# Fukaya - Seidel category :

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{C} \end{array}$$

reg. fiber

Lefschetz fibration $\rightarrow FS(\pi)$

- Construction 1 : inside $F$
- Construction 2 : inside a branched double cover of $E$
- Construction 3 : as a "partially wrapped" Fukaya categ.
- Construction 4 : Andriy's construction

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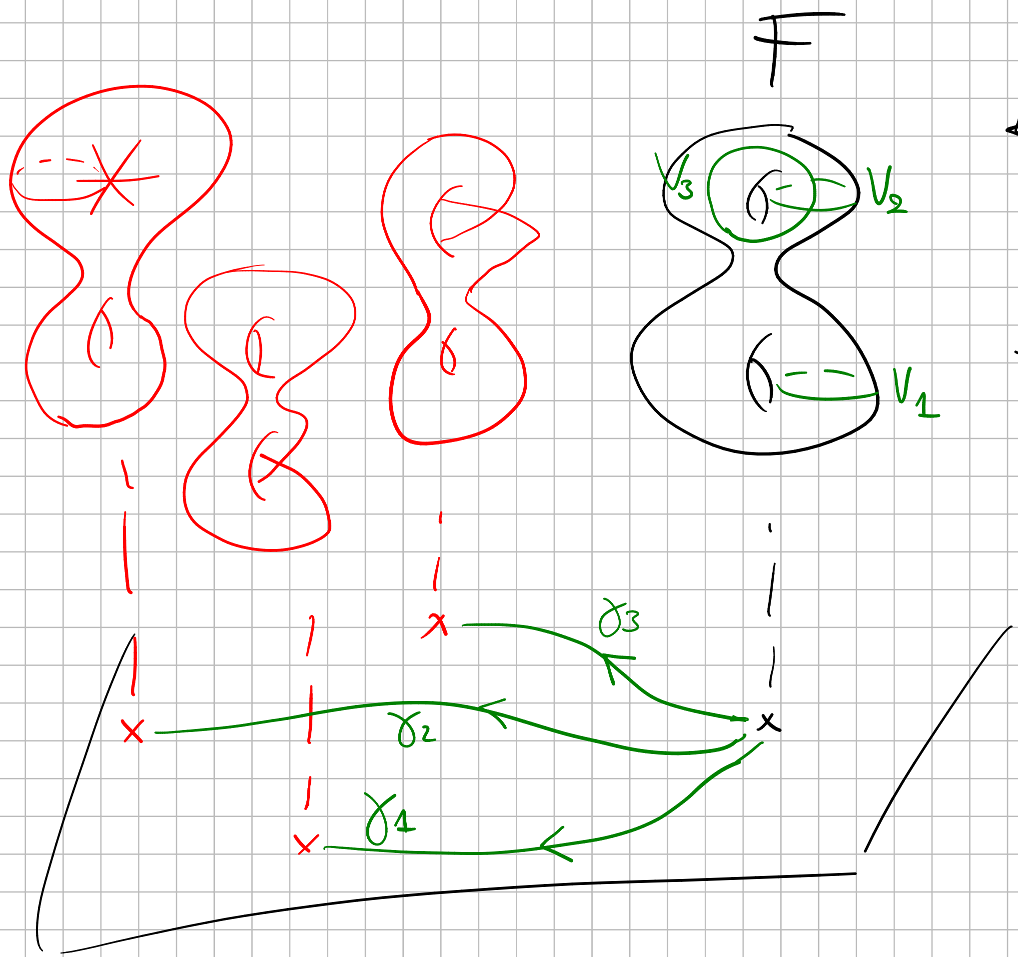
- Construction 4 : Andriy's construction ) → Viktor, next time

