Morse homology

M : smooth compact n-manifold

$f : M \longrightarrow \mathbb{R}$ Morse function (= critical points are non-degenerated)

$\Longrightarrow$ (Morse lemma) near a critical point x, in a chart:

$$f(x_1, \dots, x_n) = f(x) - x_1^2 \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

$k =: i(x)$: Morse index

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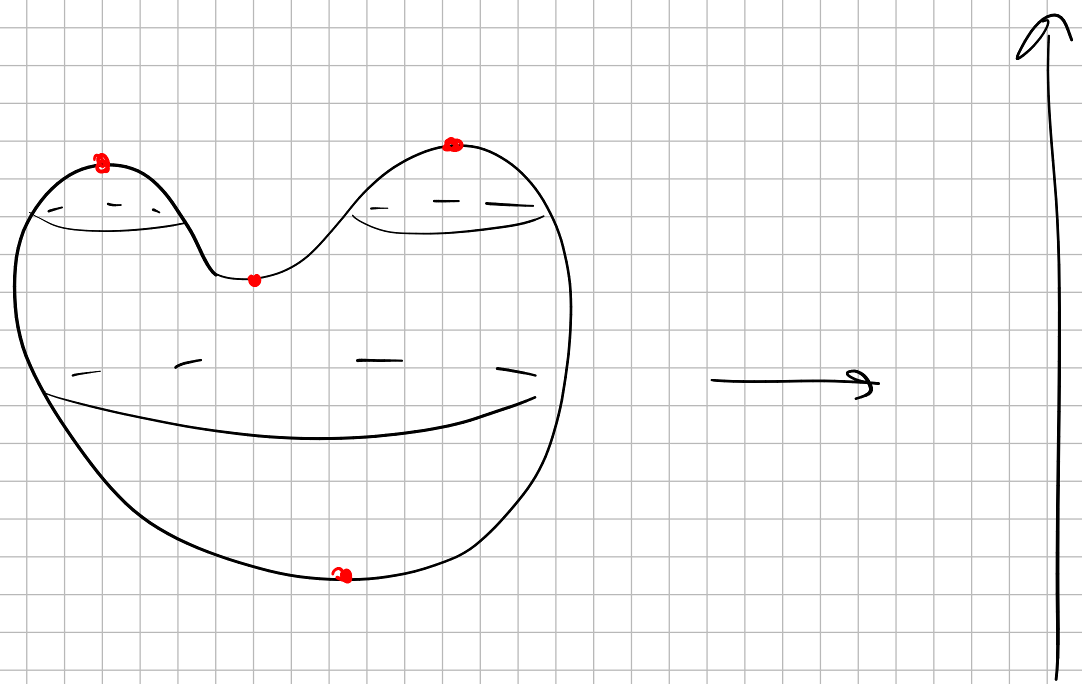
ex:



g: Riemannian metric / M

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$S_x = \left\{ y \mid \lim_{t \to +\infty} \phi_t(y) = x \right\}$ "Stable manifold" $\dim S_x = n - i(x)$

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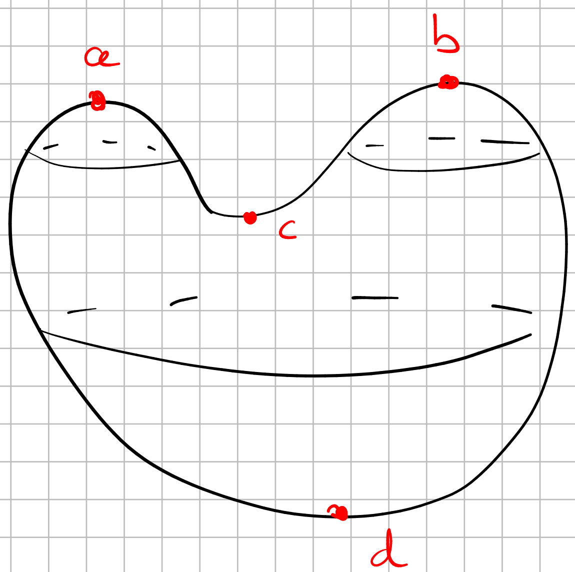

