Lefschetz fibrations and Fukaya-Seidel categ.

Def: A Lefschetz fibration (E, π, S) is:

symplectic form

compatible almost complex str.

$$\begin{cases} \cdot\ (E^{2n+2}, \omega, J) \\ \cdot\ S : \text{Riemann surface} \\ \cdot\ \pi : E \to S \text{ holomorphic map s.t.:} \end{cases}$$

near a critical point, in a holomorphic chart $(z_0, \dots, z_n)$,

$$\pi(z_0, \dots, z_n) = z_0^2 + \dots + z_n^2$$

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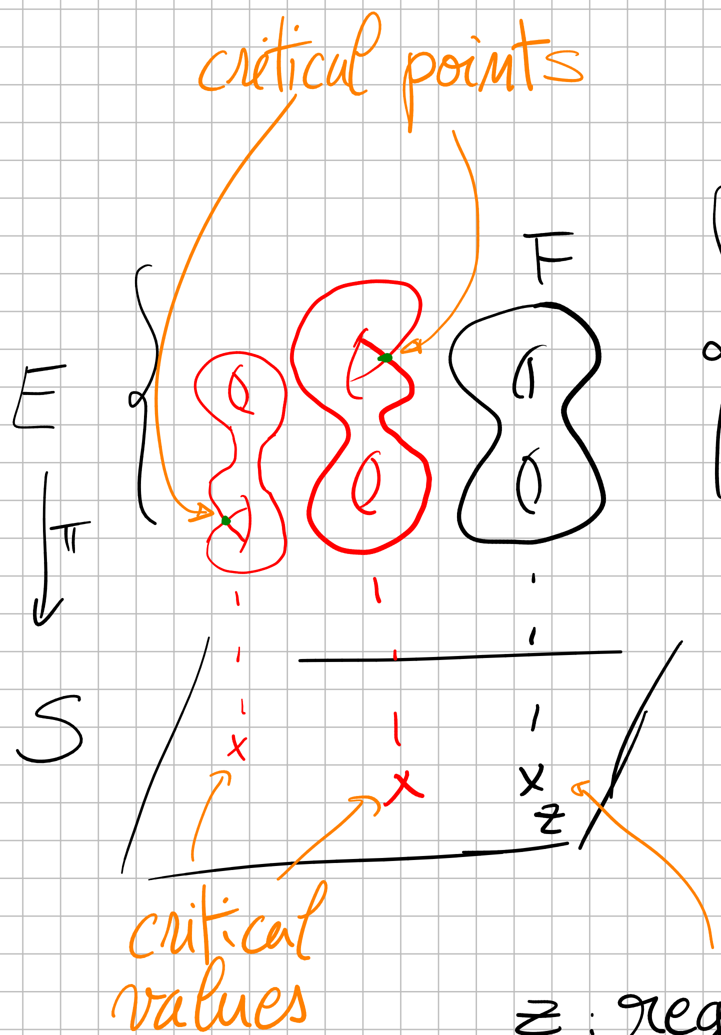

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- (E^{2n+2}, ω, J)
- S : Riemann surface
- $\pi : E \to S$ holomorphic map s.t.:

near a critical point, in a holomorphic chart $(z_0, \dots, z_n)$,

$$\pi(z_0, \dots, z_n) = z_0^2 + \dots + z_n^2$$

z : regular value

$F = \pi^{-1}(z)$ regular fiber

Relevance to symplectic geometry:

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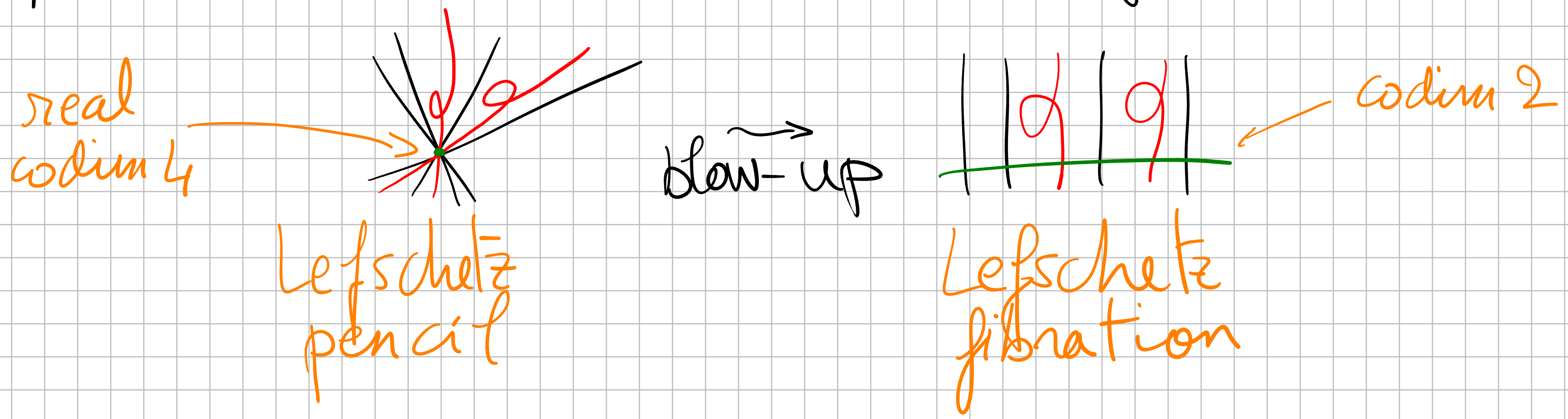