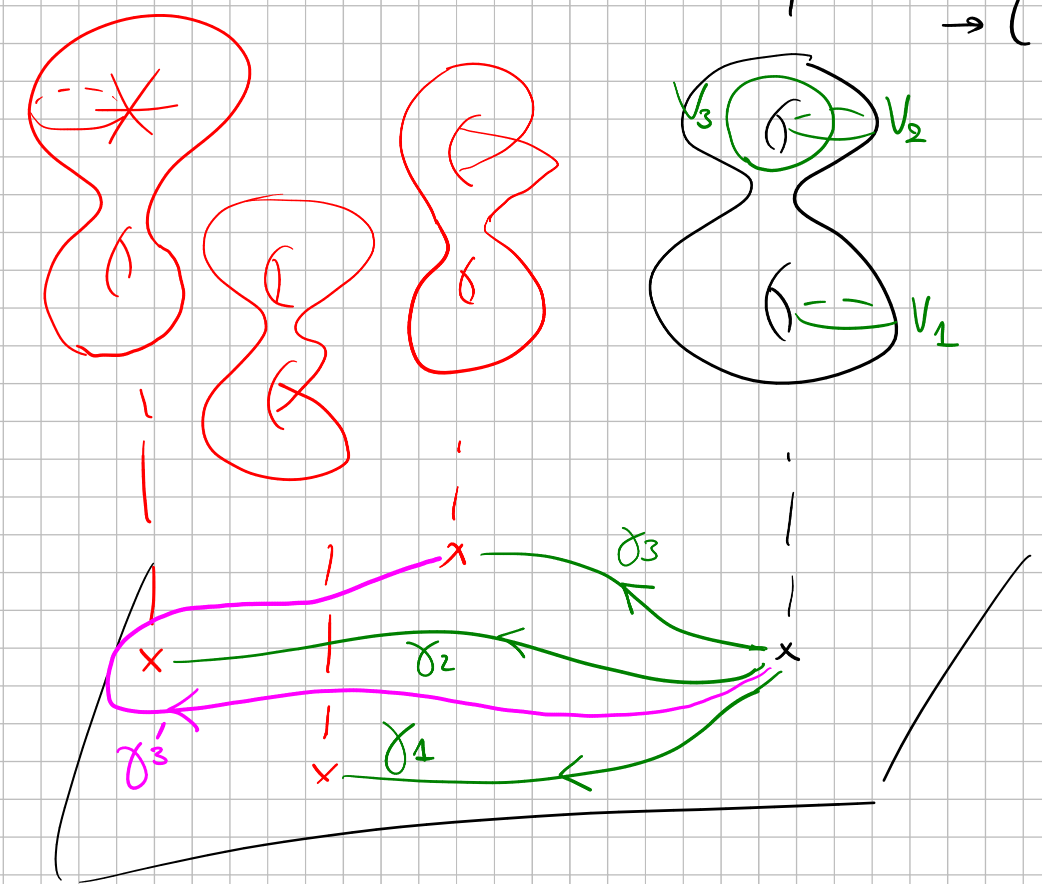
# Construction 1: inside $F$




* $\gamma_1, \dots, \gamma_m$ vanishing paths
* $\underbrace{V_1, \dots, V_m}_{\Gamma}$ vanishing cycles
* $\overrightarrow{Lag}(\Gamma)$ : $\underline{ob}$ : $V_1, \dots, V_m$
* $hom(V_i, V_j) = \begin{cases} CF(V_i, V_j) & i < j \\ \mathbb{Z}/2 & i = j \\ 0 & i > j \end{cases}$
* $\mu^k$ as in $Fuk(F)$ (when makes sense)

$$FS_1(\pi) := D^b(\overrightarrow{Lag}(\Gamma))$$

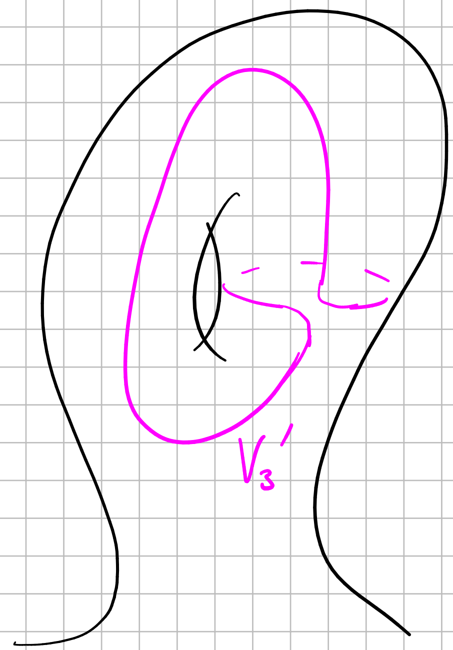
# Dependence on the paths $\gamma_2, \dots, \gamma_m$




$\to$ Check invariance under "Hurwitz moves"

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \delta_1 = \gamma_1 \\ \delta_2 = \gamma_3' \\ \delta_3 = \gamma_2 \end{pmatrix}$$

$$(V_1, V_2, V_3) \rightsquigarrow (V_1, V_3', V_2)$$



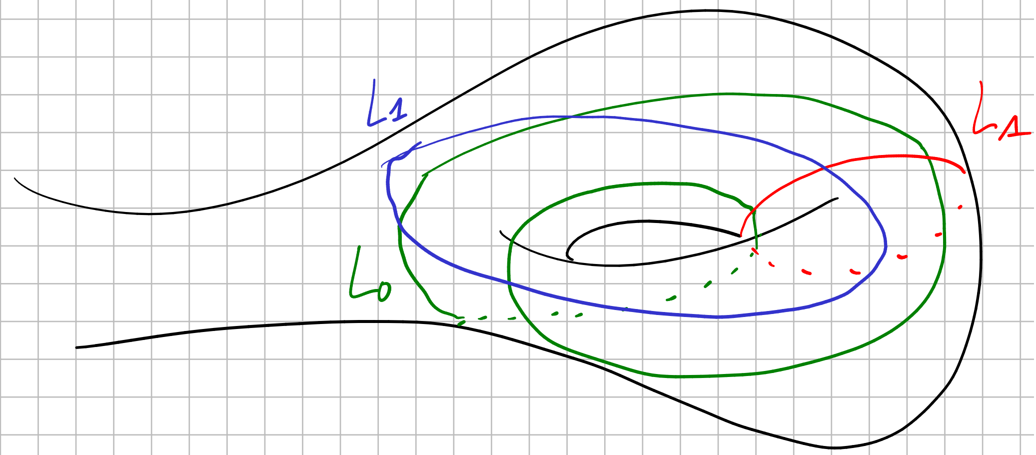

$$V_3' = \tau_{V_2}(V_3) \quad \left(\text{or } \tau_{V_2}^{-1}(V_3) \dots\right)$$

## Key Ingredient: Seidel's long exact sequence

- $(M,\omega)$ symplectic, $L_0, L_1 \subset M$ Lagrangians
  $S \subset M$ Lagrangian sphere (+ Assumptions...)
  $\leadsto \tau_S : M \to M$ Dehn-Seidel twist.

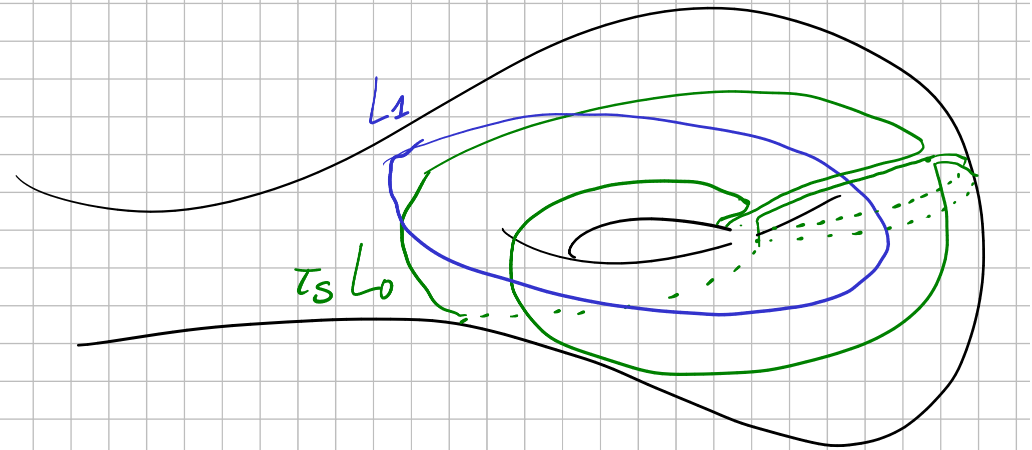
Then,

$$
\begin{tikzcd}
HF(\tau_S L_0, L_1) \arrow[rr] & & HF(L_0, L_1) \arrow[dl] \\
& HF(S, L_1) \otimes HF(L_0, S) \arrow[ul] &
\end{tikzcd}
$$




