Ground states for the Choquard equation in the Thomas–Fermi limit

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Choquard-type equation

$$-\Delta w + \epsilon w + |w|^{q-2}w = (I_\alpha * |w|^p)|w|^{p-2}w \quad \text{in } \mathbb{R}^N$$

where I_α is the Riesz potential.

- $p = 2, \alpha = 2$ in most practical applications, like the Schrodinger-Newton system

$$\begin{cases} -\Delta w + \epsilon w + |w|^{q-2}w = \psi w, \\ -\Delta \psi = |w|^2. \end{cases}$$

- $N = 3, \alpha = 2, p = 2, q = 4$: Gross-Pitaevskii-Poisson equation for cold dark matter or self-gravitating Bose-Einstein condensates
- Qualitative behaviours depend mainly on the value of p [1]:

$$\begin{cases} \lim_{|x|\to\infty} w_\epsilon(x)|x|^{\frac{N-1}{2}} e^{\sqrt{\epsilon}|x|} = C, & p > 2 \\ \lim_{|x|\to\infty} w_\epsilon(x)|x|^{\frac{N-\alpha}{2-p}} = \left(\epsilon^{-1} A_\alpha \|w_\epsilon\|_p^p\right)^{\frac{1}{2-p}}, & p < 2. \end{cases}$$

[1] Z. Liu and V. Moroz, Calc. Var. Partial Differential Equations 61 (2022)

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Different limits of this Choquard-type equation

$$-\Delta w + \epsilon w + |w|^{q-2} w = (I_\alpha * |w|^p)|w|^{p-2} w \quad \text{in } \mathbb{R}^N$$

- The formal limit $\epsilon \to 0$:

$$-\Delta w + \cancel{\epsilon w} + |w|^{q-2} w = (I_\alpha * |w|^p)|w|^{p-2} w;$$

- Choquard limit with $v_\epsilon(x) = \epsilon^{-\frac{2+\alpha}{4(p-1)}} w_\epsilon(\epsilon^{-1/2} x)$:

$$-\Delta v + v + \cancel{\epsilon^{\frac{(2+\alpha)(q-2)}{4(p-1)} - 1} |v|^{q-2} v} = (I_\alpha * |v|^p)|v|^{p-2} v;$$

- Thomas-Fermi limit $u_\epsilon(x) = \epsilon^{-\frac{1}{q-2}} w_\epsilon(\epsilon^{-\frac{4-q}{\alpha(q-2)}} x)$:

$$\cancel{-\epsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \Delta u} + u + |u|^{q-2} u = (I_\alpha * |u|^p)|u|^{p-2} u.$$

Outline

1. **Variational setup and basic properties**

2. **Qualitative properties: reularity, decay and support**

3. **Thomas-Fermi limit of the Choquard equation**

4. **Limiting ground states for $\alpha \to 0^+$ or $\alpha \to N^-$**

5. **Conclusion and open problems**

Outline for section 1

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Related equations for $p = 2$

The governing equation

$$u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$

for $p = 2$ reduces to (with $\rho = |u|^2 \geq 0$)

$$1 + \rho^{q/2-1} = I_\alpha * \rho, \qquad \text{on } \{\rho > 0\},$$

with many results in the context of Keller-Segel equations about existence [2] or uniqueness [3].

[2] V. Calvez, J. A. Carrillo, and F. Hoffmann, Nonlinear Anal. (2017); J. A. Carrillo, F. Hoffmann, E. Mainini, and B. Volzone, Calc. Var. Partial Differential Equations (2018)

[3] J. A. Carrillo and Y. Sugiyam, Indiana Univ. Math. J. (2018); V. Calvez, J. A. Carrillo, and F. Hoffmann, Nonlinear Anal. (2021); H. Chan, M. del Mar González, Y. Huang, E. Mainini, and B. Volzone, Calc. Var. Partial Differential Equations (2020)

Variational setup for the Thomas-Fermi equation

$$u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$

with $p > \frac{N+\alpha}{N}, q > \frac{2Np}{N+\alpha}$.