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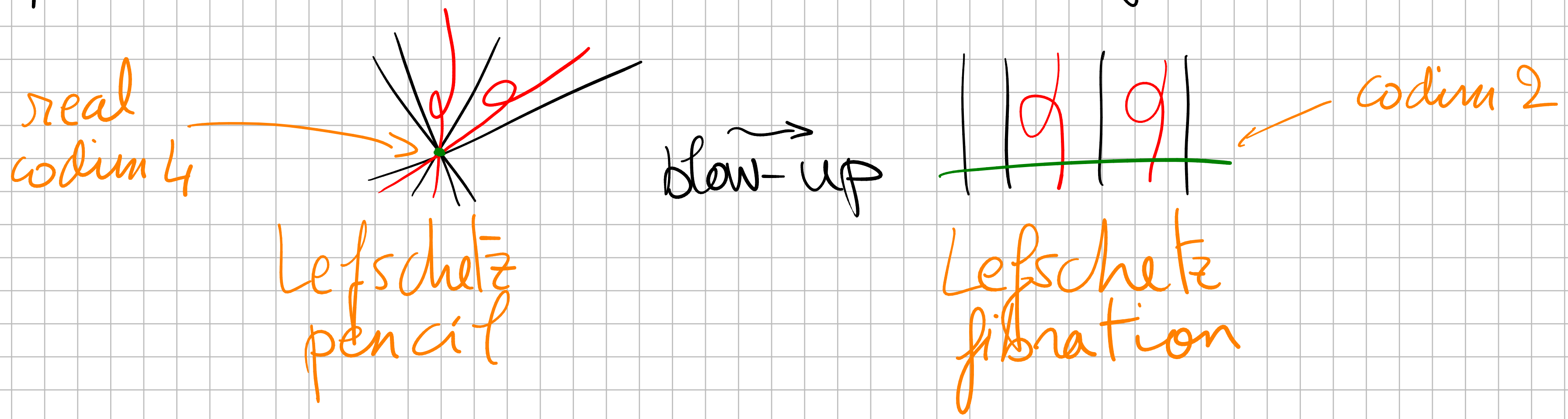

<u>Goal:</u> compute Fukaya categories inductively:

$$\left.\begin{array}{l} \mathrm{Fuk}(F) \\ + \\ FS(\pi) \end{array}\right\} \Rightarrow \text{reconstruct } \mathrm{Fuk}(E)$$

Fukaya-Seidel category

(roughly ...)

Ex: $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}$

$(z_0, z_1) \mapsto z_0^2 + z_1^2$

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$$(z_0, z_1) \mapsto z_0^2 + z_1^2 = z_0^2 - (i z_1)^2$$

$$= \underbrace{(z_0 - i z_1)}_{U} \underbrace{(z_0 + i z_1)}_{V}$$

$$\pi^{-1}(0) = \{ UV = 0 \} = \{U = 0\} \cup \{V = 0\} \simeq \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$$

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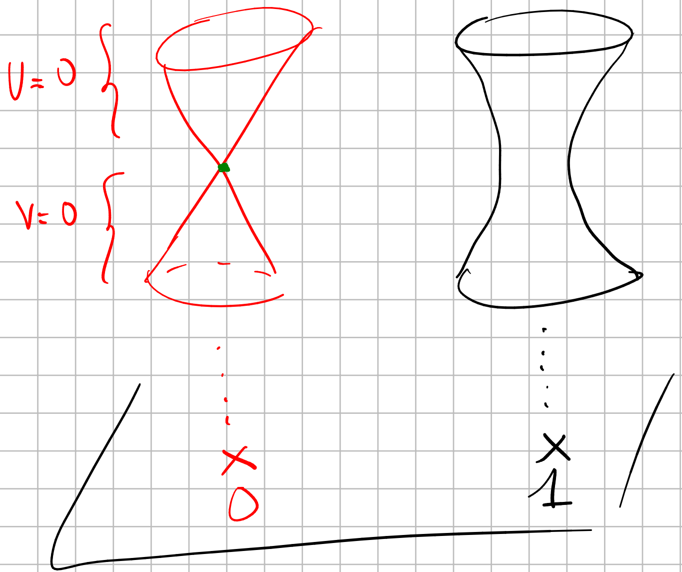