In $Tw(Fuk\, M)$:

$$\tau_S L_0 \simeq Cone\big(L_0 \to S \otimes HF(L_0, S)\big)$$



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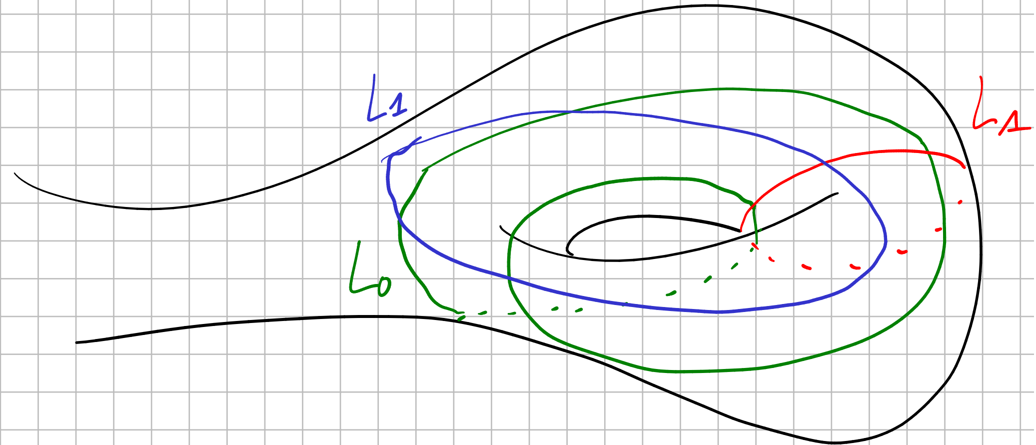


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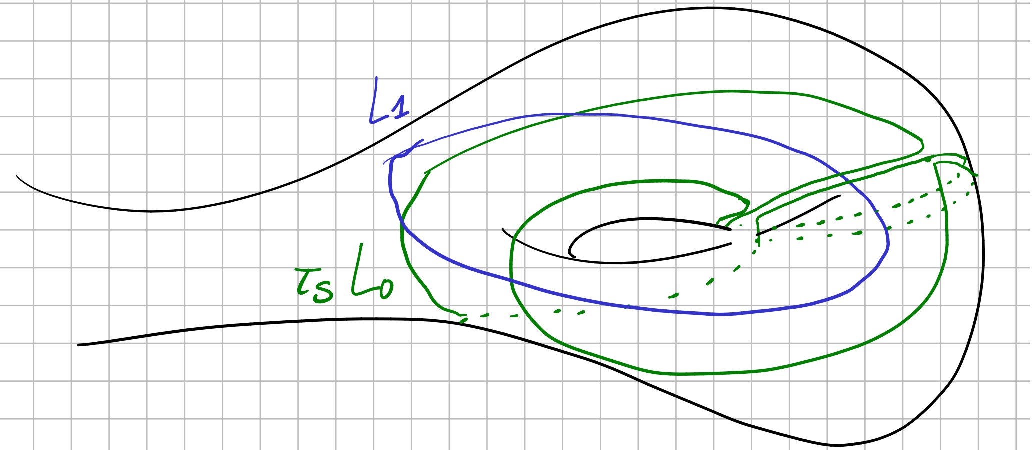


$$\Longrightarrow V_3' \simeq Cone\left[ V_3 \to V_2 \otimes HF(V_3, V_2) \right]$$

in $Tw(Fuk\, F)$

## Key ingredient: Seidel's long exact sequence

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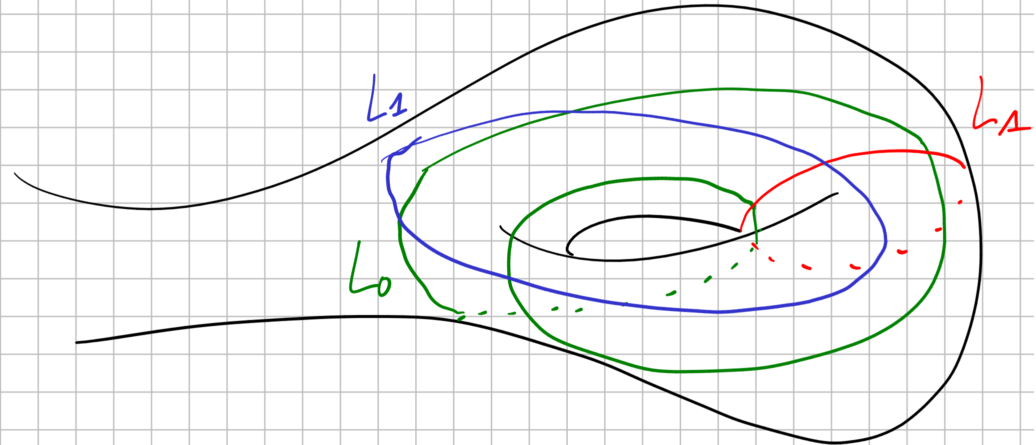
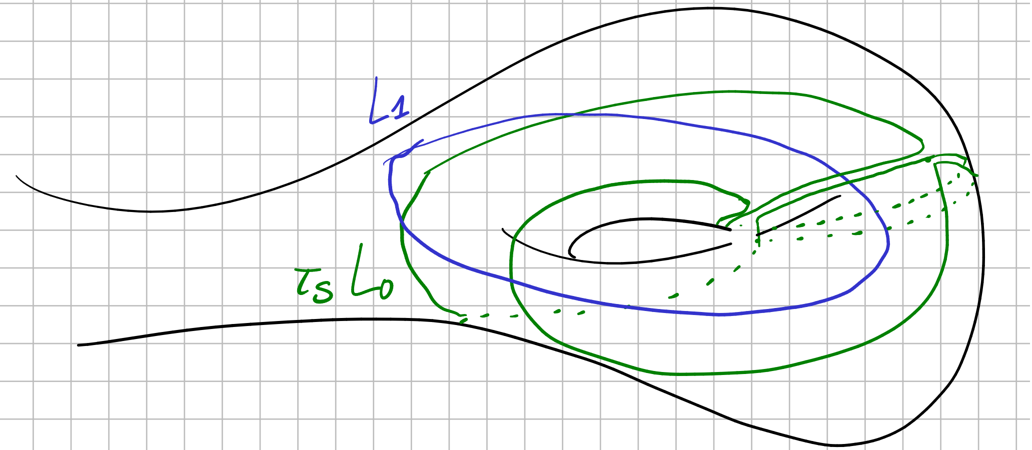



