infinite dim generalizat^{o}s: ⇝ symplectic geometry

⇝ gauge theory

infinite dim generalizat's: ⇝ <u>symplectic geometry</u>

↓

<u>gauge theory</u>

$$\begin{tikzcd} SU(2) \subset P \arrow[d] \\ Y^3 \end{tikzcd} \qquad CS : \mathcal{A}(P)/\mathcal{G} \longrightarrow \mathbb{R}/8\pi^2\mathbb{Z}$$

$\rightsquigarrow I_*(Y) \text{ "="} \ HM_*\left(\left(\mathcal{A}(P)/\mathcal{G}\right)^*, CS\right)$

instanton Floer homology

infinite dim generalizat^{o}s: $\rightsquigarrow$ **symplectic geometry**

$(M,\omega) \supset L_0, L_1$

symplectic mfd — pair of (connected) Lagrangian submanifolds

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path space

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universal cover

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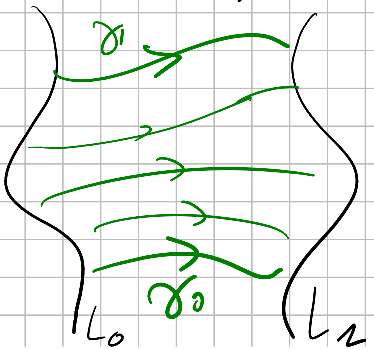
$\mathcal{A} : \widetilde{\mathcal{P}(L_0, L_1)} \longrightarrow \mathbb{R}$ action functional

$([\gamma_t], \gamma) \longmapsto \int_{[0,1]^2} \gamma_t^* \omega$



infinite dim generalizat$^{\text{o}}$s: ⇝ symplectic geometry

gauge theory

$SU(2) \curvearrowright P$
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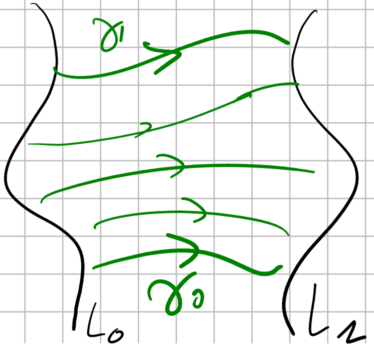
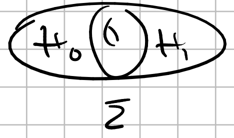


$HF_*(M; L_0, L_1)$ "=" $HM_*(\tilde{\mathcal{P}}(L_0, L_1), \mathcal{A})$

Lagrangian Floer homology.

Relation between $I_*(Y)$ and $HF_*(M; L_0, L_1)$:

$$\begin{tikzcd} P \arrow[d, dash] \\ Y \end{tikzcd} \quad =$$



$$\mathcal{M}(\Sigma) = \{A \text{ flat on } x^{U}/\Sigma\} / \mathcal{G} \quad : \text{symplectic.}$$

$$\cup$$

$$\mathcal{L}(H_i) = \{[A], \ A \text{ extends flatly to } H_i\}$$

Relation between $I_*(Y)$ and $HF_*(M; L_0, L_1)$:

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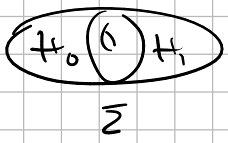

$\mathcal{M}(\Sigma) = \{ A \text{ flat on } \times^{U}/\Sigma \} \big/ \mathcal{G}$: symplectic.

$\cup$

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Conj: (Atiyah-Floer): $I_*(Y) \simeq HF_*(\mathcal{M}(\Sigma); \mathcal{L}(H_0), \mathcal{L}(H_1))$.

Lagrangian Floer homology

$(M,\omega) \supset L_0, L_1$ (+ assumptions)

assume $L_0 \pitchfork L_1$

$$CF(L_0,L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{Z}_2 \cdot x$$

$$\partial : CF(L_0,L_1) \to CF(L_0,L_1)$$

$$\partial x = \sum_{y}' n_{xy} \cdot y$$

$$n_{x,y} = \# \mathcal{M}(x,y) = \# \widetilde{\mathcal{M}}(x,y) / \mathbb{R}$$

$$\tilde{\mathcal{M}}(x,y) = \left\{ u : \overset{(s\,,\;t)}{\mathbb{R}\times[0,1]} \longrightarrow M \;\middle|\; \begin{array}{l} ,\ u(s,0)\in L_0 \\ ,\ u(s,1)\in L_1 \\ .\ du + J du\circ j = 0 \\ \lim\limits_{s\to-\infty} u(s,t) = x \\ \lim\limits_{s\to+\infty} u(s,t) = y \end{array} \right\} \circlearrowleft \mathbb{R}$$

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Rk: * for transversality reasons, need to use t-dependent almost-complex structures $\{J_t\}_{t\in[0,1]}$

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$$du + J du\circ j = 0 \;\longleftrightarrow\; \partial_s u + J\partial_t u = 0$$

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* if $L_0\cap L_1$ not transverse, can perturb the equation with a Hamiltonian vector field $X_H = \nabla^\omega H$

$H : M\times[0,1]\to\mathbb{R}$ with compact support

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$H: M\times[0,1]\to \mathbb{R}$

$\to \boxed{\partial_s u + J_t(\partial_t u - X_H) = 0}$ Floer's eq. with compact support

prop: $\partial^2 = 0 \rightarrow HF(L_0, L_1) = \dfrac{\ker \partial}{\operatorname{im} \partial}$