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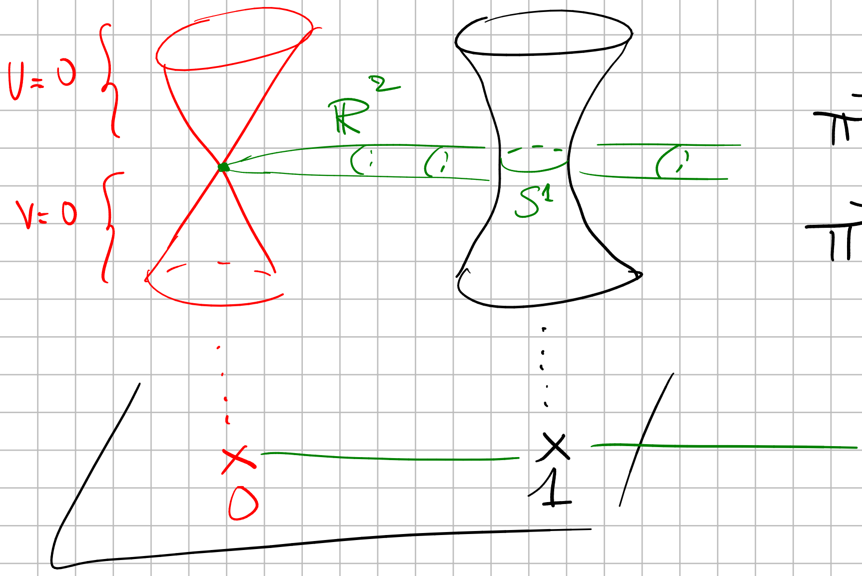

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Rk: $\mathbb{R}^2 \subset \mathbb{C}^2$

$\mathbb{R}^2 \cap \pi^{-1}(1) = S^1$ unit circle

Ex 2 : $\mathbb{C}^{n+1} \longrightarrow \mathbb{C}$

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$$F = \pi^{-1}(1) = \left\{ z = x + iy \;\middle|\; \begin{array}{l} |x|^2 - |y|^2 = 1 \\ \langle x, y \rangle = 0 \end{array} \right\}$$

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$$\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} \,/\, |x|^2 = 1 \}$$

$$\begin{array}{c} T^*\mathbb{S}^n \\ \cong \\ T\mathbb{S}^n \end{array} = \{ (q, p) \in \mathbb{R}^{n+1} \,/\, |q|^2 = 1 \,,\, \langle q, p \rangle = 0 \}$$

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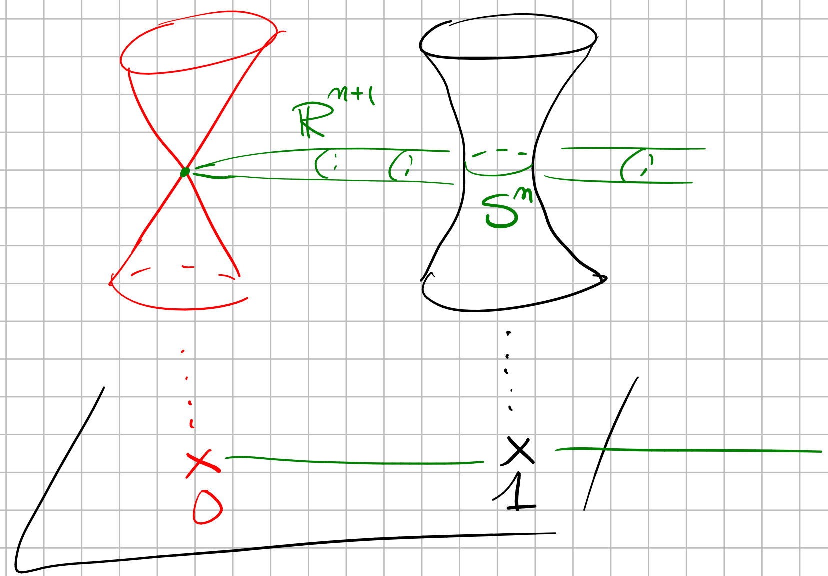
$$\cong T\mathbb{S}^n$$