In $Tw(\mathrm{Fuk}\, M)$:

$$\tau_S L_0 \simeq \mathrm{Cone}\left( L_0 \to S \otimes HF(L_0, S) \right)$$

$$\Rightarrow V_3' \simeq \mathrm{Cone}\left[ V_3 \to V_2 \otimes HF(V_3, V_2) \right]$$

in $Tw(\mathrm{Fuk}\, F)$

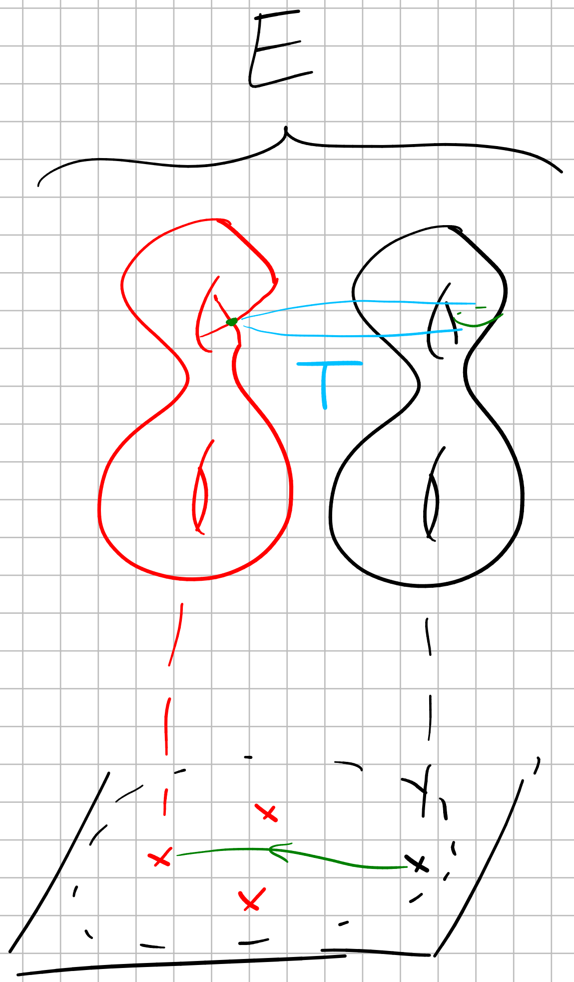



$\Rightarrow \dots$

$$\Rightarrow D^b \vec{\mathrm{Lag}}(V_1, V_2, V_3) \simeq D^b \vec{\mathrm{Lag}}(V_1, V_3', V_2)$$

Construction 2: Use the Vanishing Thimbles instead

$\longrightarrow$ not closed ....




$\mathbb{D}^{n+1} \simeq T \subset E$

$\mathbb{S}^n \simeq \partial T = V \subset F$

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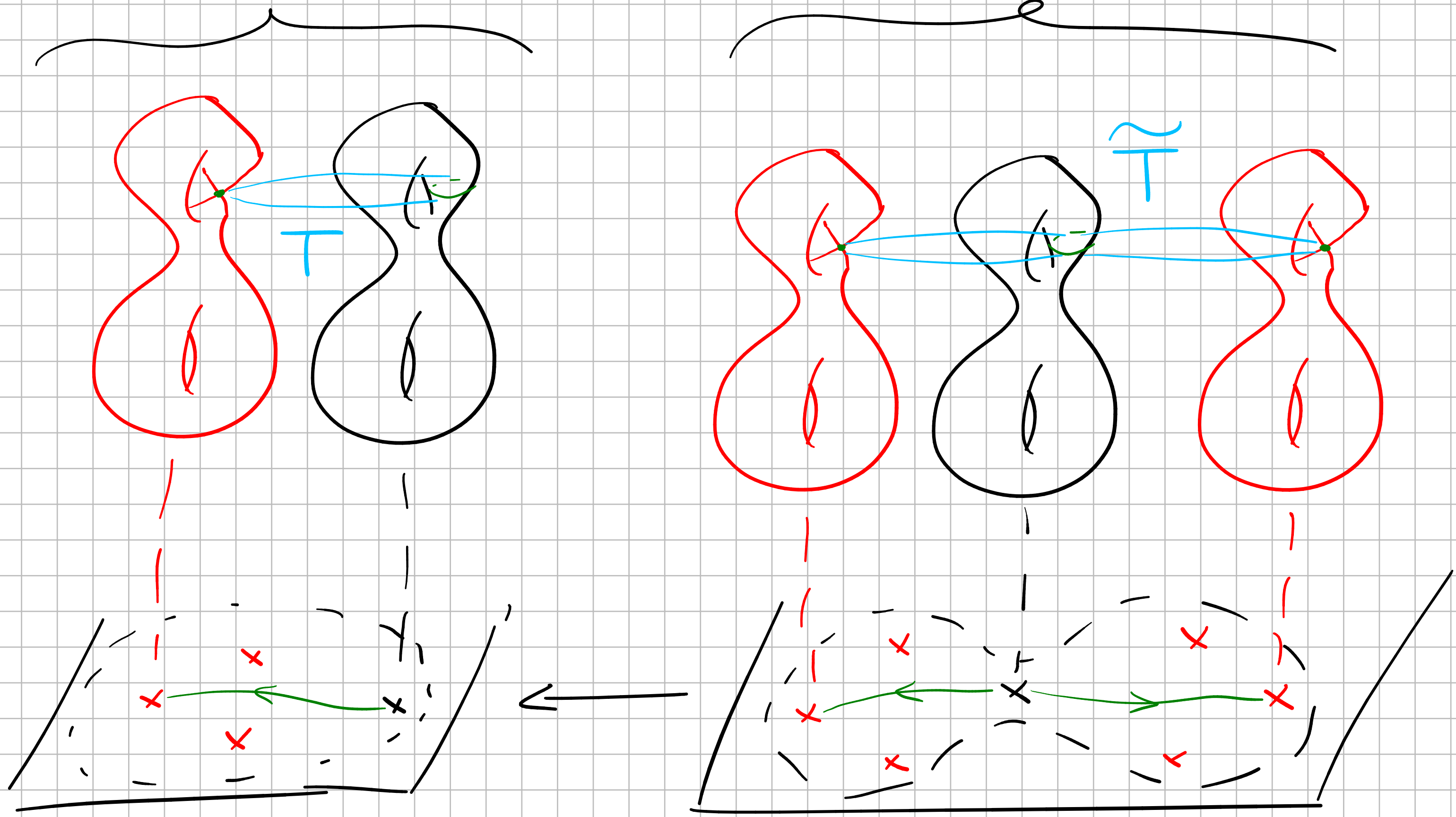
$E$ $\qquad\qquad$ $\tilde{E} \circlearrowleft \mathbb{Z}/2$