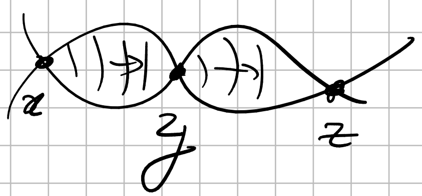
proof: $$\partial^2 x = \partial\Big(\sum_{y \in L_0 \cap L_1}' n_{x,y} \cdot y \Big)$$

$$= \sum_y' n_{xy} \sum_z' n_{yz} \cdot z$$

$$= \sum_z' \underbrace{\Big(\sum_y n_{x,y}\, n_{yz} \Big)}_{\substack{= \text{count of "broken strips"} \\ \subset \partial \overline{\mathcal{M}}(x,z)}} \cdot z$$

Gromov compactness + Gluing:

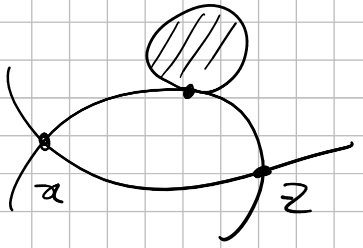
$\partial\overline{\mathcal{M}}(x,z) = \{$



$\}$

$\bigcup_y \mathcal{M}(x,y)\times\mathcal{M}(y,z)$

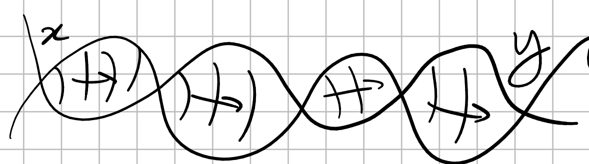
$\cup\ \{$



$\}$

disc bubbling

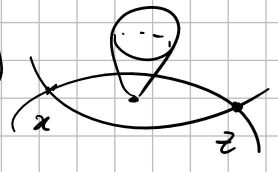
$\cup\ \{$



$\}$

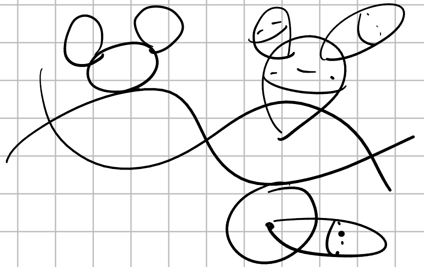
strip breaking

$\cup\ \{$



$\}$ sphere bubbling

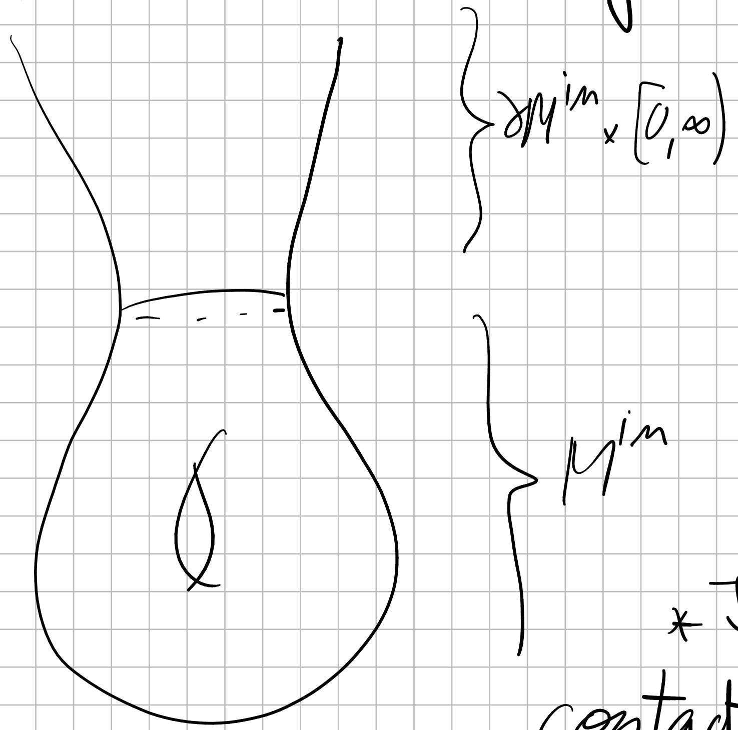
$\cup\ \{$



$\}$ more complicated things

example of assumptions on (M, L_0, L_1) that makes HF well-defined:

- (M, ω) exact : $\omega = d\lambda$, $\lambda \in \Omega^1(M)$ ($\Rightarrow M$ noncompact)
- M "convex at infinity" : $M = M^{in} \cup \partial M^{in} \times [0, +\infty)$



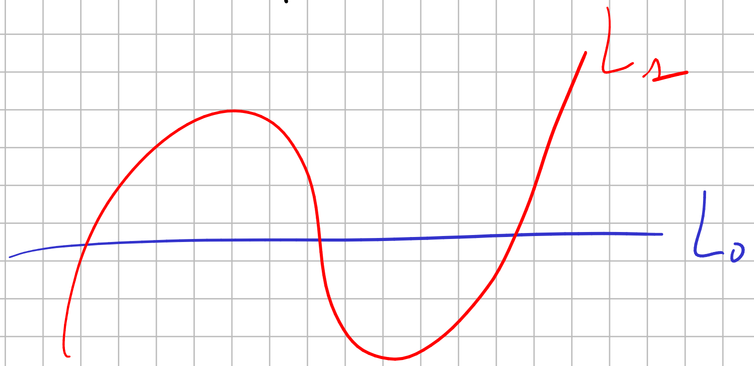
. M^{in} : compact, with contact boundary: $\Leftrightarrow \alpha = \lambda_{|\partial M^{in}}$ contact form