Fact: $F \xrightarrow[\sim]{\text{symplecto}} T^*\mathbb{S}^n$

$$x+iy \longmapsto \left(q = \frac{x}{|x|} \,,\, p = |x|\, y\right)$$

$\pi^{-1}(0) \simeq T^*\mathbb{S}^n / S^n$ $\leftarrow$ zero section



· Symplectic connexion : $x \in E \setminus E^{crit}$, $T_x E = V_x \oplus H_x$

$$\left| \begin{array}{l} \cdot\, V_x = \operatorname{Ker} d\pi_x \subset T_x E \\ \cdot\, H_x = (V_x)^{\perp\omega} = (V_x)^{\perp g_J} \end{array} \right.$$

$g_J(v, w) = \omega(v, Jw)$

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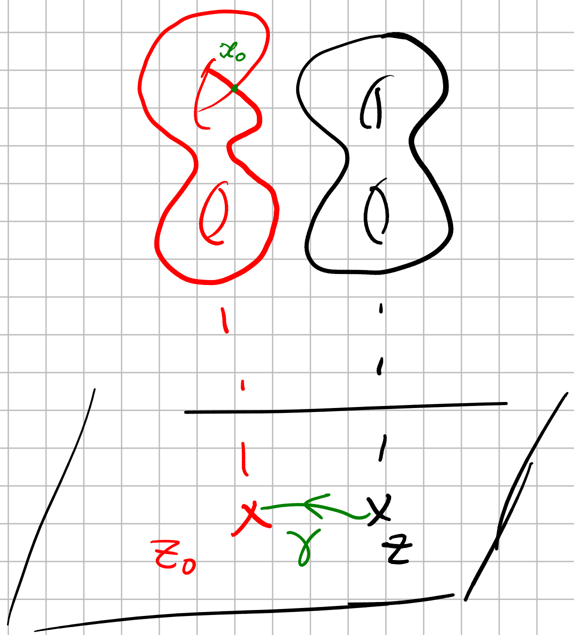

$\gamma : [0,1] \to S$ vanishing path

$\gamma(0) = z$, $\gamma(1) = z_0$ "$\gamma : z \longrightarrow z_0$"

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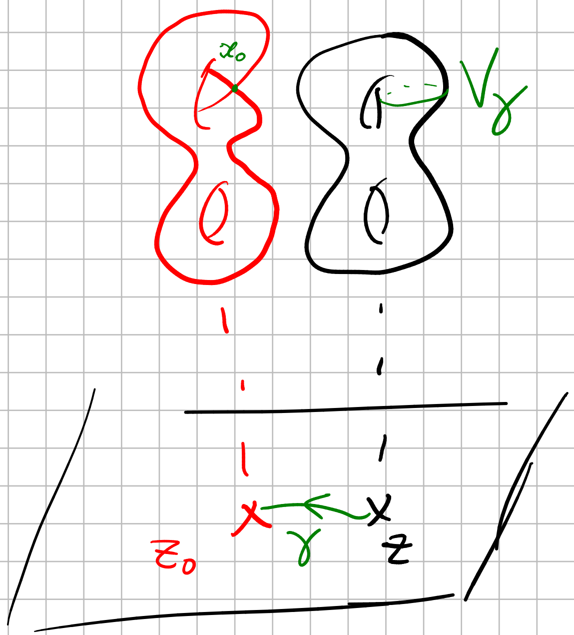

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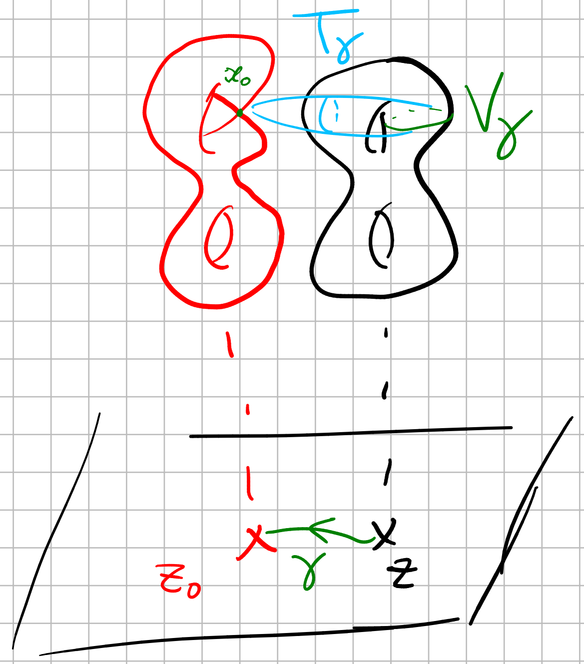