$z^2 \leftarrow\!\shortmid z$

$\mathbb{C} \leftarrow \mathbb{C}$

$\mathbb{Z}/2$

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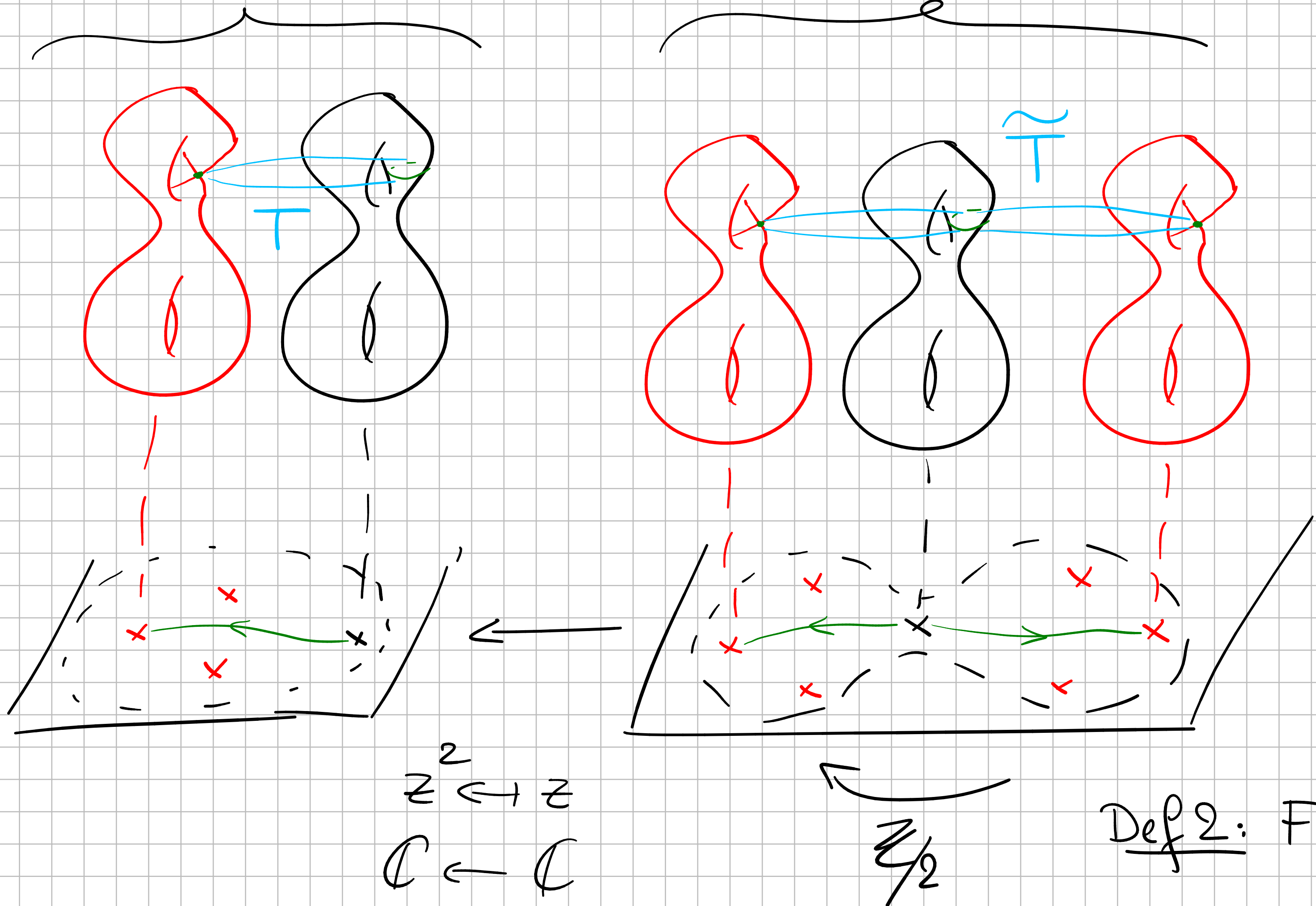


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$\mathbb{S}^{n+1} \simeq \tilde{T} \subset \tilde{E}$ "matching cycle"

$\rightsquigarrow \mathcal{A} \subset \mathrm{Fuk}(\tilde{E})$
$\circlearrowleft$
$\mathbb{Z}/2$