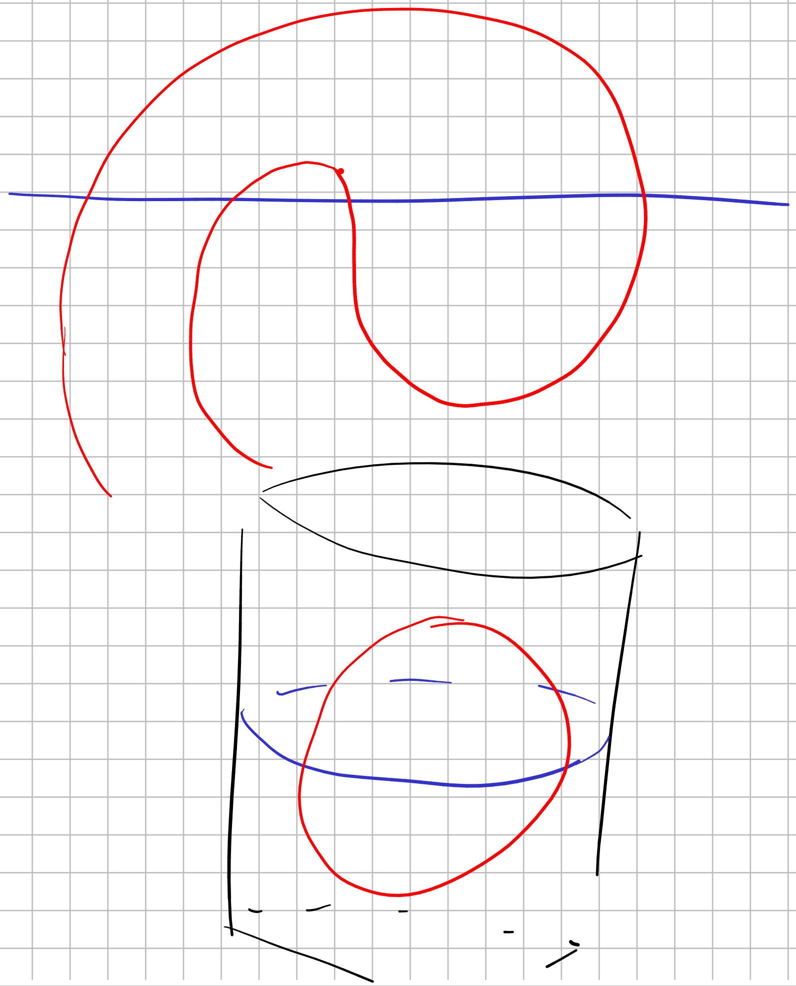
. $\partial M^{in} \times [0,\infty)$: positive symplectization

$\overset{\Downarrow}{t}$ $\quad \underline{\lambda = e^t \alpha}$

- J almost complex structure / M "of contact type": $\begin{cases} * \text{ invariant under } \partial_t \\ * \ (e^t dt) \circ J = -\lambda \end{cases}$
- $L_0, L_1 \subset M$ compact Lagrangians, exact ($\lambda_{|L_i} = df_i$)

Examples on surfaces





Some properties of $HF(L_0, L_1)$:

- * Hamiltonian isotopy invariance:

$$HF(L_0, L_1) = HF(L_0, \phi_t(L_1)) \quad , \ \phi_t : \text{Hamilt. isotopy}$$

→ can define $HF(L_0, L_1)$ if $L_0 \not\pitchfork L_1$

- * $HF(L, L) = H_*(L)$ (in the exact setting ...)

Rk: • Instanton theory $\approx$ (3+1)-TQFT

$$\mathrm{Cob}_{3+1} \longrightarrow \mathrm{Vect}$$

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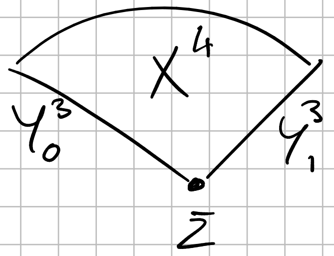
$$\mathrm{Cob}_{3+1} \longrightarrow \mathrm{Vect}$$

• Σ closed surface $\rightsquigarrow$ $\mathrm{Cob}_{3+1}^{\Sigma} \longrightarrow$ ~~Vect~~ $\mathrm{Don}(\mathcal{M}(\Sigma))$



$$Y_0 \xrightarrow{X} Y_1$$

The Donaldson category:

$\mathrm{Don}(M,\omega)$: * objects $L \subset M$ Lagrangians (+ assumptions ...)

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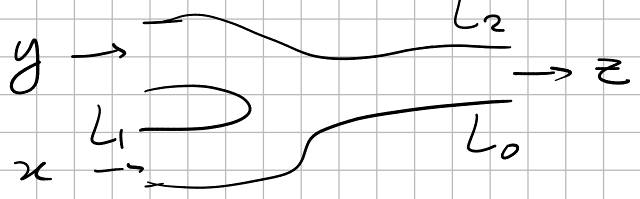
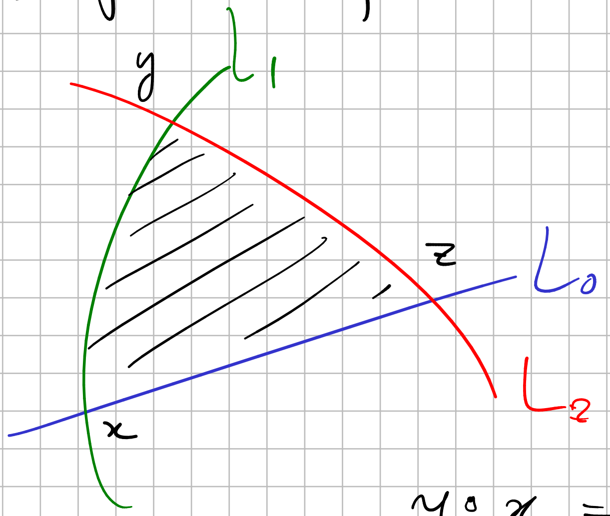
"pair of pants product"



ex:



$y \circ x = z$

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Don(M,ω) :
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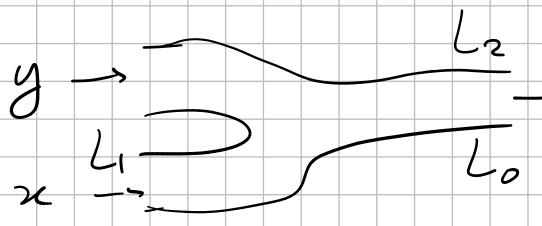

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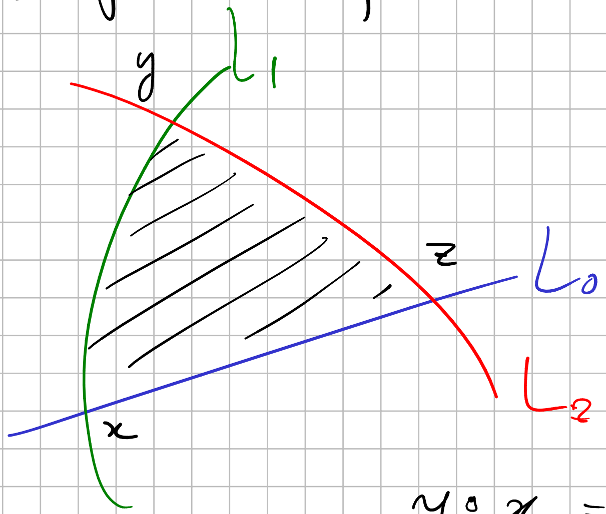

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Pb: Don(M) not triangulated

$\Rightarrow$ hard to compute

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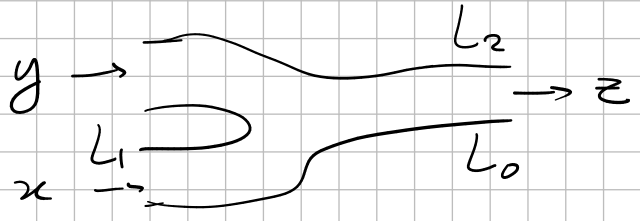

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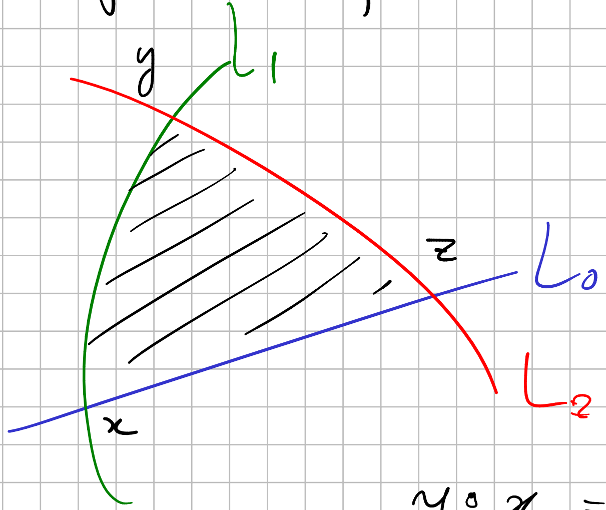

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. Solution 1: (Fukaya) work with $D^b\mathrm{Fuk}(M)$ instead.

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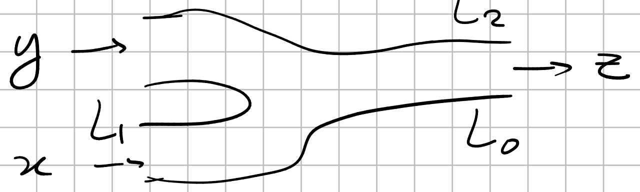

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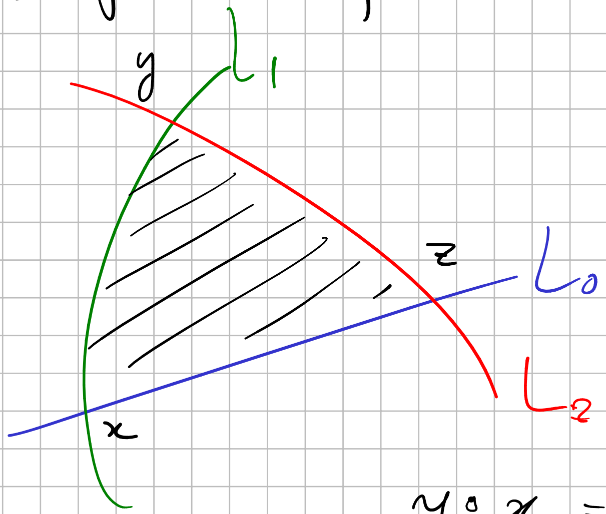

$y \circ x = z$

Pb: Don(M) not triangulated

$\Rightarrow$ hard to compute

- Solution 1: (Fukaya) work with $D^b\mathrm{Fuk}(M)$ instead.
- Solution 2: (Joyce) allow Lagrangian immersions. "Joy(M)"

$\mathrm{Don}(M, \omega) \rightsquigarrow \mathrm{Fuk}(M, \omega)$

$\hom(L_0, L_1) = HF(L_0, L_1) \rightsquigarrow \hom(L_0, L_1) = CF(L_0, L_1)$

"A_∞-category"

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"A_∞-category"

* "everything holds up to homotopy"

⊕

* "homotopies are part of the structure"

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"$\underbrace{A_\infty}$-category"

Assoc. algebra $\rightsquigarrow$ A_∞ algebra
module $\rightsquigarrow$ A_∞ module
morphism $\rightsquigarrow$ A_∞-morphism
linear category $\rightsquigarrow$ A_∞-category

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taking homology.

