Derived categories

Goal: work with triangulated categories

Def: (Verdier) very roughly:

A linear category $\mathcal{C}$ is triangulated if every morphism $X \xrightarrow{f} Y$ belongs to an exact sequence $X \xrightarrow{f} Y \to Z \to X[-1]$

"shift"

(+ axioms ...)

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & Z \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

Derived categories

Goal: work with triangulated categories

Def: (Verdier) very roughly:

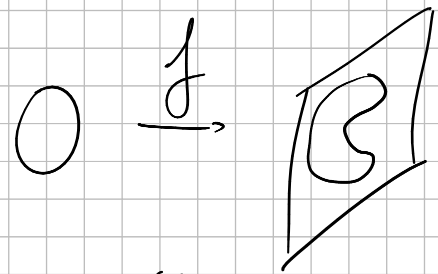
A linear category $\mathcal{C}$ is triangulated if every morphism $X \xrightarrow{f} Y$ belongs to an exact sequence $X \xrightarrow{f} Y \to Z \to X[-1]$ "shift"

(+ axioms ...)

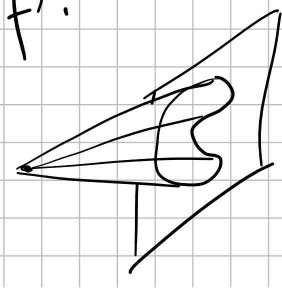
$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & Z \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

ex: The cat. of chain complexes / R

$$\begin{tikzcd} (C^0_*, \partial^0) \arrow[rr, "f"] & & (C^1_*, \partial^1) \arrow[dl] \\ & \mathrm{Cone}(f) = \left(C^0_*[1] \oplus C^1_*, \begin{pmatrix} \partial^0 & 0 \\ f & \partial^1 \end{pmatrix} \right) \arrow[ul] & \end{tikzcd}$$

$O \xrightarrow{f}$ 

Cone(f):



- $(M,\omega) \rightsquigarrow \mathrm{Fuk}(M,\omega) \rightsquigarrow \mathrm{Don}(M) = H_*(\mathrm{Fuk}(M))$

A_∞-cat.

Not triangulated.

• $$\begin{tikzcd} (M,\omega) \arrow[r, rightsquigarrow] & \mathrm{Fuk}(M,\omega) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \mathrm{Don}(M) = H_*(\mathrm{Fuk}(M)) \\ & \mathrm{Tw}(\mathrm{Fuk}(M,\omega)) \arrow[r, rightsquigarrow] & D^b\mathrm{Fuk}(M) = H_*(\mathrm{Tw}(\mathrm{Fuk}(M))) \end{tikzcd}$$

A_∞-cat. (under $\mathrm{Fuk}(M,\omega)$)

Not triangulated. (under $\mathrm{Don}(M)$)

triangulated A_∞-cat. (under $\mathrm{Tw}(\mathrm{Fuk}(M,\omega))$)

triangulated category. (under $D^b\mathrm{Fuk}(M)$)

$$\bullet\ (M,\omega) \rightsquigarrow \mathrm{Fuk}(M,\omega) \rightsquigarrow \mathrm{Don}(M) = H_*(\mathrm{Fuk}(M))$$

A_∞-cat. Not triangulated.

$$\Big\downarrow$$

$$\mathrm{Tw}(\mathrm{Fuk}(M,\omega)) \rightsquigarrow D^b\mathrm{Fuk}(M) = H_*(\mathrm{Tw}(\mathrm{Fuk}(M)))$$

triangulated A_∞-cat. triangulated category.

$$\mathcal{A} \rightsquigarrow \Sigma\mathcal{A} \rightsquigarrow \mathrm{Tw}\,\mathcal{A}$$

A_∞-cat. additive enlargement twisted complexes.

- $\Sigma\mathcal{A}$: additive enlargement of $\mathcal{A}$:
  - objects: $X = (I, \{X^i\}, \{V^i\})$ "$=$" $\bigoplus_{i \in I} V^i \otimes X^i$

    finite set

    objects in $\mathcal{A}$

    graded vector spaces

- $$\hom_{\Sigma\mathcal{A}}\left(\bigoplus_i V^i \otimes X^i, \bigoplus_j W^j \otimes Y^j\right) := \bigoplus_{i,j} \hom(V^i, W^j) \otimes \hom_{\mathcal{A}}(X^i, Y^j)$$

- $$X_0 = \bigoplus_{i_0} V_0^{i_0} \otimes X_0^{i_0}, \quad X_1 = \bigoplus_{i_1} V_1^{i_1} \otimes X_1^{i_1}, \quad \dots, \quad X_k = \bigoplus_{i_k} V_k^{i_k} \otimes X_k^{i_k}$$

  $$\phi_{01}^{i_0 i_1} \otimes u_{01}^{i_0 i_1} \qquad \phi_{12}^{i_1 i_2} \otimes u_{12}^{i_1 i_2}$$

$$\mu^k_{\Sigma\mathcal{A}}\left(\phi_{k-1\,k}^{i_{k-1} i_k} \otimes u_{k-1\,k}^{i_{k-1} i_k}, \dots, \phi_{01}^{i_0 i_1} \otimes u_{01}^{i_0 i_1}\right)$$

$$= (\pm 1) \cdot \phi_{k-1\,k}^{i_{k-1} i_k} \circ \dots \circ \phi_{01}^{i_0 i_1} \otimes \mu^k_{\mathcal{A}}\left(u_{k-1\,k}^{i_{k-1} i_k}, \dots, u_{01}^{i_0 i_1}\right)$$