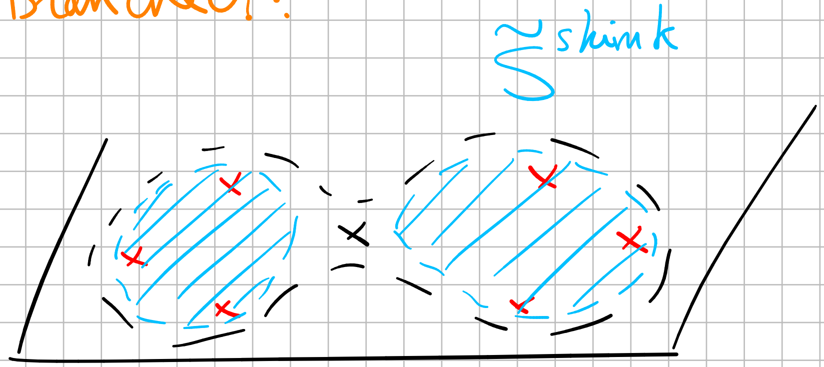
Def 2: $FS_2(\pi) := \mathcal{A}^{inv}$:

- $ob(\mathcal{A}^{inv})$ = object fixed by the $\mathbb{Z}/2$-action
- $hom_{\mathcal{A}^{inv}}(L, L') = hom_{\mathcal{A}}(L, L')^{\mathbb{Z}/2}$

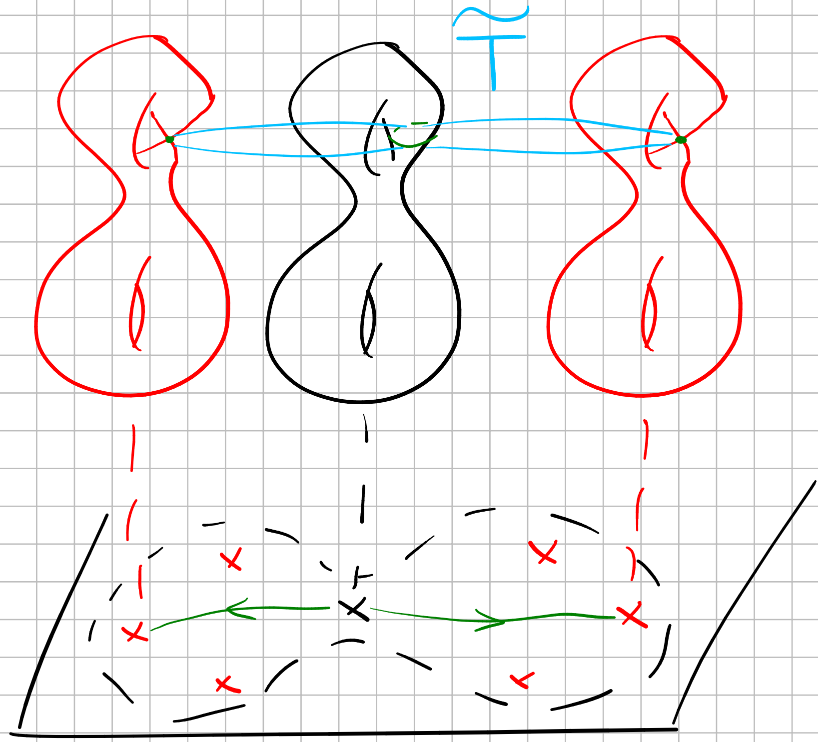
**Warning**: Doesn't work with $\mathbb{Z}/2\mathbb{Z}$ - coefficients $\Rightarrow$ need to work over $\mathbb{Z}$, use extra structures & orient moduli spaces of J-hol curves ...

$\rightarrow \tilde{\mathcal{A}} \subset Fuk(\tilde{E})$ : full $A_\infty$ - subcat. with objects:

- "type (U) (pre-) equivariant Lagrangian branes", i.e. over "$\tilde{S}^{shrink}$":
  $\uparrow$ untranched.




- "type (B) matching cycles $\tilde{T}$"



$\mathbb{Z}_2 \circlearrowright \tilde{E}$ induces a "naive $\mathbb{Z}/2$-action" on $\tilde{\mathcal{A}}$, i.e.

an $A_\infty$-functor $\epsilon : \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$ st

- terms of $\epsilon$ of order 2 vanish
- $\epsilon \circ \epsilon = \mathrm{id}_{\tilde{\mathcal{A}}}$

just have:

$\longrightarrow$

- $\epsilon_{ob} : Ob(\tilde{\mathcal{A}}) \longrightarrow Ob(\tilde{\mathcal{A}})$
- $hom_{\tilde{\mathcal{A}}}(a,b) \longrightarrow hom_{\tilde{\mathcal{A}}}(\epsilon_{ob}(a), \epsilon_{ob}(b))$

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Rk: this action involves some choices of Hamiltonian perturbations that rotate around the branched locus F in a prescribed (non-symmetric) way.

Prop: $\Gamma = (\gamma_1, \dots, \gamma_m)$ vanishing paths $\rightsquigarrow$ $\tilde{T}_1, \dots, \tilde{T}_m \subset \tilde{E}$ matching cycles

$$HF(\tilde{T}_i, \tilde{T}_j)^{\mathbb{Z}/2} = \begin{cases} HF(V_i, V_j) & \text{if } i<j \\ \mathbb{Z} & i=j \\ 0 & i>j \end{cases}$$

($V_i, V_j \subset F$ vanish. cycles)

$\to$ $\tilde{T}_1, \dots, \tilde{T}_m$ is an "exceptional collection" of objects in $\tilde{\mathcal{A}}^{inv}$

prop:

$$\begin{tikzcd} \mathrm{Fuk}(E) \arrow[r] & FS_2(\pi) = \mathcal{A}^{inv} \\ & FS_1(\pi) = D^b \mathrm{Lag}^{\to}(\Gamma) \arrow[u] \end{tikzcd}$$