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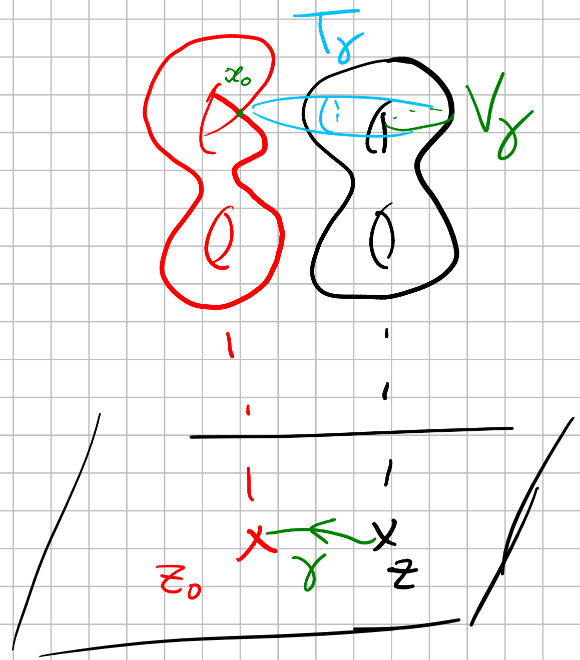

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Rk : in the loc. model $\mathbb{C}^{n+1} \to \mathbb{C}$

$T_\gamma = B^{n+1} \subset \mathbb{R}^{n+1}$

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· Monodromy

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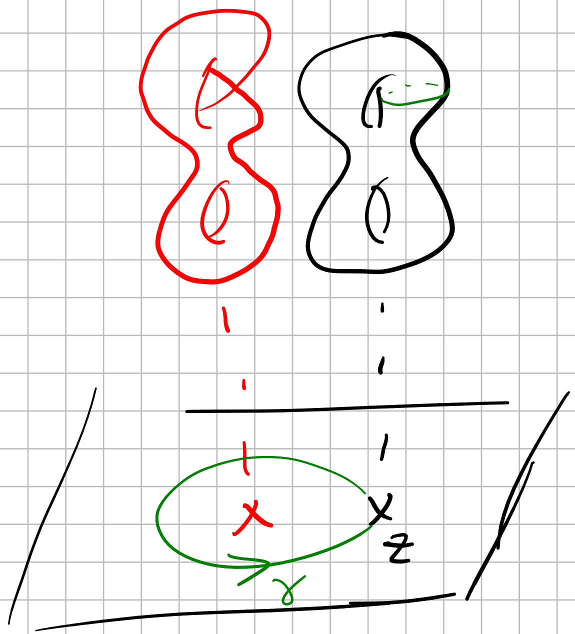

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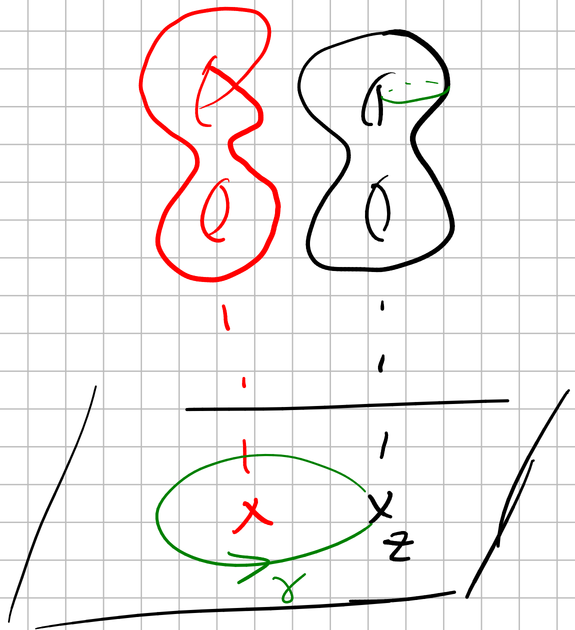