Solutions are critical points of the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

with $\mathcal{D}_\alpha(f, g) = A_\alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dxdy, \quad A_\alpha = \frac{\Gamma((N-\alpha)/2)}{\pi^{N/2} 2^\alpha \Gamma(\alpha/2)}.$

Constrained optimisation for ground states of $u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$

$$\inf E(u) = \inf \left\{ \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p) \right\}$$

for $u \in L^2(\mathbb{R}^N) \bigcap L^q(\mathbb{R}^N)$, subject to natural constraints like the *Nehari manifold*

$$\mathscr{N} = \left\{ u \neq 0 : \int_{\mathbb{R}^2} |u|^2 dx + \int_{\mathbb{R}^N} |u|^q dx = \mathcal{D}_\alpha(|u|^p, |u|^p) \right\},$$

or the *Pohozaev manifold*

$$\mathscr{P} = \left\{ u \neq 0 : \frac{N}{2}\int_{\mathbb{R}^2} |u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} |u|^q dx = \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p) \right\}.$$

Other constrains like $\mathscr{D} = \{u \neq 0 : d = \mathcal{D}_\alpha(|u|^p, |u|^p)\}$, with rescaling.

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Formulation with quotient of norms

$$\mathscr{C}_{N,\alpha,p,q} = \sup\left\{\mathcal{R}(u) = \frac{\mathcal{D}_\alpha(|u|^p,|u|^p)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} : u \in L^q(\mathbb{R}^N)\cap L^2(\mathbb{R}^N), u \neq 0\right\}$$

Easy to see the range of exponents via HLS inequality:

$$\mathcal{D}_\alpha(|u|^p,|u|^p) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le \mathscr{C}_{N,\alpha}\|u\|^{2p}_{2Np/(N+\alpha)},$$

i.e., $2 < \frac{2Np}{N+\alpha} < q$ or equivalently

$$p > \frac{N+\alpha}{N}, \qquad q > \frac{2Np}{N+\alpha},$$

with $\mathscr{C}_{N,\alpha,p,q} \le \mathscr{C}_{N,\alpha}$.

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Alternative interpolation inequality

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\rho(x)|\,|\rho(y)|}{|x-y|^{d-\alpha}}dx\,dy \leq \tilde{\mathcal{C}}_{N,\alpha,m,n}\Big(\int_{\mathbb{R}^N}|\rho|^n dx\Big)^{\frac{2\theta}{n}}\Big(\int_{\mathbb{R}^N}|\rho|^m dx\Big)^{\frac{2(1-\theta)}{m}}$$

with $\rho = |u|^p, m = q/p, n = 2/p$, for all $\rho \in L^n(\mathbb{R}^N)\bigcap L^m(\mathbb{R}^N)$ and $0 < n < \frac{2N}{N+\alpha} < m$.

Existing works mainly on $n = 1 (p = 2)$. Besides those more recent on the Keller-Segel models, earlier works include HLS inequality [4].

[4] P.-L. Lions, Ann. Inst. H. Poincaré Anal. Non Linéaire (1984)

Explicit ground states for $p < 2$

$$u + |u|^{q-2}u = \big(I_\alpha * |u|^p\big)|u|^{p-2}u$$

There is one-parameter family of explicit ground states

$$p = \frac{N+\alpha+2}{N+1}, \quad q = \frac{2(N+2)}{N+1}, \quad v(x) = A(R^2+|x|^2)^{-(N+1)/2},$$

for suitable $A > 0$ and $R > 0$.

Existence of ground states via quotient

$$\mathscr{C}_{N,\alpha,p,q} = \sup\left\{\mathcal{R}(u) = \frac{\mathcal{D}_\alpha(|u|^p,|u|^p)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} : u \in L^q(\mathbb{R}^N)\cap L^2(\mathbb{R}^N), u \neq 0\right\}$$

Sketch of the proof for the existence of optimisers:

- Radial optimisers via Schartz symmetrization: $u_n \to u_n^*$
- Normalisation $\|u_n^*\|_2 = \|u_n^*\|_q = 1$ (for compactness)
- Strauss's L^s-bounds for radial non-increasing functions:

$$u_n^*(x) \leq U(x) := C\max\{|x|^{-N/2}, |x|^{-N/q}\}.$$

- $u_n^* \to u$ and non-local Brezis-Lieb lemma:

$$\lim_{n\to\infty} \mathcal{D}_\alpha(u_n^p, u_n^p) = \mathcal{D}_\alpha(u^p, u^p) = \mathscr{C}_{N,\alpha,p,q}.$$

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From optimisers to ground states via scaling

Maximizers of the quotient $\frac{\mathcal{D}_\alpha(|u|^p,|u|^p)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}}$ satisfy

$$Au + B|u|^{q-2}u = C(I_\alpha * |u|^p)|u|^{p-2}u$$

where

$$A = \frac{2p\theta}{\|u\|_2^2}, \quad B = \frac{2p(1-\theta)}{\|u\|_q^q}, \quad C = \frac{2p}{\mathcal{D}_\alpha(|u|^p, |u|^p)}$$