* "everything holds up to homotopy"

$\oplus$

* "homotopies are part of the structure"

Def An **A_∞-algebra** $(A, \mu^1, \mu^2, \mu^3, \dots)$ (over $\mathbb{Z}/2$)

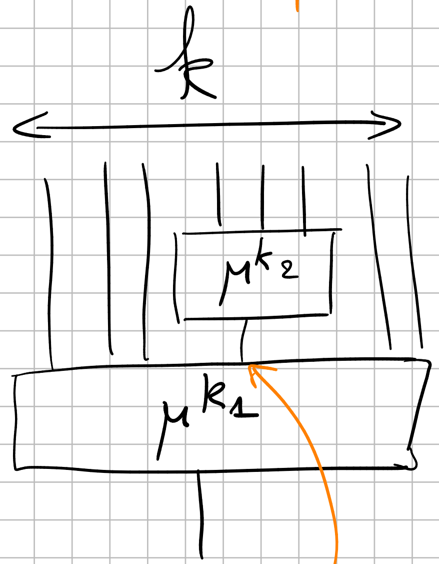
is a family of maps $\mu^k : A^{\otimes k} \longrightarrow A$

satisfying the A_∞ relations:

- $\mu^1 \circ \mu^1 = 0$ (μ^1 differential)
- $\mu^1 \circ \mu^2 + \mu^2 \circ (\mu^1 \otimes id + id \otimes \mu^1) = 0$ (μ^2 chain map)
- $\mu^2 \circ (\mu^2 \otimes id) + \mu^2 \circ (id \otimes \mu^2) + \mu^1 \circ \mu^3 + \mu^3 \circ (\mu^1 \otimes id \otimes id + id \otimes \mu^1 \otimes id + id \otimes id \otimes \mu^1)$

  (μ^2 associative up to homotopy)
- ...

$$\forall k \geq 1, \quad \sum_{\substack{k_1+k_2=k+1 \\ 1 \leq \ell \leq k_1}}{}'$$



$= 0$

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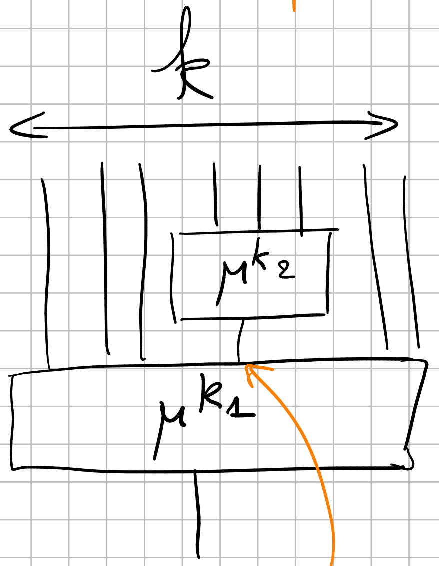
- $\mu^2 \circ (\mu^2 \otimes id) + \mu^2 \circ (id \otimes \mu^2) + \mu^1 \circ \mu^3 + \mu^3 \circ (\mu^1 \otimes id \otimes id + id \otimes \mu^1 \otimes id + id \otimes id \otimes \mu^1)$ (μ^2 associative up to homotopy)
- $\dots$

$\Rightarrow (H_*(A), [\mu^2])$ is an associative algebra

$$\forall k \geq 1, \quad \sum_{\substack{k_1+k_2=k+1 \\ 1 \leq \ell \leq k_1}}'$$



$$= 0$$

Def: An *A_∞-category* $\mathcal{A}$ consists in:

- a collection of objects
- $L_0, L_1 \in Ob(\mathcal{A}) \rightsquigarrow hom_{\mathcal{A}}(L_0, L_1)$ morphisms (vector space / $\mathbb{F}_2$)
- $\forall k \geqslant 1$, $\forall L_0, \dots L_k \in Ob(\mathcal{A})$ composition maps:

$$\mu^k : hom_{\mathcal{A}}(L_{k-1}, L_k) \otimes \cdots \otimes hom_{\mathcal{A}}(L_0, L_1) \longrightarrow hom_{\mathcal{A}}(L_0, L_k)$$

satisfying the A_∞-relations.

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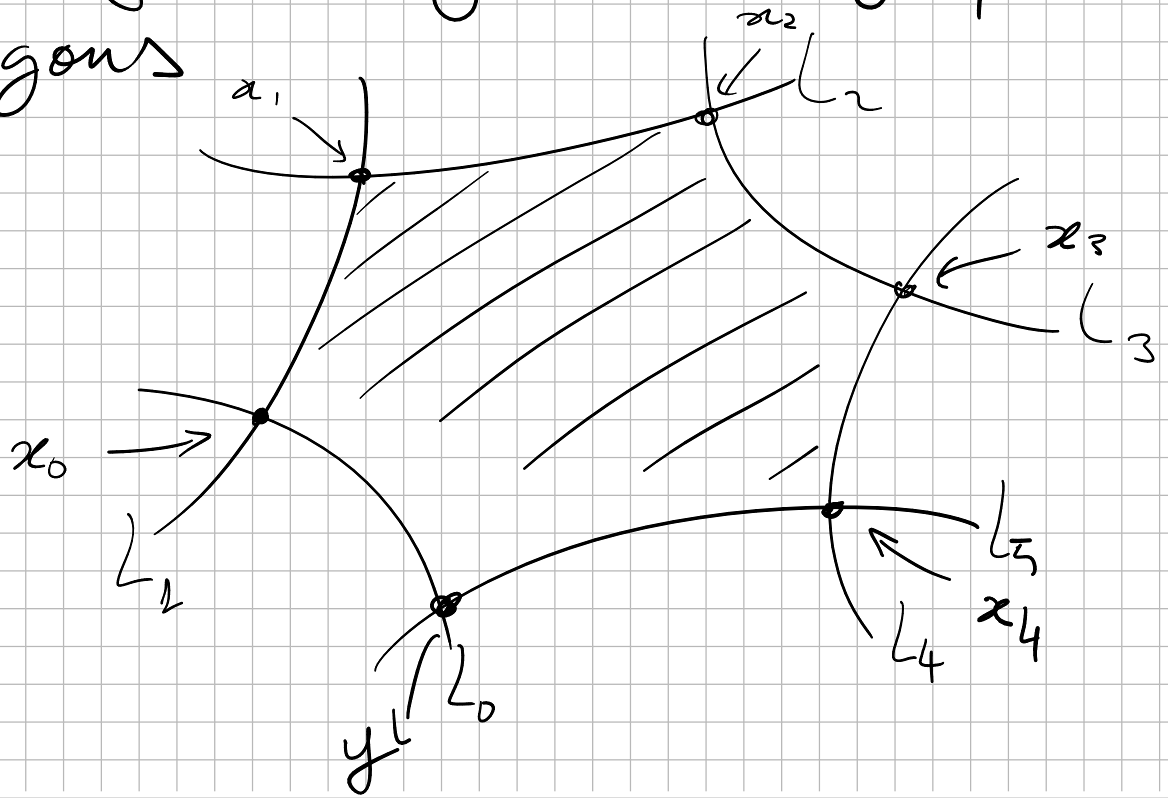
Rk: $\mathcal{A}$ A_∞-cat $\rightsquigarrow A = \bigoplus_{L_0, L_1} hom_{\mathcal{A}}(L_0, L_1)$: A_∞-algebra