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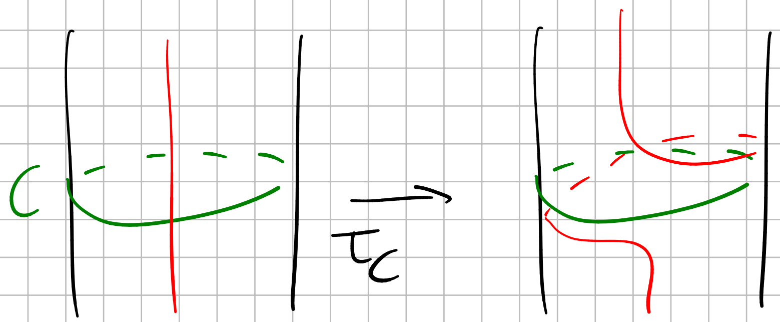
Th [Seidel] ϕ_γ is a Dehn-Seidel twist around the vanishing cycle V_γ

Dehn twist on a surface :



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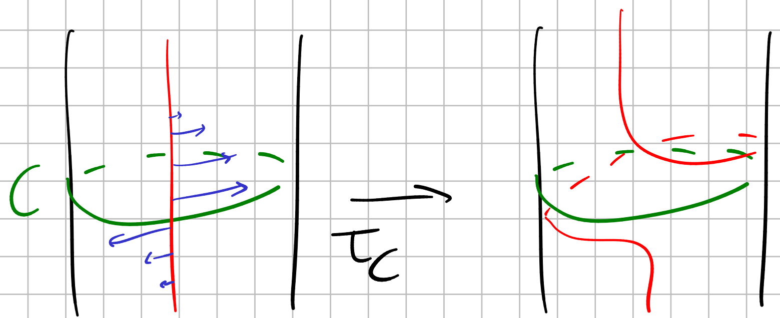

<u>Obs</u>: time one flow of a discontinuous vector field

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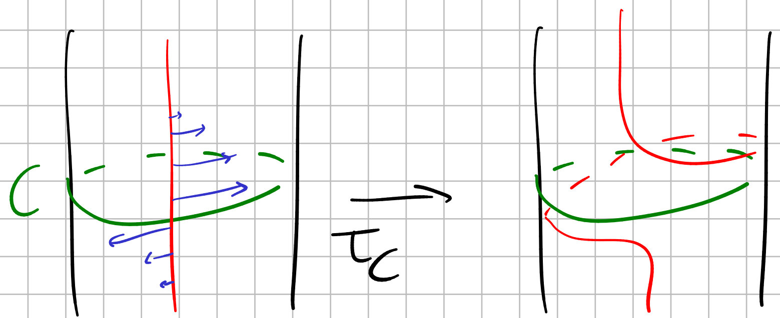

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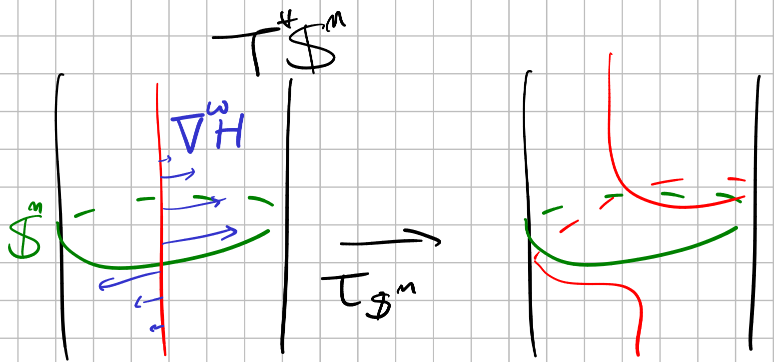
Seidel: picture generalizes to $S \subset (M, \omega)$

Lagrangian sphere

symplectic mfd

$\rightsquigarrow \tau_S : M \to M$ symplectomorphism

Dehn-Seidel twist:



S^n + round metric

$\Rightarrow$ all geodesics are closed, and have same length.

$\to$ $T^*S^n \simeq TS^n \ni (q,p)$

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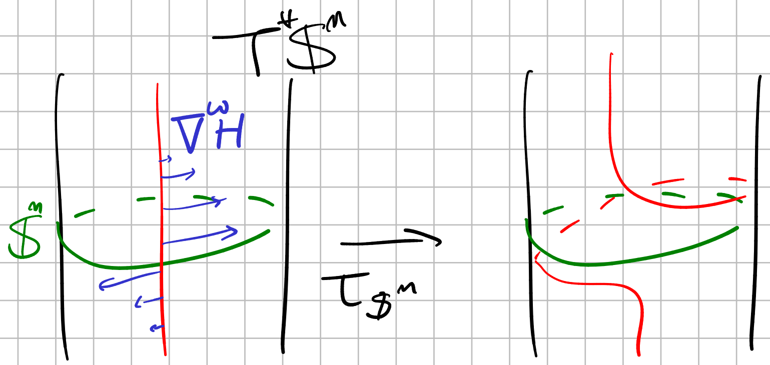