The coefficients A, B, C can be made equal via $u \mapsto u_* = \lambda u(\mu \cdot)$:

$$\lambda^{q-2} = \frac{1-\theta}{\theta} \cdot \frac{\|u\|_2^2}{\|u\|_q^q}, \quad \mu^\alpha = \frac{1-\theta}{\lambda^{q-2p}} \cdot \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|_2^q}.$$

Equivalence between formulations

Constrained minimization problem:

$$\sigma_* = \inf_{u \in \mathscr{P}} \left\{ E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p) \right\}$$

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Maximization of the quotient:

$$\mathcal{C}_{N,\alpha,p,q} = \sup_{u \neq 0} \left\{ R(u) = \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} \right\}$$

The key is the following identity linking the optimal constants:

$$\boxed{\sigma_* = \alpha(2Np)^{\frac{N}{\alpha}} \left(\frac{\theta_*}{N+\alpha}\right)^{\frac{N+\alpha}{\alpha}} \mathscr{C}_{N,\alpha,p,q}^{-\frac{N}{\alpha}}, \quad \theta_* = \left(\frac{1-\theta}{\theta}\right)^{\frac{q\theta}{(2(1-\theta)+q\theta)}} \cdot \frac{N+\alpha}{2Np(1-\theta)}.}$$

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Equivalence between formulations

The intermidate optimisation problem:

$$\inf \mathcal{E}(w) = \frac{1}{2}\int |u|^2 + \frac{1}{q}\int |u|^q$$

over the set $\mathcal{A} = \{u \in L^2(\mathbb{R}^N) \bigcap L^q(\mathbb{R}^N) : \mathcal{D}_\alpha(|u|^p, |u|^p) = 1\}$.

- $w_t(\cdot) = t^{-(N+\alpha)/2p} w(\cdot/t)$ is invariant, and

$$\mathcal{E}(w) \geq \mathcal{E}(w_{t*}) = \left(\|w\|_2^{2p\theta} \|w\|_q^{2p(1-\theta)}\right)^{N/(N+\alpha)} \theta_* \geq (\mathscr{C}_{N,\alpha,p,q})^{-N/(N+\alpha)} \theta_*.$$

- If $u \in \mathscr{P}$, $u(\cdot/\tau_u) \in \mathcal{A}$ for

$$\tau_u = (\mathcal{D}_\alpha(|u|^p, |u|^p))^{-1/(N+\alpha)} = \left(\frac{N+\alpha}{2Np\mathcal{E}(u)}\right)^{1/\alpha}$$

  and if $w \in \mathcal{A}$, $w(\cdot/t_w) \in \mathscr{P}$ with $t_w = (2Np\mathcal{E}(w)/(N+\alpha))^{1/\alpha}$. Optmise

$$E(u) = \mathcal{E}(u) - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p) = \frac{\alpha}{N+\alpha}\left(\frac{2Np}{N+\alpha}\right)^{N/\alpha} \mathcal{E}(w)^{(N+\alpha)/\alpha}.$$

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Numerical gradient flow

The (usual) gradient flow for

$$\inf_u \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 dx + \frac{1}{q} \int_{\mathbb{R}^n} |u|^q dx \right\}, \qquad \text{s. t.} \quad \mathcal{D}_\alpha(|u|^p, |u|^p) = C,$$

is given by

$$\frac{d}{dt} u = -\left(u + |u|^{q-2}\right) + \lambda (I_\alpha * |u|^p) |u|^{p-2} u,$$

where the parameter λ is dermterined by $\frac{d}{dt} \mathcal{D}_\alpha(|u|^p, |u|^p) = 0$, or

$$\lambda = \left[\int (I_\alpha * |u|^p)^2 |u|^{2p-2} dx \right]^{-1} \int I_\alpha * |u|^p (|u|^p + |u|^{p+q-2}) dx.$$

The solution has to be rescaled (spatially) again, to satisfy the Nehari or Pohozaev identity.

Outline for section 2

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2. Qualitative properties: reularity, decay and support

3. Thomas-Fermi limit of the Choquard equation

4. Limiting ground states for $\alpha \to 0^+$ or $\alpha \to N^-$

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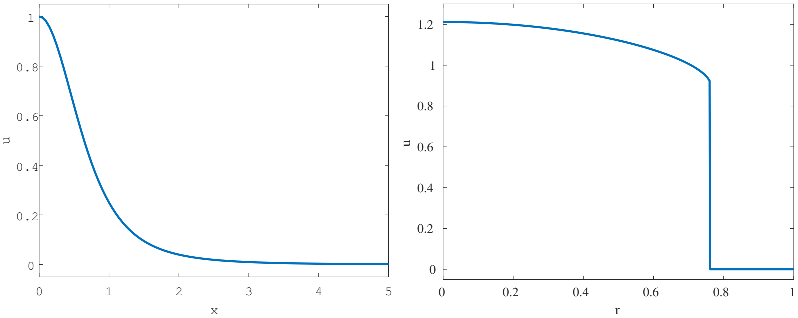
General behavious of ground states

$$u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$



Both in three dimension with $p = 1.5(< 2), \alpha = 1, q = 2.5$ (left) and $p = 4 > 2, \alpha = 2.5, q = 8$ (right).