Def: (oversimplified) $\mathrm{Fuk}(M)$ is the A_∞-category whose:

- objects: $L \subset M$ Lagrangians
- $\hom(L_0, L_1) = CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{F}_2 \cdot x$
- compositions $\mu^k : \hom(L_{k-1}, L_k) \otimes \cdots \otimes \hom(L_0, L_1) \to \hom(L_0, L_k)$

are defined by counting pseudo-holomorphic polygons



⚠ when $k \geq 3$, need to allow the complex structure of the domain to vary in the "Deligne-Mumford moduli space" $\mathcal{R}^{k+1}$

Deligne-Mumford moduli space of discs

$$\mathcal{R}^{k+1} = \left\{ \text{isom. classes of disks with } k+1 \text{ boundary marked points} \right\}$$

$$\simeq \mathrm{Conf}_{k+1}(\partial D^2) \Big/ \underbrace{\mathrm{Aut}\, D^2}_{PSL_2(\mathbb{R})}$$

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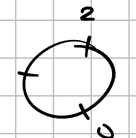
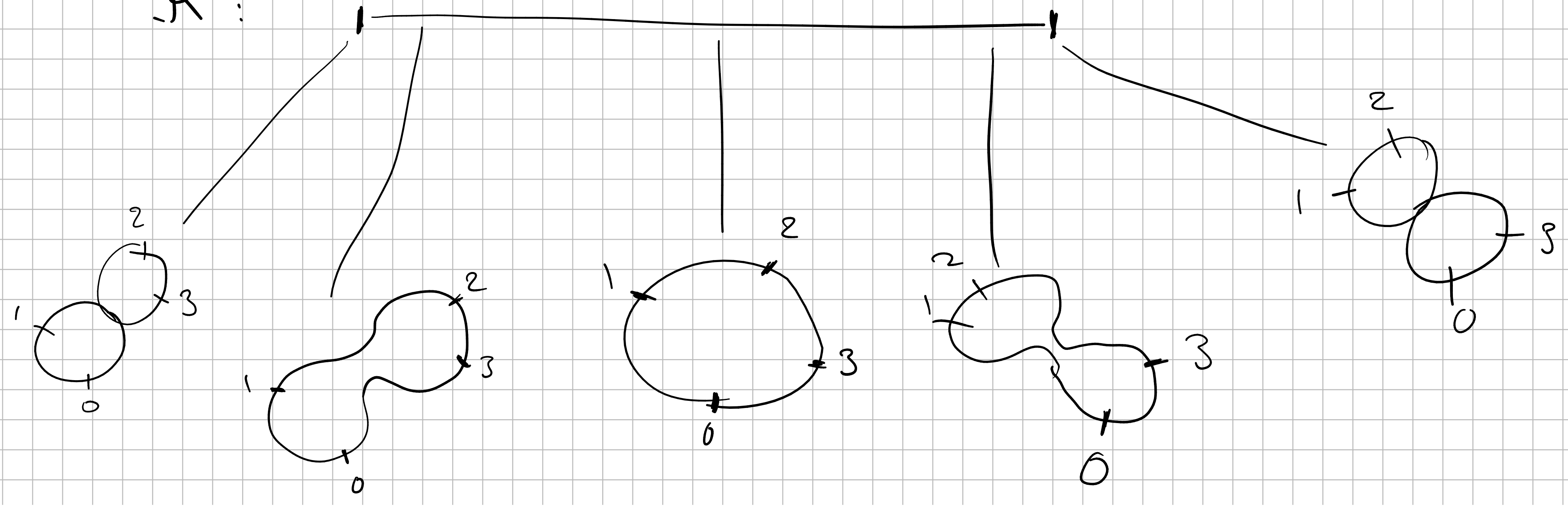
ex $\mathcal{R}^3 = \overline{\mathcal{R}}^3 = \{pt\} = \{$