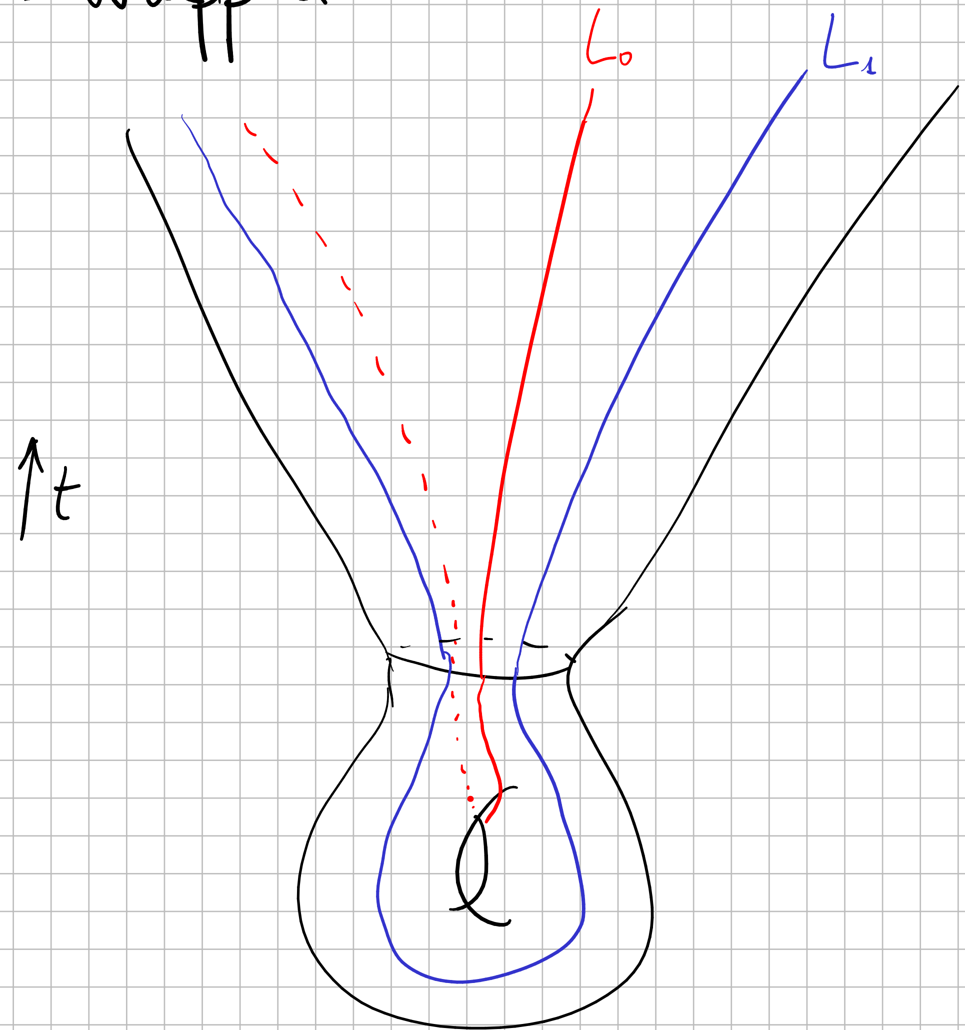
"cohomologically full & faithful $A_\infty$-functors"

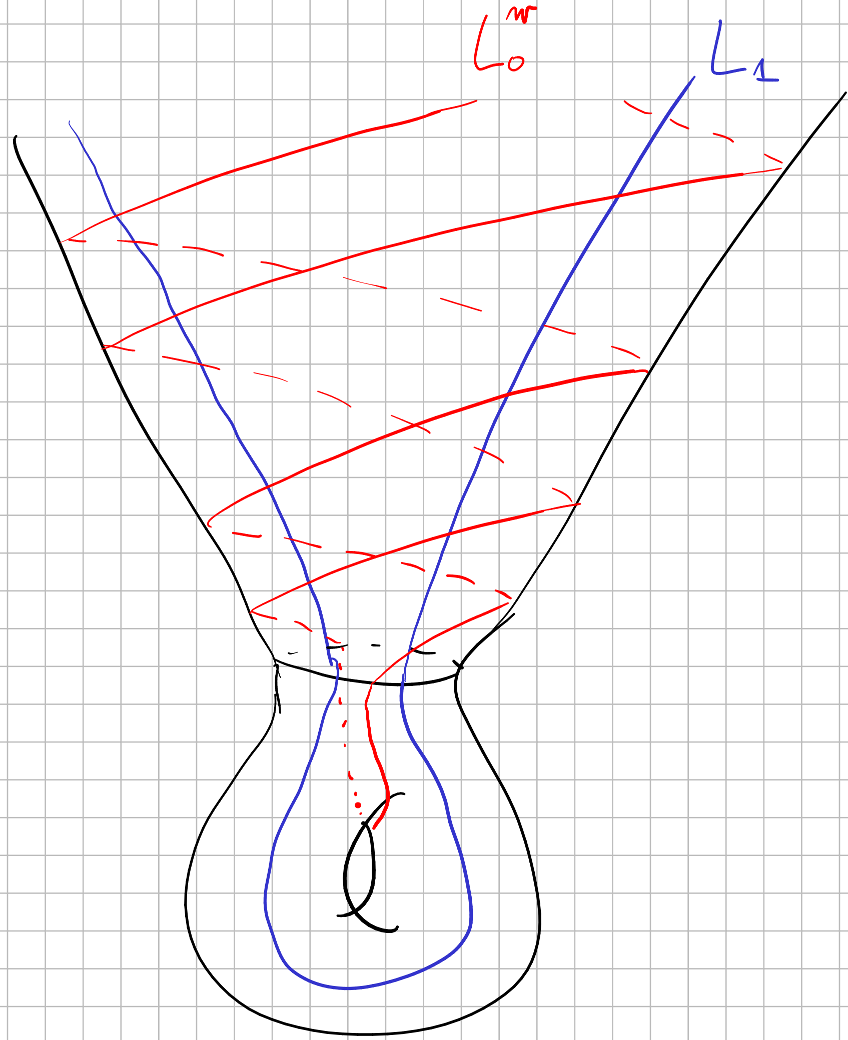
$F : \mathcal{A} \to \mathcal{B}$ is

if $H(F) : H(\mathcal{A}) \to H(\mathcal{B})$ is full & faithful.