Full support for $p < 2$

Lemma

Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. If $p < 2$ then $\mathrm{Supp}(u) = \mathbb{R}^N$.

If there is a set A such that $A \bigcap \mathrm{Supp}(u) = \emptyset$, then the perturbed function $u_\epsilon(x) = u(x) + \epsilon\chi_A(x)$ has a larger ratio $\mathcal{R}(u_\epsilon)$:

$$\mathcal{R}(u_\epsilon) = \frac{\mathcal{D}_\alpha(|u+\epsilon\chi_A|^p, |u+\epsilon\chi_A|^p)}{\left(\int_{\mathbb{R}^N}(u^2+\epsilon^2\chi_A)dx\right)^{p\theta}\left(\int_{\mathbb{R}^N}(u^q+\epsilon^q\chi_A)dx\right)^{\frac{2p(1-\theta)}{q}}}$$

$$\geq \mathcal{R}(u) + \frac{2\epsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A) - C\epsilon^2}{1+C\epsilon^2} > \mathcal{R}(u)$$

if ϵ is small enough.

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Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. If $p < 2$ then $\mathrm{Supp}(u) = \mathbb{R}^N$.

If there is a set A such that $A \bigcap \mathrm{Supp}(u) = \emptyset$, then the perturbed function $u_\epsilon(x) = u(x) + \epsilon \chi_A(x)$ has a larger ratio $\mathcal{R}(u_\epsilon)$:

$$\mathcal{R}(u_\epsilon) = \frac{\mathcal{D}_\alpha(|u + \epsilon\chi_A|^p, |u + \epsilon\chi_A|^p)}{\left(\int_{\mathbb{R}^N}(u^2 + \epsilon^2\chi_A)dx\right)^{p\theta}\left(\int_{\mathbb{R}^N}(u^q + \epsilon^q\chi_A)dx\right)^{\frac{2p(1-\theta)}{q}}}$$

$$\geq \mathcal{R}(u) + \frac{2\epsilon^p\mathcal{D}_\alpha(|u|^p, \chi_A) - C\epsilon^2}{1 + C\epsilon^2} > \mathcal{R}(u)$$

if ϵ is small enough.

Compactly supported ground states for $p > 2$

$$u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$

If $\{u > 0\} = \mathbb{R}^N$, then the governing equation can be written as

$$1 + u^{q-2} = (I_\alpha * u^p)u^{p-2}.$$

This leads to a contradiction when $|x|$ is large.

Far field decay of u and $I_\alpha * u^p$

Lemma

Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2 \bigcap L^q$ be a nonnegative radial nonincreasing ground state of $u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$. Then there exists $\epsilon > 0$ such that $u \in L^{p-\epsilon}(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} \dfrac{I_\alpha * u^p}{I_\alpha(x) \int_{\mathbb{R}^N} u^p dx} = 1.$

- Trivial for $p > 2$ (the solution is compactly supported)
- For $p \in (\frac{N+\alpha}{N}, 2)$, if $|u|^p$ decays slightly faster than functions in L^1, then $I_\alpha * |u|^p$ shows the same decay as I_α.
- To show $u \in L^{p-\epsilon}(\mathbb{R}^N)$ by iteration: if $u \in L^{s_n}(\mathbb{R}^N)$, then we can show that $u \in L^{s_{n+1}}(\mathbb{R}^N)$, for $s_{n+1} < s_n$.