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$$f: T^*S^n \longrightarrow \mathbb{R}$$
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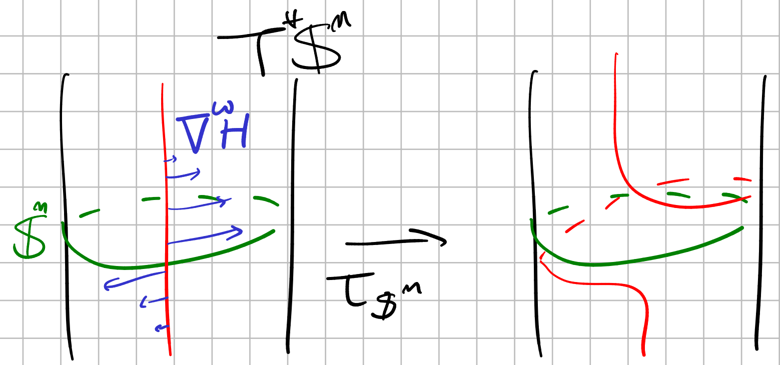

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$H: T^*S^n \to \mathbb{R}$

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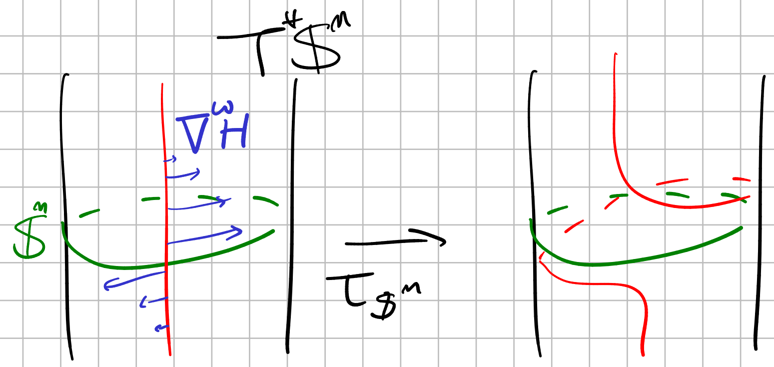

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$\rightarrow \phi_H^t: T^*S^n \setminus S^n \to T^*S^n \setminus S^n$

extends to S^n when $t = 2\pi$

$$\boxed{\rightarrow \tau_{S^n} := \phi_H^{2\pi}: T^*S^n \to T^*S^n}$$

$$H : M \to \mathbb{R}$$

$$dH \stackrel{\pm}{=} \omega(\nabla^{\omega} H, \cdot)$$

$$\nabla^{\omega} H = \pm J \, \nabla^{g_J} H$$

Lagrangian Floer homology

(M, ω) symplectic manifold
$\overset{\cup}{L_0}, L_1$ pair of Lagrangian submanifolds $\Big\} \longrightarrow HF(L_0, L_1)$

Warning: Need to make assumptions on M, L_0, L_1, Def. is more or less complicated depending on those assumpt's

Assume that $L_0 \pitchfork L_1$

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{Z}_2 \cdot x$$

$\partial : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$ defined by counting pseudo-holomorphic strips

- take J a.c.s on M compatible with ω

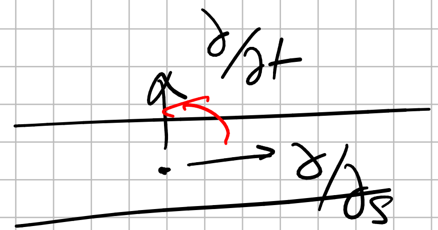
Σ : Riemann surface

$u : (\Sigma, j) \to (M, J)$ is pseudo-holomorphic if

du is $\mathbb{C}$-linear $\iff du \circ j = J \circ du$

$\iff \boxed{du + J \circ du \circ j = 0}$ Cauchy-Riemann equation

$$\Sigma = \underset{\substack{\in \\ s}}{\mathbb{R}} \times \underset{\substack{\in \\ t}}{[0,1]}$$