$Tw(\mathcal{A})$: twisted complexes

- <u>objects</u> (X, δ_X) such that:

  $X \in \mathrm{ob}\, \Sigma\mathcal{A}$, $\delta_X \in \hom^1_{\Sigma\mathcal{A}}(X, X)$

  - δ_X is strictly lower-triangular
  - $\sum_{k \geq 1} \mu^k_{\Sigma\mathcal{A}}(\delta_X, \dots, \delta_X) = 0$ (Maurer-Cartan equation)

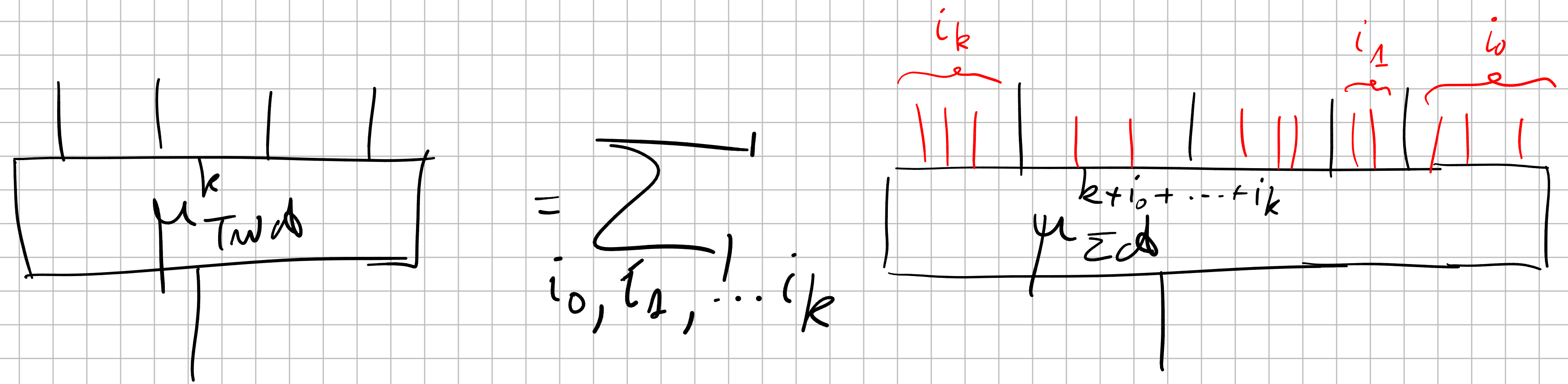
- $\hom_{Tw\mathcal{A}}((X, \delta_X), (Y, \delta_Y)) = \hom_{\Sigma\mathcal{A}}(X, Y)$

Compositions:

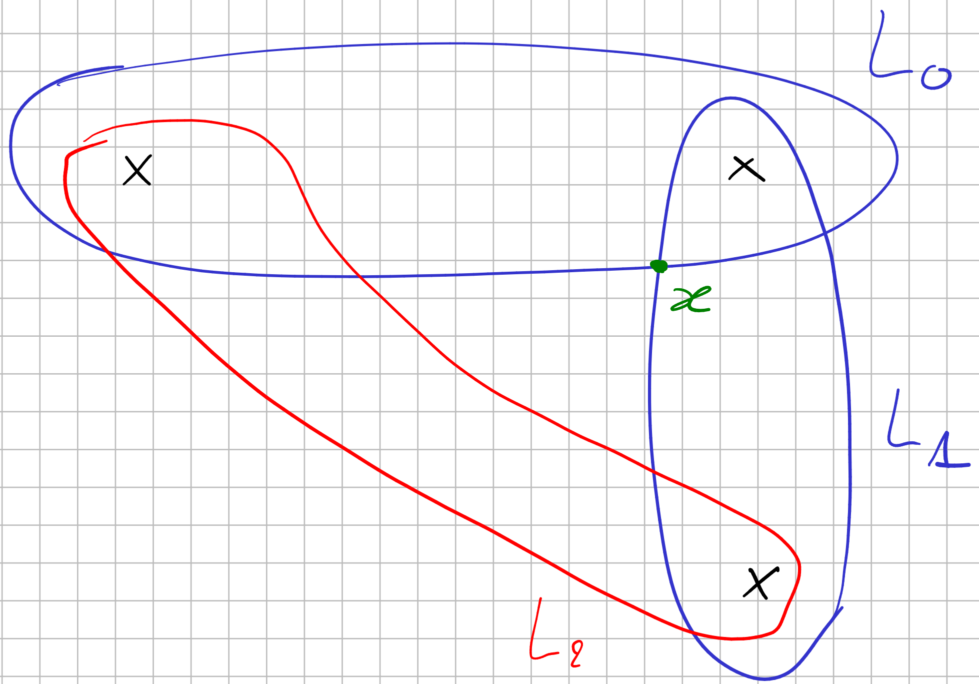
- $(X_0, \delta_{X_0}) \xrightarrow{a_1} (X_1, \delta_{X_1}) \xrightarrow{a_2} \cdots \xrightarrow{a_k} (X_k, \delta_{X_k})$