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Decay for $p \in (\frac{N+\alpha}{N}, 2)$ via iteration

Sketch of the proof: From $(1+|u|^{q-2})u = (I_\alpha * |u|^p)|u|^{p-2}u$ and Hölder inequality, we have

$$\int_{\mathbb{R}^N} |u|^\sigma dx \leq \int_{\mathbb{R}^N} |(I_\alpha * u^p)u^{p-1}|^\sigma dx \leq \Big(\int_{\mathbb{R}^N} |(I_\alpha * u^p)|^{\sigma t} dx\Big)^{\frac{1}{t}} \Big(\int_{\mathbb{R}^N} u^{(p-1)\sigma r} dx\Big)^{\frac{1}{r}},$$

with $1/t + 1/r = 1$. Now choose

$$\sigma = s_{n+1}, \qquad (p-1)\sigma r = s_n, \qquad \frac{1}{\sigma t} = \frac{p}{s_n} - \frac{\alpha}{N},$$

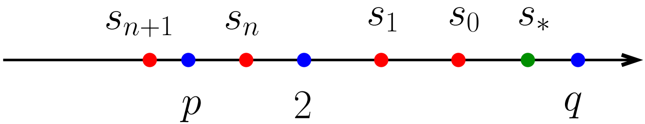
which is equivalent to

$$\frac{1}{s_{n+1}} = \frac{1}{\sigma t} + \frac{1}{\sigma r} = \frac{p}{s_n} - \frac{\alpha}{N} + \frac{p-1}{s_n} = \frac{2p-1}{s_n} - \frac{\alpha}{N}.$$

That is $\|u\|_{s_{n+1}} \leq C_n \|u\|_{s_n}$, provided that $\frac{\alpha}{N} < \frac{p}{s_n} < 1$ for HLS inequality to hold.

Decay for $p \in (\frac{N+\alpha}{N}, 2)$ via iteration

$$\frac{1}{s_n} = (\underbrace{2p-1}_{>1})^n \underbrace{\left(\frac{1}{s_0} - \frac{1}{s_*}\right)}_{>0} + \frac{1}{s_*}, \quad s_* = \frac{2N(p-1)}{\alpha} > 2.$$



Choose s_0 appropriately so that $s_n > p$ and $s_{n+1} < p$ (they depend continuously on s_0, as long as $s_0 < s_*$ and $\alpha/N < p/s_n < 1$).

L^∞-bounds and Hölder continuity

If q is larger than Np/α and $u \in L^q(\mathbb{R}^N)$, then $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$ and is Hölder continuous [5].

If $q \in \left(\frac{2Np}{N+\alpha}, \frac{Np}{\alpha}\right)$, then we can use similar argument as in the previous case (with $|u|^{q-2}u \approx (I_\alpha * |u|^p)|u|^{p-2}u$):

$$\int_{\mathbb{R}^N} |u|^{(q-1)\sigma_n} dx \leq \int_{\mathbb{R}^N} |(I_\alpha * u^p) u^{p-1}|^{\sigma_n} dx \leq \Big(\int_{\mathbb{R}^N} |(I_\alpha * u^p)|^{\sigma_n t} dx\Big)^{\frac{1}{t}} \Big(\int_{\mathbb{R}^N} u^{(p-1)\sigma_n r} dx\Big)^{\frac{1}{r}},$$

by choosing

$$\frac{1}{s_{n+1}} = \frac{1}{(q-1)\sigma_n} = \underbrace{\frac{2p-1}{q-1}}_{<1} \cdot \frac{1}{s_n} - \frac{\alpha}{N(q-1)}.$$

The sequence s_n is monotonically increasing, as long as $s_n < Np/\alpha$.
The sequence terminates instead of converging to $s_* = \frac{N(2p-q)}{\alpha} > q$.

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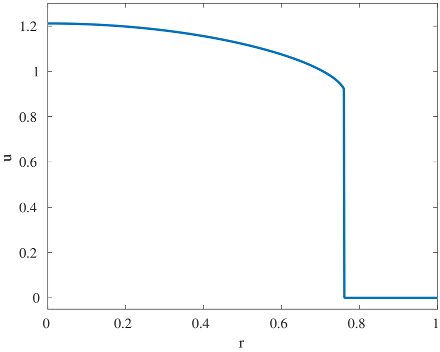
Other properties for ground states $p > 2$



- Lower bound $\lambda_* = \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}}$ of the jump near the boundary
- C^∞ inside the support and Hölder continuous up to the boundary
- $R \to \infty$ as $\alpha \to 0$ and $R \to 0$ as $\alpha \to N$

Outline for section 3

1. Variational setup and basic properties

2. Qualitative properties: reularity, decay and support

3. **Thomas-Fermi limit of the Choquard equation**

4. Limiting ground states for $\alpha \to 0^+$ or $\alpha \to N^-$

5. Conclusion and open problems

The Choquard equation and the rescaled one

$$-\Delta w + \epsilon w + |w|^{q-2} w = (I_\alpha * |w|^p)|w|^{p-2} w$$

The rescaled function $u(x) = \epsilon^{-\frac{1}{q-2}} w(\epsilon^{-\frac{2p-q}{\alpha(q-2)}} x)$ satisfies

$$-\epsilon^\nu \Delta u + u + |u|^{q-2} u = (I_\alpha * |u|^p)|u|^{p-2} u,$$

with $\nu := \frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}$.