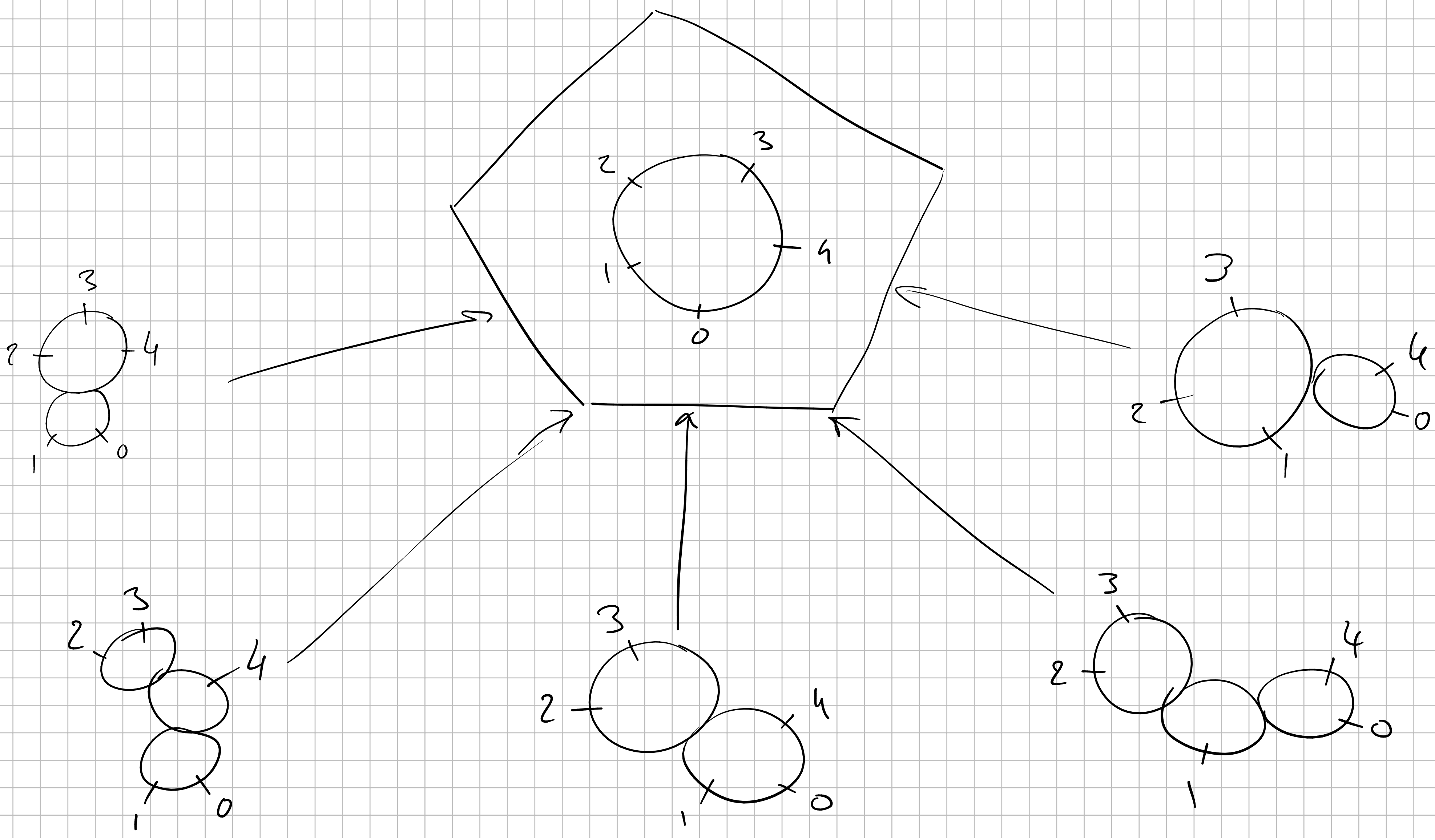


$\}$

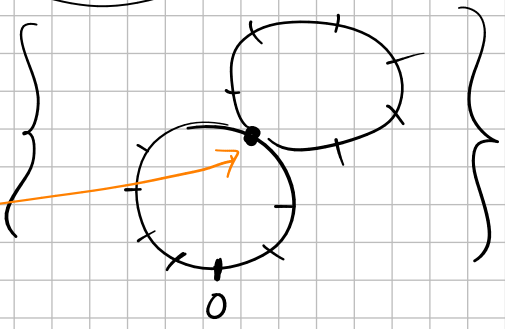
$\overline{\mathcal{R}}^4$:



$\overline{\mathcal{R}}^5$:



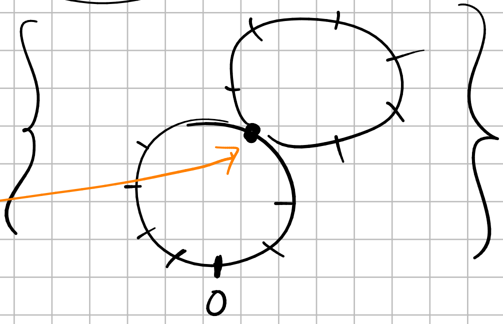
- $\overline{R}^{k+1}$ is a polytope of dimension $k-2$
- $$\partial^1 \overline{R}^{k+1} = \bigsqcup_{\substack{k_1+k_2=k+1 \\ 1 \leq \ell \leq k_1}} \underbrace{\overline{R}^{k_1+1} \times \overline{R}^{k_2+1}} \simeq$$