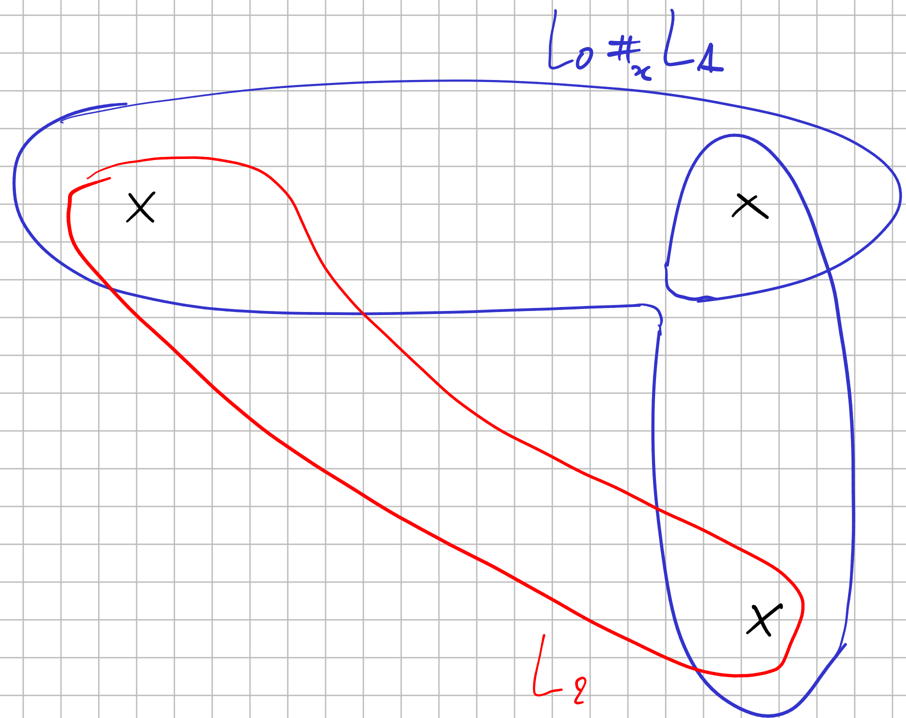
$$\mu^k_{Tw\,d}(a_k, \dots, a_1)$$

$$= \sum{}' \mu_{\Sigma d}\big(\delta_{X_k}, \dots, \delta_{X_k}, a_k, \delta_{X_{k-1}}, \dots, \delta_{X_{k-1}}, a_{k-1}, \dots, a_1, \delta_{X_0}, \dots\big)$$



Ex:





$$\mathrm{Cone}\left(L_0 \xrightarrow{x} L_1 \right) \simeq L_0 \#_x L_1$$

Fukaya-Seidel category:

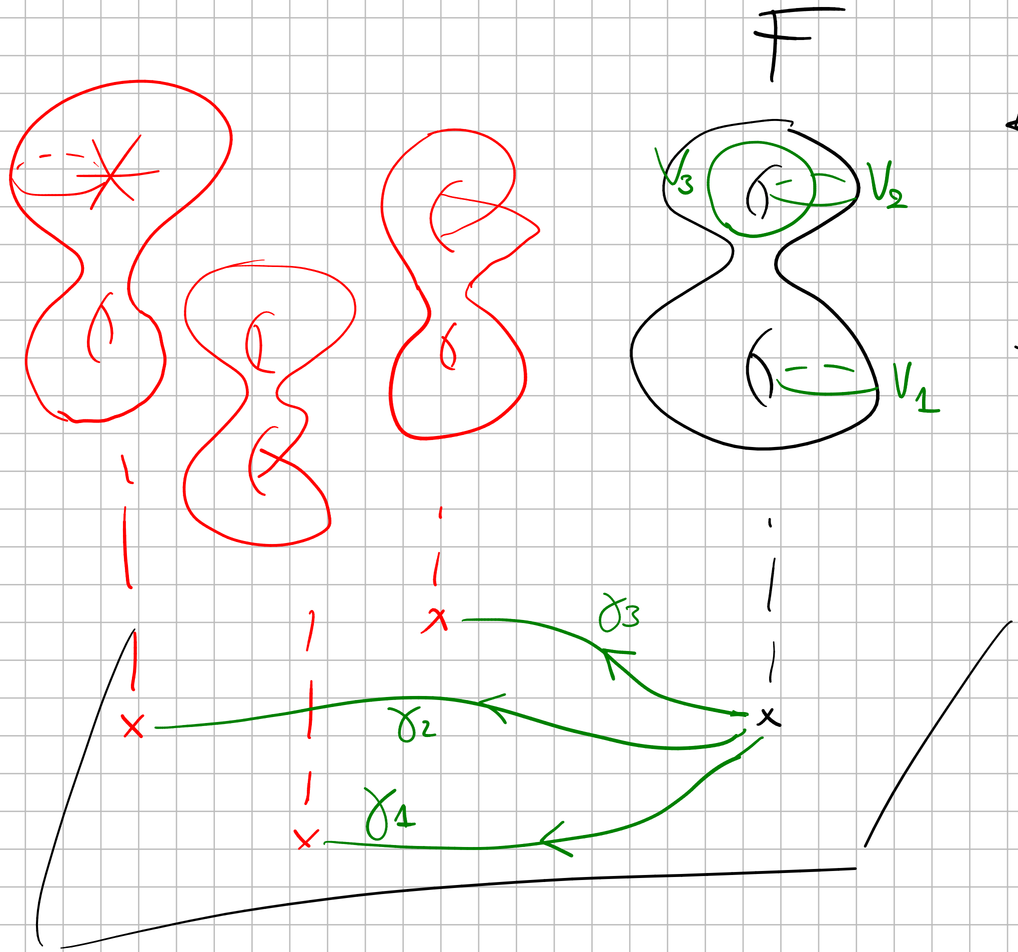
$$\begin{tikzcd} F \arrow[r, hookrightarrow] & E \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

reg. fiber

Lefschetz fibration $\rightarrow FS(\pi)$

- Construction 1: inside F
- Construction 2: inside a branched double cover of E
- Construction 3: as a "partially wrapped" Fukaya categ.
- Construction 4: Andriy's construction