# Construction 3: Partially wrapped Fukaya category

→ "wrapped"




$H \sim t^2$ at $\infty$

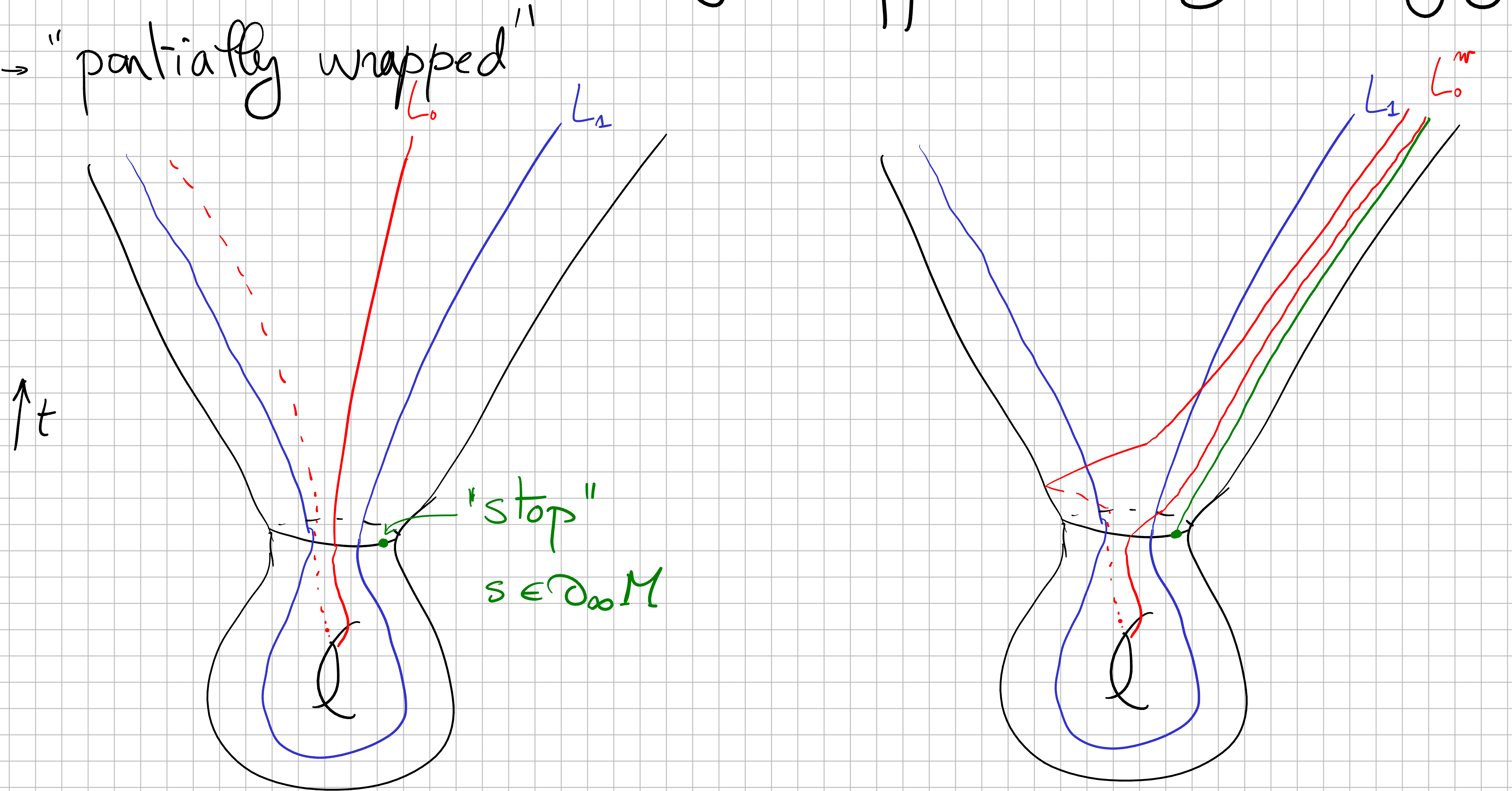
$L_0^w = \phi_H^1(L_0)$

$CW(L_0, L_1) = CF(L_0^w, L_1)$

$\rightsquigarrow \mathcal{W}(M)$ wrapped Fukaya category

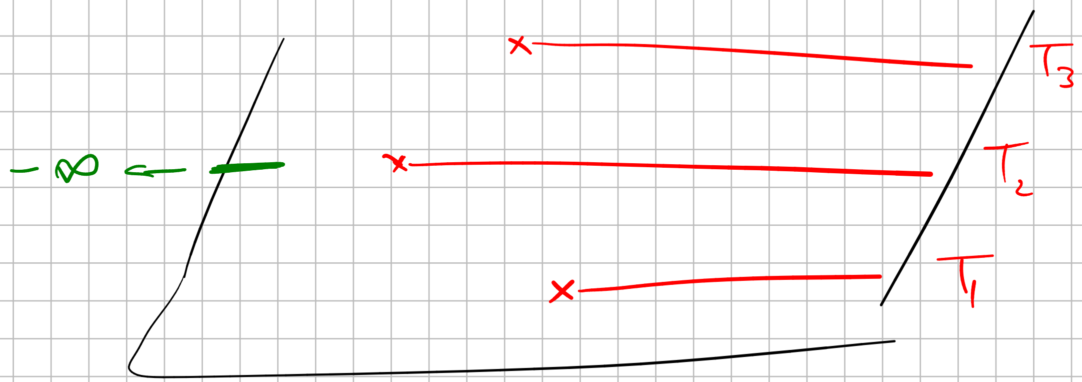
# Construction 3: Partially wrapped Fukaya category

→ "partially wrapped"




$H \sim t^2$ at $\infty$ + $H \equiv 0$ near $s$ $\quad CW(L_0, L_1) = CF(L_0^w, L_1)$

$L_0^w = \phi_H^1(L_0)$ $\quad \rightsquigarrow \mathcal{W}(M, s)$ partially wrapped Fukaya category




Third def.: $FS_3(\pi) = W(E, s)$ $\quad s = \pi^{-1}(-\infty)$

Ganatra-Pardon-Shende: $W(E,s)$ generated by the thimbles.