Question: What is the limiting ground states of $u(x)$ as $\epsilon^\nu \to 0$?

The Thomas-Fermi limit regime

$$-\epsilon^\nu \Delta u + u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$

In the variational framework,

$$\sigma_\epsilon = \inf_{\mathcal{P}_\varepsilon(u)=0} \mathcal{J}_\epsilon(u) = \mathcal{J}_\epsilon(u_\epsilon) \quad \xrightarrow[\epsilon^\nu \to 0^+]{?} \quad \sigma_* = \inf_{\mathcal{P}_0(u)=0} \mathcal{J}_0(u) = \mathcal{J}_0(u_*)$$

where

$$\mathcal{J}_\epsilon(u) = \frac{1}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

$$\mathcal{P}_\varepsilon(u) = \frac{N-2}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p).$$

$$\mathcal{J}_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

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The Thomas-Fermi limit regime

$$-\epsilon^{\nu}\Delta u + u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u$$

In the variational framework,

$$\sigma_\epsilon = \inf_{\mathcal{P}_\varepsilon(u)=0} \mathcal{J}_\epsilon(u) = \mathcal{J}_\epsilon(u_\epsilon) \quad \xrightarrow[\epsilon^\nu \to 0^+]{?} \quad \sigma_* = \inf_{\mathcal{P}_0(u)=0} \mathcal{J}_0(u) = \mathcal{J}_0(u_*)$$

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The Thomas-Fermi limit regime

Theorem (Greco-H.-Liu-Moroz(2025))

Let $N \geq 3$ and $\alpha \in (0, N)$. Assume that either of the following holds:

(*i*) $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > 2\frac{2p+\alpha}{2+\alpha}$, *or* $p > \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha}$;

(*ii*) $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $\frac{2Np}{N+\alpha} < q < 2\frac{2p+\alpha}{2+\alpha}$.

Then there exists a sequence of parameters $(\epsilon_k)_{k\in\mathbb{N}}$ *with the corresponding ground states* (u_{ε_k}) *of* (P_{ε_k}) *such that* $\epsilon_k^{\nu} \to 0$ *and the rescaled sequence of ground states* $u_{\epsilon_k}(x)$ *converges in* $L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *to a nonnegative ground state of the Thomas-Fermi equation with* $\varepsilon_k^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \|\nabla u_{\varepsilon_k}\|_2^2 \to 0$ *as* $k \to \infty$.

The T-F limit regime for $\inf_{\mathcal{P}_\epsilon(u)=0} \mathcal{J}_\epsilon(u)$

$$\mathcal{J}_\epsilon(u) = \frac{1}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

$$\mathcal{P}_\varepsilon(u) = \frac{N-2}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p).$$

The key is to control the *vanishing* term $\epsilon^\nu \int |\nabla u|^2 dx$ along the limit: relatively easy if $u_\epsilon \to u_*$ with $\|\nabla u\|_2$ bounded (for $p < 2$), and more difficult if not (for $p > 2$).

Easy case if $\|\nabla u_*\|_2$ is bounded

$$\mathcal{J}_\epsilon(u) = \frac{1}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

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Let $u_\epsilon \in \mathscr{P}_\epsilon = \{u \in L^2(\mathbb{R}^N) \bigcap \mathbb{R}^q(\mathbb{R}^N) : \mathcal{P}_\epsilon(u) = 0\}$ be a ground state of $\mathcal{J}_\epsilon$.

- From $\mathcal{P}_0(u_\epsilon) = \mathcal{P}_\epsilon(u_\epsilon) - \frac{N-2}{2}\epsilon^\nu \|\nabla u\|_2^2 < 0$, $\mathcal{P}_0(u_\epsilon(\cdot/t_\epsilon)) = 0$ for some $t_\epsilon \in (0,1)$. Then

$$\begin{aligned}\sigma_* = \inf_{u\in\mathcal{P}_0} \mathcal{J}_0(u) &= \frac{\alpha t_\epsilon^{N+\alpha}}{2Np}\mathcal{D}_\alpha(|u_\epsilon|^p, |u_\epsilon|^p) \\ &< \frac{\alpha}{2Np}\mathcal{D}_\alpha(|u_\epsilon|^p, |u_\epsilon|^p) + \frac{\epsilon^\nu}{N}\|\nabla u_\epsilon\|_2^2 = \mathcal{J}_\epsilon(u_\epsilon) = \sigma_\epsilon.\end{aligned}$$

- Similarly, from u_*, we get $u_*(\cdot/t_\epsilon) \in \mathcal{P}_\epsilon$ for some $t_\epsilon > 1$. Together with $t_\epsilon \to 1$, we can show $\lim_{\epsilon\to 0} \sigma_\epsilon \le \sigma_*$.