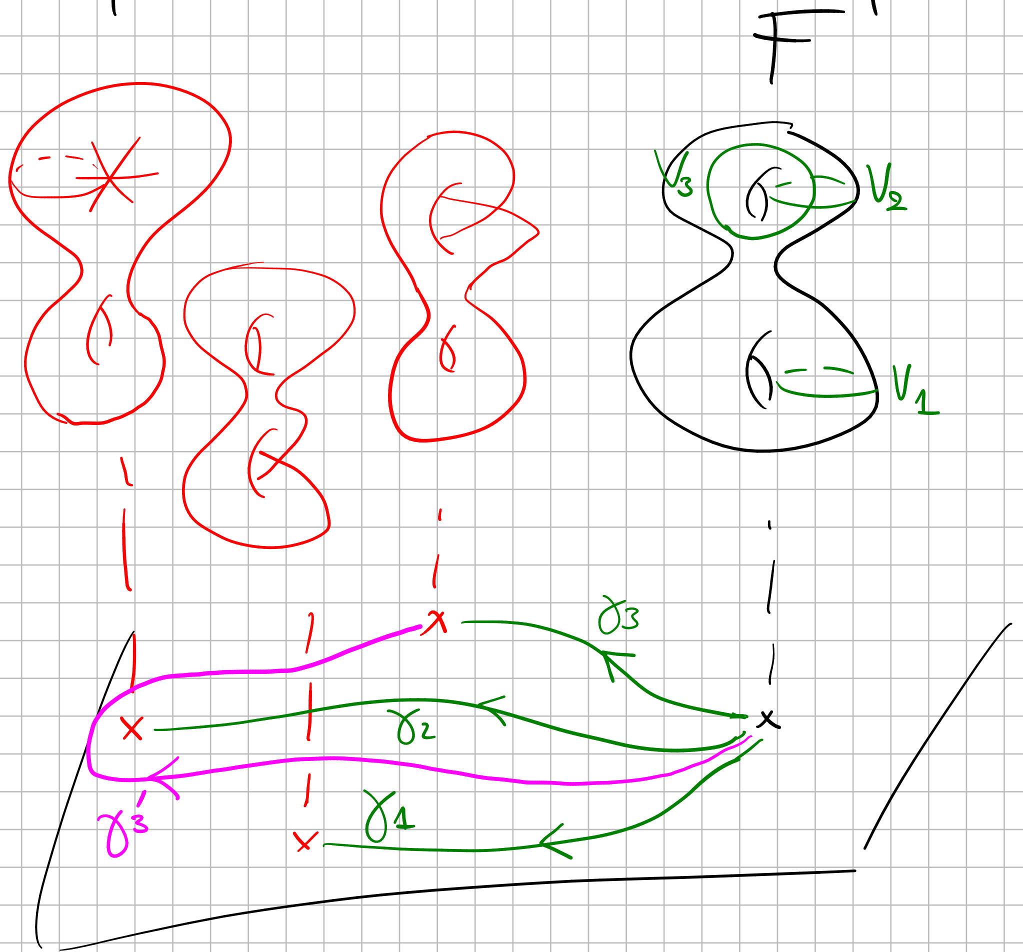
Construction 1 : inside F

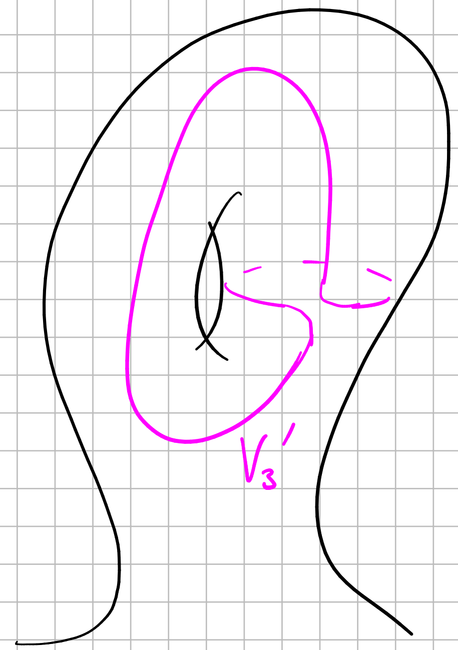


* $\gamma_1, \dots, \gamma_m$ vanishing paths
* $\underbrace{V_1, \dots, V_m}_{\Gamma}$ vanishing cycles
* $\overrightarrow{Lag}(\Gamma)$: ob : $V_1, \dots, V_m$
* $$\hom(V_i, V_j) = \begin{cases} CF(V_i, V_j) & i < j \\ \mathbb{Z}/2 & i = j \\ 0 & i > j \end{cases}$$