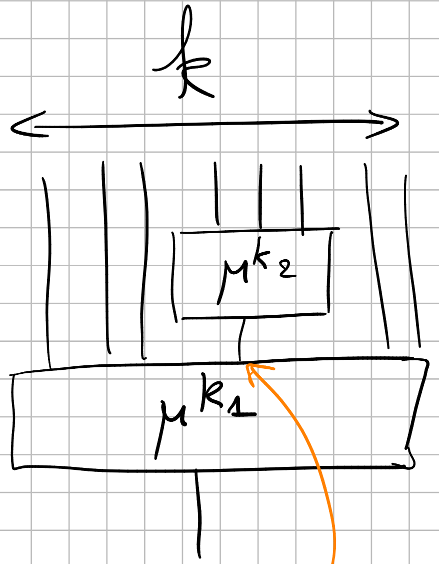
$k_1, k_2 \geq 2$

- $\overline{R}^{k+1}$ is a polytope of dimension $k-2$

- $$\partial^1 \overline{R}^{k+1} = \coprod_{\substack{k_1+k_2=k+1 \\ 1 \leqslant \ell \leqslant k_1 \\ \underline{\underline{k_1, k_2 \geqslant 2}}}} \underbrace{\overline{R}^{k_1+1} \times \overline{R}^{k_2+1}} \simeq$$



A_∞- relations: $\forall k \geqslant 1$, $$\sum_{\substack{k_1+k_2=k+1 \\ 1 \leqslant \ell \leqslant k_1 \\ \underline{\underline{k_1, k_2 \geqslant 1}}}}{}'$$



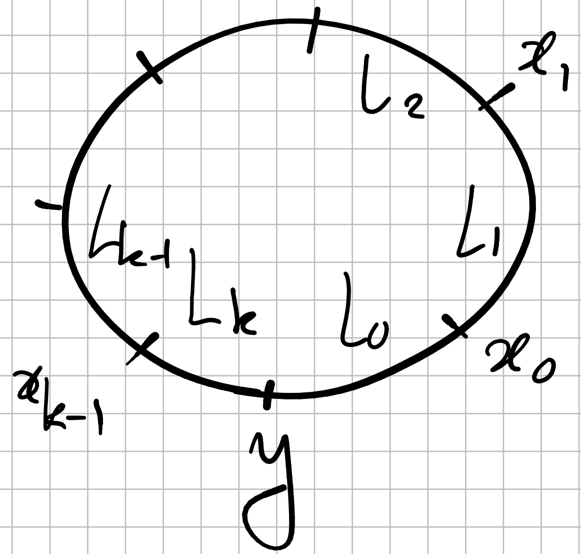
$= 0$

Define $\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k)$

by: $\mu^k(x_{k-1} \otimes \cdots \otimes x_0) = \sum_y{}' \; \# \mathcal{M}(x_{k-1}, \dots, x_0; y) \cdot y$

$\uparrow$ count the zero-dim. part.

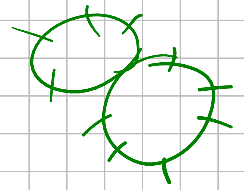
with
$$\mathcal{M}(x_{k-1}, \dots, x_0; y) = \left\{ (D, u) \;\middle|\; \begin{array}{l} \cdot\, D \in R^{k+1} \\ \cdot\, u : D \to M \; : \\ \cdot\, \bar\partial u = 0 \\ \cdot\, \text{Lagrangian boundary cond.} \\ \cdot\, \text{limits} = x_{k-1}, \dots x_0, y \text{ at punctures} \end{array} \right\}$$



prop: The μ^k's satisfy the A_∞- relations.

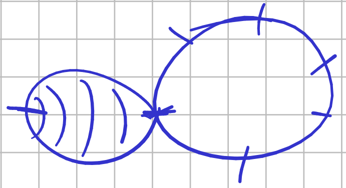
proof: Compactify the 1- dimensional parts of $\mathcal{M}(x_k \cdots ; y)$

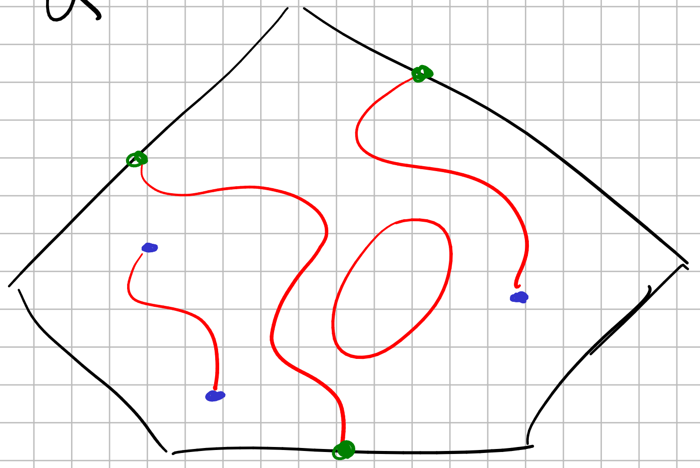
$$\overline{\mathcal{M}}_1(x_{k-1}, \ldots, x_0; y) = \mathcal{M}_1(x_{k-1}, \ldots, x_0; y) \cup \{ D \in \partial' \overline{R}^{k+1} \}$$



$$\cup \{ \text{strip breaking at punctures} \}$$

$$\begin{tikzcd} {} \arrow[d] & (D,u) \arrow[d, mapsto] \\ \overline{\mathcal{R}}^{k+1} & D \end{tikzcd}$$





$$\Rightarrow \# \text{blue} + \# \text{green} = 0$$

□

Rk: (in Seidel's book)

* Actual objects of $\mathrm{Fuk}(M)$ are "Lagrangian branes" $L^\# = (L, \alpha^\#, P^\#)$

↳ Pin structure ⇒ work over $\mathbb{Z}$ instead of $\mathbb{Z}/2$

↳ Grading = Have $CF(L_0^\#, L_1^\#)$ $\mathbb{Z}$-graded

* Need to set up a system of "coherent perturbations" (domain-dependent Hamiltonian isotopies, almost-complex structures) so to have $CF(L_0^\#, L_1^\#)$ well-defined if $L_0 \cap L_1$ not transverse (ex. $L_0 = L_1$ ...)