Easy case if $\|\nabla u_*\|_2$ is bounded

$$\mathcal{J}_\epsilon(u) = \frac{1}{2}\epsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

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Let $u_\epsilon \in \mathscr{P}_\epsilon = \{u \in L^2(\mathbb{R}^N) \bigcap \mathbb{R}^q(\mathbb{R}^N) : \mathcal{P}_\epsilon(u) = 0\}$ be a ground state of $\mathcal{J}_\epsilon$.

- From $\mathcal{P}_0(u_\epsilon) = \mathcal{P}_\epsilon(u_\epsilon) - \frac{N-2}{2}\epsilon^\nu \|\nabla u\|_2^2 < 0$, $\mathcal{P}_0(u_\epsilon(\cdot/t_\epsilon)) = 0$ for some $t_\epsilon \in (0,1)$. Then

$$\sigma_* = \inf_{u \in \mathcal{P}_0} \mathcal{J}_0(u) = \frac{\alpha t_\epsilon^{N+\alpha}}{2Np}\mathcal{D}_\alpha(|u_\epsilon|^p, |u_\epsilon|^p)$$
$$< \frac{\alpha}{2Np}\mathcal{D}_\alpha(|u_\epsilon|^p, |u_\epsilon|^p) + \frac{\epsilon^\nu}{N}\|\nabla u_\epsilon\|_2^2 = \mathcal{J}_\epsilon(u_\epsilon) = \sigma_\epsilon.$$

- Similarly, from u_*, we get $u_*(\cdot/t_\epsilon) \in \mathcal{P}_\epsilon$ for some $t_\epsilon > 1$. Together with $t_\epsilon \to 1$, we can show $\lim_{\epsilon \to 0} \sigma_\epsilon \leq \sigma_*$.

More difficult case if $\|\nabla u_*\|_2$ unbounded

Choose a cut-off inside the support

$$\psi_s(x) = \eta_s(x) u_*(x),$$

where

$$\eta_s(x) = \begin{cases} 1, & |x| \le R_* - s, \\ \in (0,1), & R_* - s < |x| < R_* - s/2, \\ 0, & |x| > R_* - s/2, \end{cases}$$

and $|\eta_s'(x)| \le C/s$. It is easy to show

$$D_\alpha(|\psi_s|^p, |\psi_s|^p) = D_\alpha(|u_*|^p, |u_*|^p) - O(s^{\frac{N+\alpha}{2N}}),$$
$$\|\psi_s\|_q^q = \|u_*\|_q^q - O(s), \quad \|\psi_s\|_2^2 = \|u_*\|_2^2 - O(s).$$

More difficult to bound $\epsilon^\nu \|\nabla \psi_s\|_2^2$ (need more delicate relation between ϵ and s, so that $\|\nabla \psi_s\|_2^2$ blows up slower than ϵ^ν).

Outline for section 4

1. Variational setup and basic properties

2. Qualitative properties: reularity, decay and support

3. Thomas-Fermi limit of the Choquard equation

4. Limiting ground states for $\alpha \to 0^+$ or $\alpha \to N^-$

5. Conclusion and open problems

Optimal constant for limiting α

$$\mathscr{C}_{N,\alpha,p,q} = \sup\left\{\mathcal{R}(u) = \frac{\mathcal{D}_\alpha\big(|u|^p, |u|^p\big)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} : u \in L^q(\mathbb{R}^N)\cap L^2(\mathbb{R}^N), u \neq 0\right\}$$