$$j \frac{\partial}{\partial s} = \frac{\partial}{\partial t}$$

$$\partial x = \sum_y{}' \, n_{xy} \cdot y$$

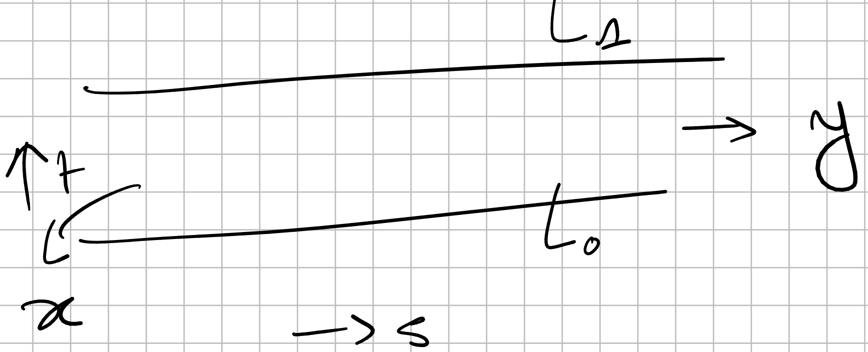
$$n_{x,y} = \# \mathcal{M}(x,y)$$

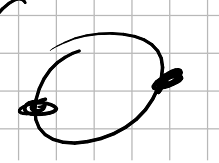
cardinality of the zero-dimensional part of $\mathcal{M}(x,y)$.

$$= \widetilde{\mathcal{M}}(x,y) / \mathbb{R}$$

$$\widetilde{\mathcal{M}}(x,y) = \left\{ u : \mathbb{R} \times \underline{[0,1]} \to M \;\middle|\; \begin{array}{l} u(s,0) \in L_0 \\ u(s,1) \in L_1 \\ du + J du \circ j = 0 \\ \lim_{s \to -\infty} u(s,t) = x \\ \lim_{s \to +\infty} u(s,t) = y \end{array} \right\} \circlearrowleft \mathbb{R}$$

$$u(s,t) \mapsto u(s+s_0,t)$$



Aut()

$$J_0 \xrightarrow[J_t]{} J_1$$

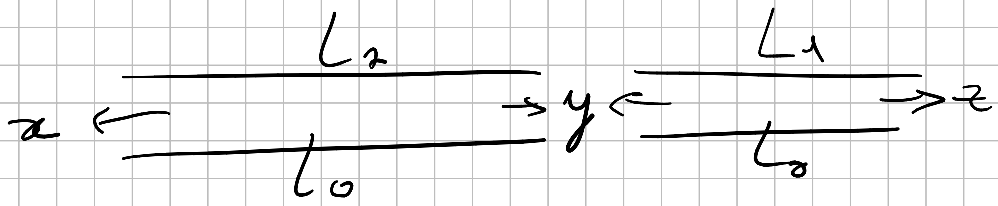
<u>prop:</u> $\partial^2 = 0 \to HF(L_0, L_1) = \dfrac{\ker \partial}{\operatorname{im} \partial}$

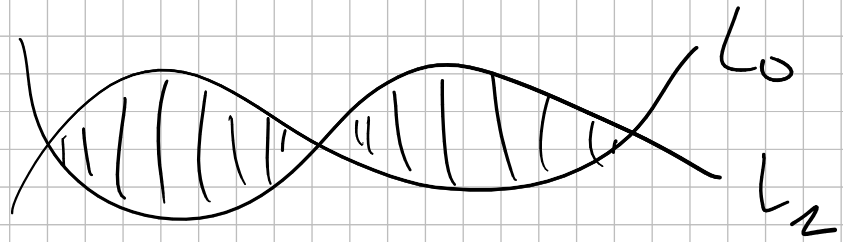
<u>proof:</u>

$$\partial^2 x = \partial\left(\sum_{y \in L_0 \cap L_1}{}' \, n_{x,y} \cdot y \right)$$

$$= \sum_{y}{}' \, n_{x,y} \sum_{z}{}' \, n_{yz} \cdot z$$

$$= \sum_{z}{}' \underbrace{\left(\sum_{y} n_{x,y} \, n_{yz} \right)}_{= \text{count of "broken strips"}} \cdot z$$



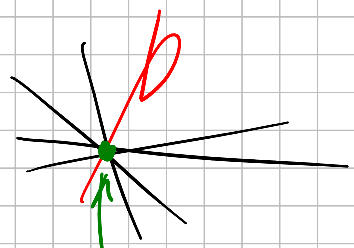


$$\{\text{broken strips}\} = \partial \underbrace{\left\{\begin{array}{c}\text{moduli space}\\ \text{of unbroken}\\ \text{strips}\end{array}\right\}}_{\text{compact 1-dim manifold}}$$

$$\begin{tikzcd} L \arrow[d, shift right] \\ M \arrow[u, shift right, "s_1, s_1"'] \end{tikzcd}$$

$L \longleftarrow$ want to have enough sections

$$\frac{s_1}{s_2} : M \longrightarrow \mathbb{CP}^1$$



$s_1 = s_2 = 0$