* μ^k as in $Fuk(F)$ (when makes sense)

$$FS(\pi) := D^b(\overrightarrow{Lag}(\Gamma))$$

Dependence on the paths $\gamma_2, \dots, \gamma_m$





$$V_3' = \tau_{V_2}(V_3) \quad \left(\text{or } \tau_{V_2}^{-1}(V_3) \dots\right)$$

$$V_3' \simeq \mathrm{Cone}\left[V_3 \to V_2 \otimes HF(V_3, V_2)\right]$$
in $Tw(\mathrm{Fuk}\,F)$

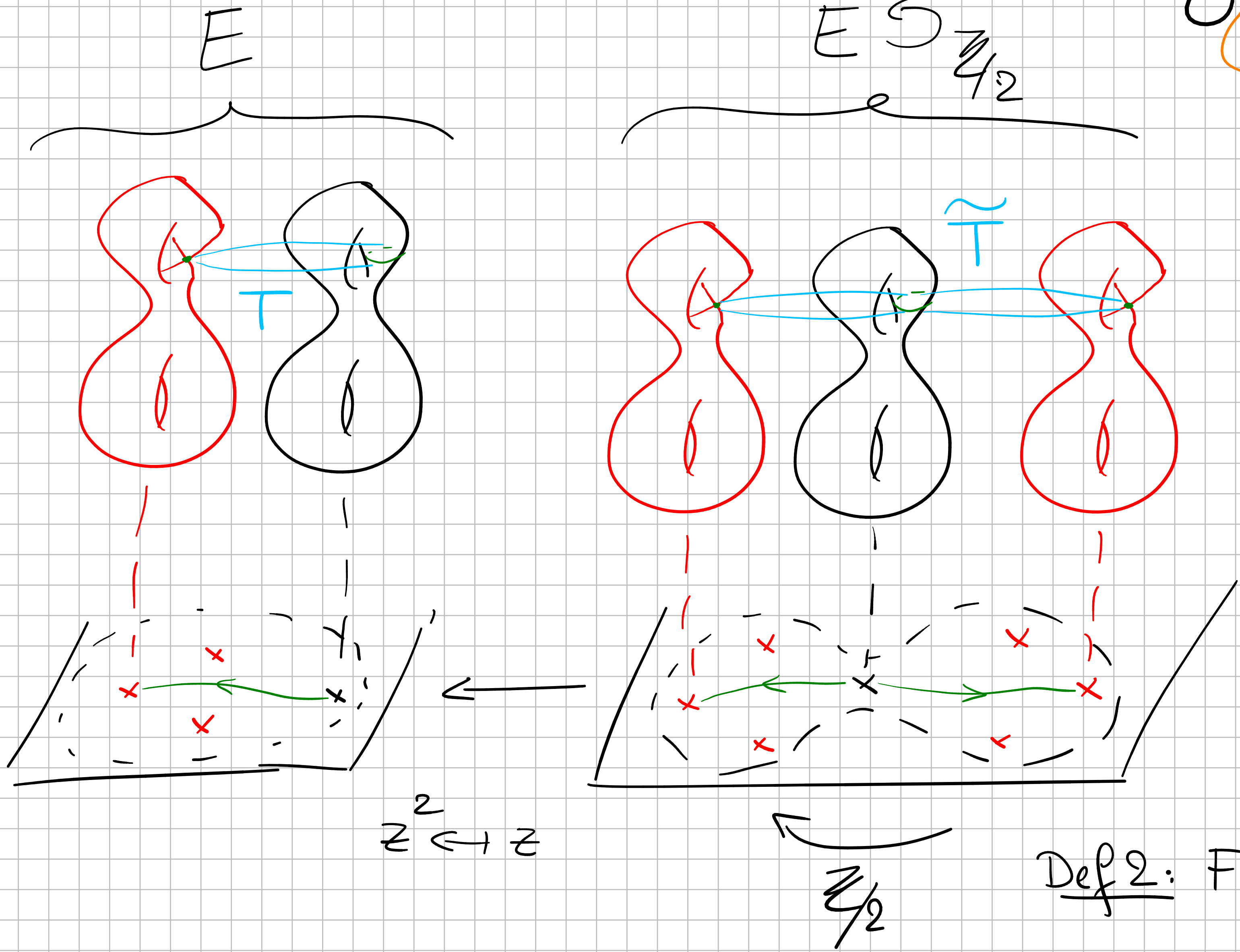
Seidel's LES:

$$\begin{tikzcd} HF(\tau_S L_0, L_1) \arrow[r] & HF(L_0, L_1) \arrow[dl] \\ HF(S, L_1) \otimes HF(L_0, S) \arrow[u] & \end{tikzcd}$$

Lag. sphere

Construction 2: Use the Vanishing Thimbles instead

$\hookrightarrow$ not closed ....



$\mathbb{D}^{n+1} \simeq T \subset E$

$S^n \simeq \partial T = V \subset F$

$S^{n+1} \simeq \tilde{T} \subset \tilde{E}$

$\rightsquigarrow \mathcal{A} \subset \mathrm{Fuk}(\tilde{E})$

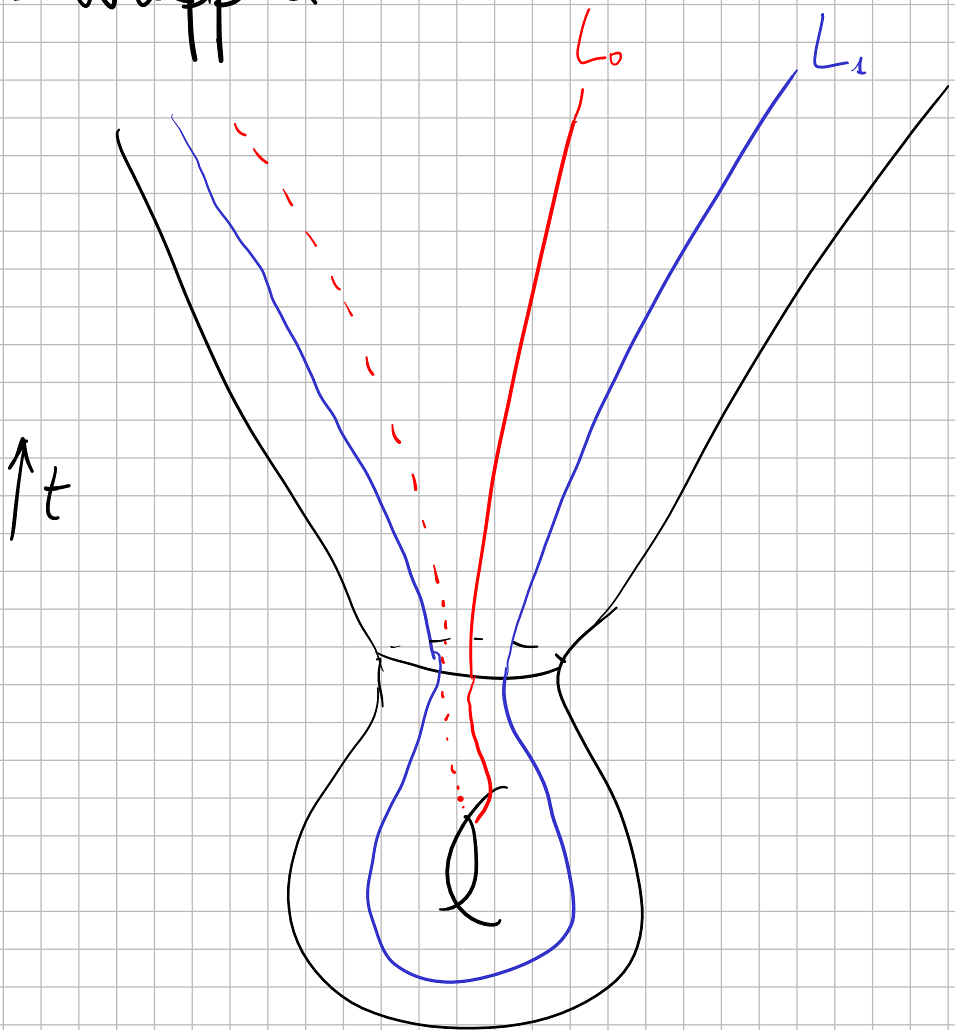
$\circlearrowleft \mathbb{Z}/2$

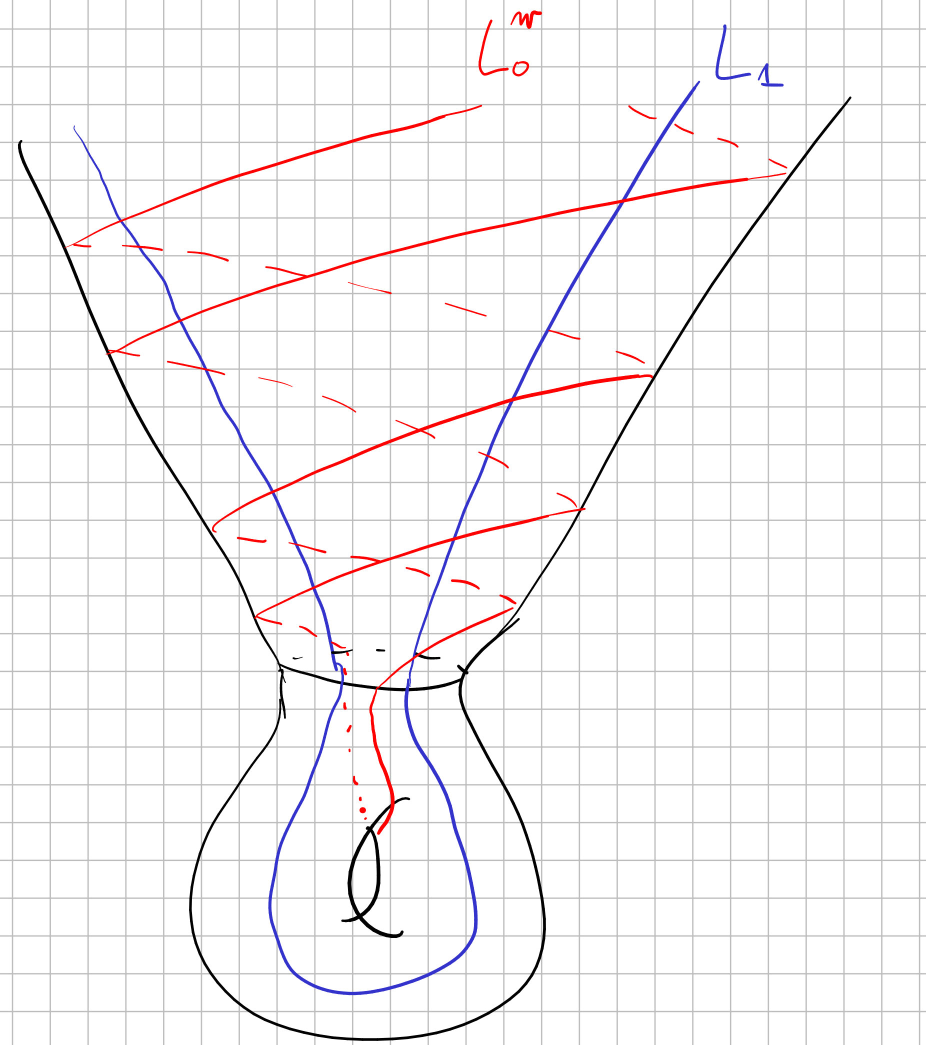
Def 2: $FS(\pi) = \mathcal{A}^{inv}$:

- $ob(\mathcal{A}^{inv})$ = object fixed by the $\mathbb{Z}/2$ - action
- $hom_{\mathcal{A}^{inv}}(L, L') = hom_{\mathcal{A}}(L, L')^{\mathbb{Z}/2}$

Construction 3: Partially wrapped Fukaya category

→ "wrapped"