The constant $\mathscr{C}_{N,\alpha,p,q}$ is bounded above by $\mathscr{C}_{N,\alpha}$ via

$$\mathscr{C}_{N,\alpha,p,q} = \frac{\mathcal{D}_\alpha\big(|u|^p, |u|^p\big)}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} \leq \mathscr{C}_{N,\alpha}\frac{\|u\|_{2Np/(N+\alpha)}^{2p}}{\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}} \leq \mathscr{C}_{N,\alpha},$$

and below by taking $u = \chi_{B_1(0)}$ with

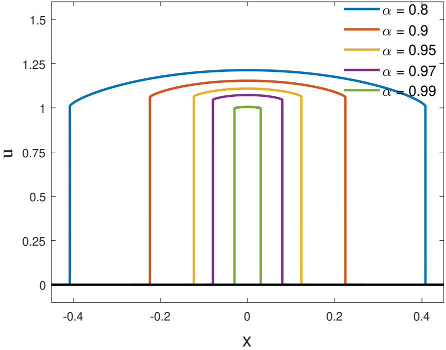
$$(I_\alpha * \chi_{B_1(0)})(x) = \frac{\Gamma((N-\alpha)/2)}{\pi^\alpha\Gamma(1+\alpha/2)\Gamma(N/2)}\,{}_2F_1\left(-\frac{\alpha}{2}, \frac{N-\alpha}{2}; \frac{N}{2}; |x|^2\right), \qquad |x| \leq 1.$$

Asymptotics for $\alpha \to 0^+$ or $\alpha \to N^-$ (working in progress)

$$I_\alpha(x) = \frac{\Gamma((N-\alpha)/2)}{\pi^{N/2} 2^\alpha \Gamma(\alpha/2)} |x|^{\alpha - N}.$$

The Riesz kernel becomes the identity operator ($\alpha \to 0^+$) or a constant ($\alpha \to N^-$), and the optmiser u seems to be a characteristic function in both limits.

Asymptotic profiles for $\alpha \to N^-$

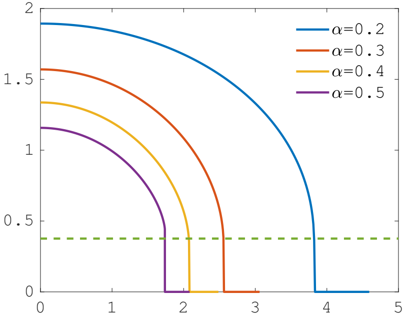


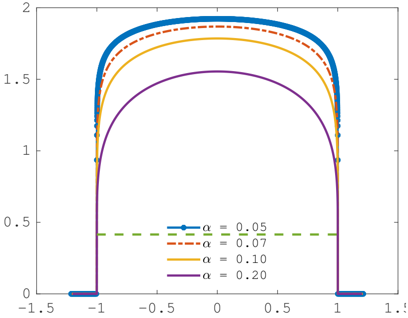
The radius of support and the solution ($\epsilon = N - \alpha$):

$$R_\alpha = c_0\epsilon^\gamma + c_1\epsilon^{\gamma+1} + c_2\epsilon^{\gamma+2} + \cdots, \quad u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots$$

We can show that $u_0(x) \to \chi_{B_{R_\alpha}(0)}$ and $u_1(x) \sim \int_{B_{R_\alpha}(0)} \ln|x-y|dy$.

Asymptotic profiles for $\alpha \to 0^+$





The radius of support and the solution ($\epsilon = \alpha$):

$$R_\alpha = \exp(c_0/\alpha + c_1 + c_2\alpha + \cdots), \quad u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots$$

We can show that $u_0(x) \to \chi_{B_{R_\alpha}(0)}$ and $u_1(x) \sim \ln(1 - |x|^2/R_\alpha^2)$.

Outline for section 5

1. Variational setup and basic properties

2. Qualitative properties: reularity, decay and support

3. Thomas-Fermi limit of the Choquard equation

4. Limiting ground states for $\alpha \to 0^+$ or $\alpha \to N^-$

5. Conclusion and open problems

Conclusion and open problems

- Uniqueness is not known in general except $p = 2$
- The regularity could be sharper
- Validity of the Pohozaev identity:
  a) variation of the domain
  b) rescaling of the optimisers of the quotient
  c) limit of $\mathscr{P}_\epsilon$

  but boundary terms appear if we multiply both sides of the governing equation with $x \cdot \nabla u$ and integrate:

$$\frac{N}{2}\int |u|^2 + \frac{N}{q}\int |u|^q - \frac{N+\alpha}{2p}\int I_\alpha(|u|^p)|u|^p$$
$$= \omega_N R^N \left(\frac{1}{2}u(R)^2 + \frac{1}{q}u(R)^q - \frac{1}{p}u(R)^p I_\alpha(|u|^p)(R)\right)$$
$$= \omega_N R^N \left(\left(\frac{1}{2} - \frac{1}{p}\right) u(R)^2 + \left(\frac{1}{q} - \frac{1}{p}\right) u(R)^q\right).$$

Conclusion and open problems

- Uniqueness is not known in general except $p = 2$
- The regularity could be sharper
- Validity of the Pohozaev identity:
  a) variation of the domain
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