorem [Ekholm-Frank-K 2016]

Let $p = d > 1$. Suppose that $V \in L^q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for some $q > 1$ and that $\int_{\mathbb{R}^d} V\, dx > 0$. Then, as $\varepsilon \to 0_+$

$$\log(-\lambda(\varepsilon)) = -d\, \omega_d^{\frac{1}{d-1}}\, \varepsilon^{\frac{1}{1-d}} \left(\int_{\mathbb{R}^d} V(x)\, dx \right)^{\frac{1}{1-d}} + o(\varepsilon^{\frac{1}{1-d}})$$

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where ω_d denotes the surface area of the unit sphere in $\mathbb{R}^d$.

- The theorem doesn't hold for $q = 1$. Indeed, there exist singular potentials $V \in L^1(\mathbb{R}^d)$ such that

$$\lambda(\varepsilon) = -\infty$$

for any $\varepsilon > 0$. For example,

$$V(x) = |x|^{-d} \big| \log |x| \big|^{-a}, \ a > 1, \qquad \text{if } \ |x| \leq e^{-1},$$

and $V(x) = 0$ elsewhere.

Main results: $d > p, \;\; W \neq 0$

We assume that the operator $-\Delta_p - W$ is critical. Hence it admits an Agmon ground state ϕ_0. In other words, a positive solution of

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) - W|u|^{p-2}\, u = 0$$

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- Notice that $\phi_0 \in L^p(\mathbb{R}^d)$ **if and only if** $d > p^2$.

Main results: $p^2 < d,\ W \neq 0$

Theorem [Das-K-Pinchover-2025]

Suppose that $W \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let ϕ_0 be the corresponding normalized ground state and let $V \in C_0(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} V\phi_0^p dx > 0$. Then, as $\varepsilon \to 0_+$

$$\lambda(\varepsilon) = -\,\varepsilon\,\frac{\int_{\mathbb{R}^d} V\phi_0^p\,dx}{\int_{\mathbb{R}^d} \phi_0^p\,dx} + o(\varepsilon).$$

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- Note that if $d \leq p$, then the normalized Agmon ground state is given by $\phi_0 = 1$.

Main results: $p < d \le p^2,\ W \neq 0$

Theorem [Das-K-Pinchover-2025]

Let $V, W \in C_0(\mathbb{R}^d)$. Suppose that $\int_{\mathbb{R}^d} V\phi_0^p dx > 0$.

- If $p < d < p^2$, then

$$\lambda(\varepsilon) \asymp -\varepsilon^{\frac{p(p-1)}{d-p}} \left(\int_{\mathbb{R}^d} V\phi_0^p\, dx\right)^{\frac{p(p-1)}{d-p}} \qquad \varepsilon \to 0_+$$

- If $d = p^2$, then

$$\lambda(\varepsilon) \asymp \frac{\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^d} V\phi_0^p\, dx \qquad \varepsilon \to 0_+$$

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- Here $\asymp$ means that exist positive constants K_1, K_2, **independent of** V, such that

$$K_1 \le \liminf_{\varepsilon \to 0_+} \frac{\varepsilon^{\frac{p(p-1)}{p-d}} |\lambda(\varepsilon)|}{\left(\int_{\mathbb{R}^d} V\phi_0^p\, dx\right)^{\frac{p(p-1)}{d-p}}} \le \limsup_{\varepsilon \to 0_+} \frac{\varepsilon^{\frac{p(p-1)}{p-d}} |\lambda(\varepsilon)|}{\left(\int_{\mathbb{R}^d} V\phi_0^p\, dx\right)^{\frac{p(p-1)}{d-p}}} \le K_2$$

if $d < p^2$ and accordingly for the case $d = p^2$.

Summary: Asymptotic order of $\lambda(\varepsilon)$.

Dimension	Leading order of $\lambda(\varepsilon)$	Critical potential
$d < p$	$\varepsilon^{\frac{p}{p-d}}$	$W = 0$
$d = p$	$\exp\left[\left(-c/\varepsilon\right)^{\frac{1}{d-1}}\right]$	$W = 0$
$p < d < p^2$	$\varepsilon^{\frac{p(p-1)}{d-p}}$	$W \in C_0(\mathbb{R}^d)$
$d = p^2$	$\dfrac{\varepsilon}{\log \varepsilon}$	$W \in C_0(\mathbb{R}^d)$
$d > p^2$	ε	$W \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

- The case $\int_{\mathbb{R}^d} V \, dx = 0$ remains open for all $p \neq 2$.

Figure: Tomas Ekholm

Figure: Rupert Frank

Figure: Ujjal Das

Figure: Yehuda Pinchover

THANK YOU!