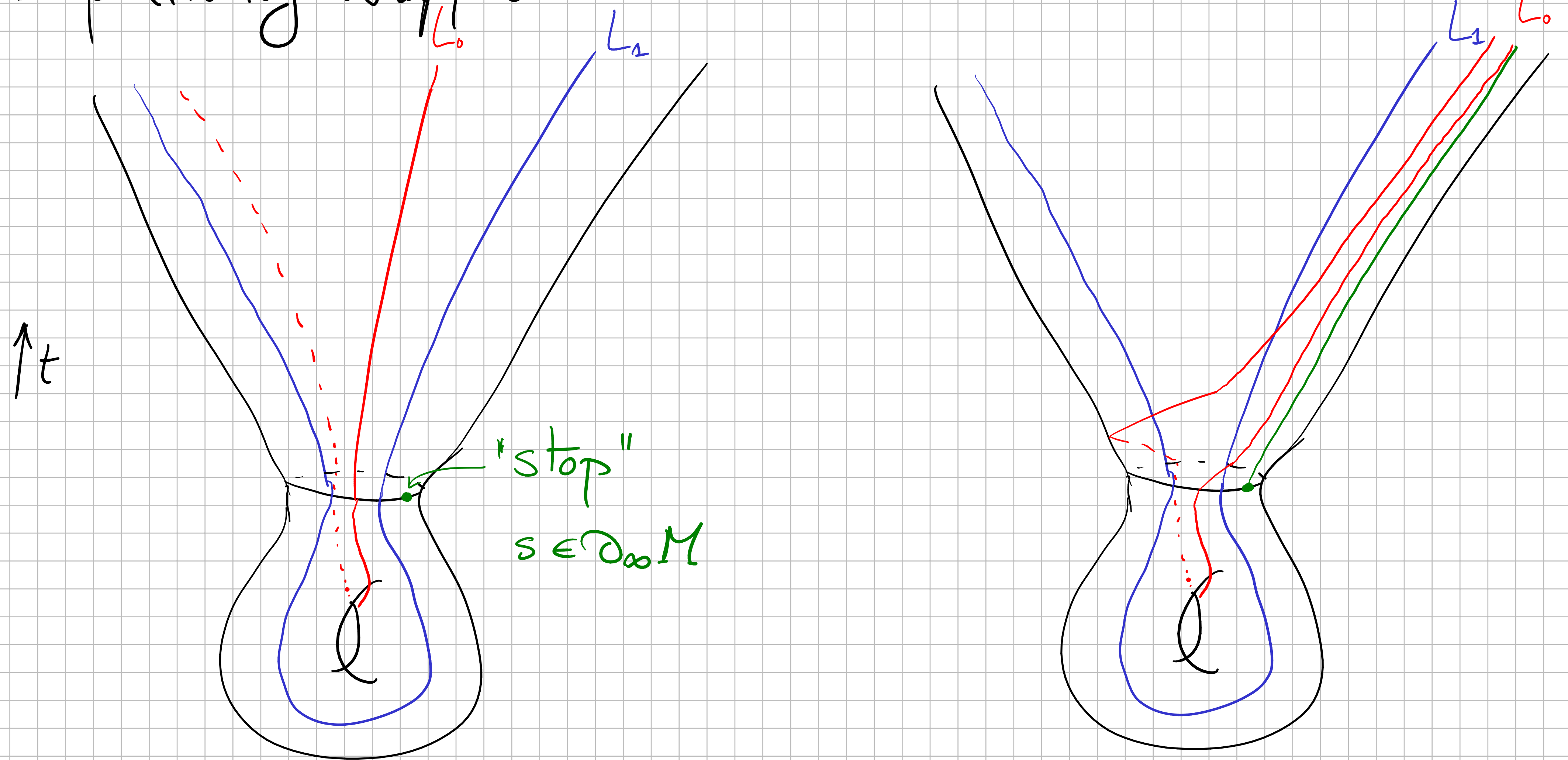


$H \sim t^2$ at ∞ $\qquad CW(L_0, L_1) = CF(L_0^w, L_1)$

$L_0^w = \phi_H^1(L_0)$ $\qquad \rightsquigarrow \mathcal{W}(M)$ wrapped Fukaya category

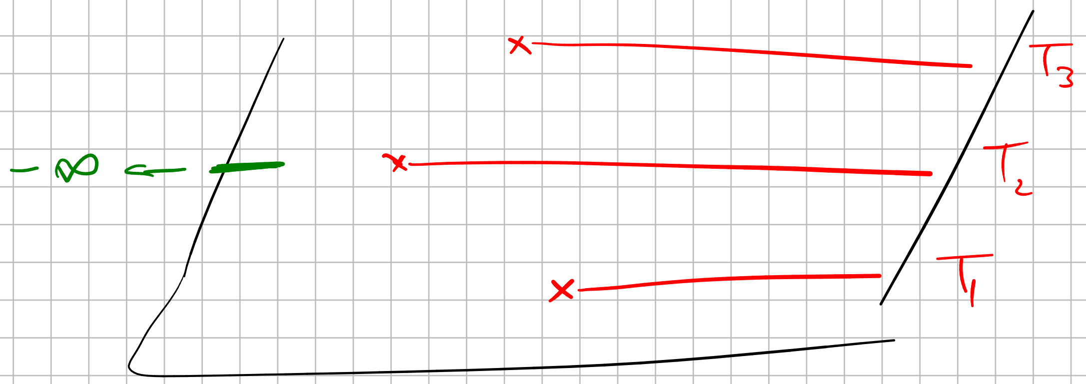
Construction 3: Partially wrapped Fukaya category

→ "partially wrapped"



$H \sim t^2$ at ∞ + $H \equiv 0$ near s $\quad CW(L_0, L_1) = CF(L_0^w, L_1)$

$L_0^w = \phi_H^1(L_0)$ $\quad \rightsquigarrow \mathcal{W}(M, s)$ partially wrapped Fukaya category



Third def.: $FS(\pi) = \mathcal{W}(E, s)$ $\quad s = \pi^{-1}(-\infty)$

Ganatra-Pardon-Shende: $\mathcal{W}(E,s)$ generated by the thimbles.