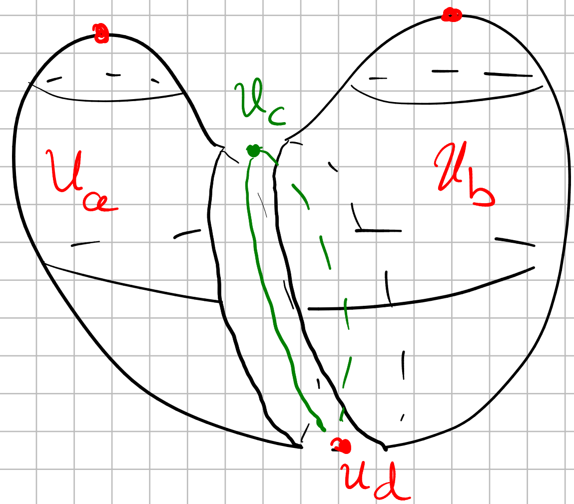

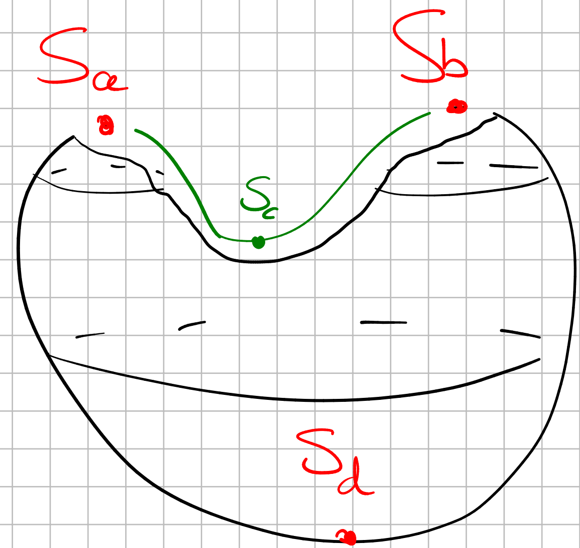

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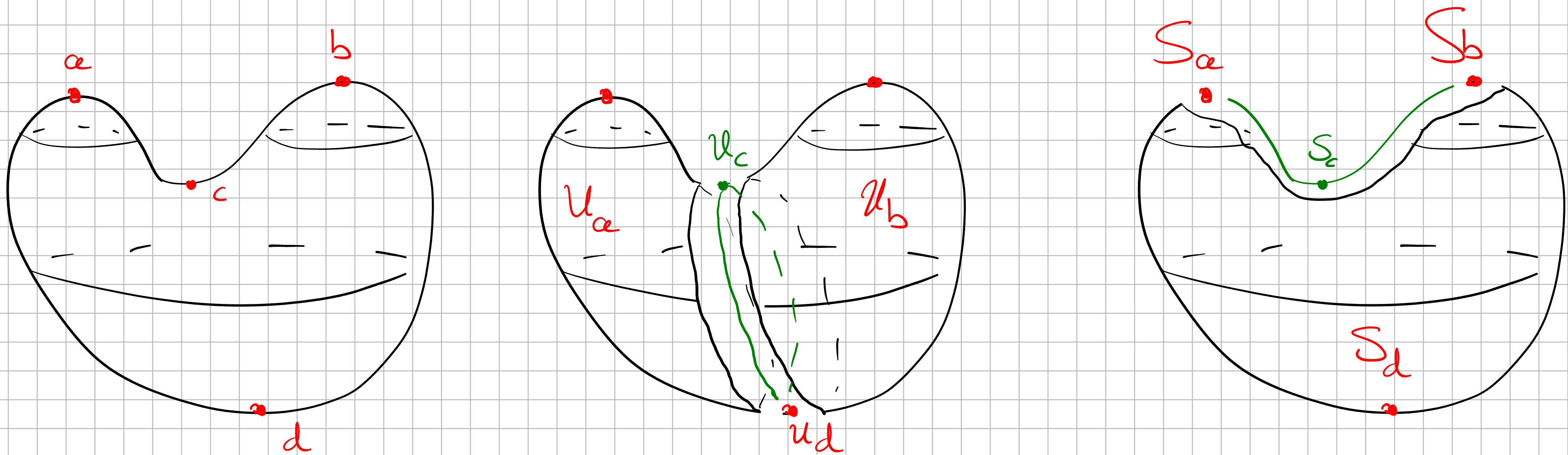

f self-indexed $\Rightarrow$ cellular decomposition

$x, y \in \mathrm{Crit}\, f$

$$\widetilde{\mathcal{M}}(x,y) = \left\{ \gamma : \mathbb{R} \to M \text{ flow line}, \lim_{-\infty} \gamma = x,\ \lim_{+\infty} \gamma = y \right\} \simeq U_x \cap S_y$$

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$\mathcal{M}(x,y) = \widetilde{\mathcal{M}}(x,y)/\mathbb{R}$:

* generically smooth, of dim $i(x) - i(y) - 1$
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<u>Def</u>: $CM_k(M,f) = \bigoplus_{\substack{x \in \mathrm{Crit} f \\ i(x) = k}} \mathbb{Z} \cdot x$

$$\partial : CM_k(M,f) \to CM_{k-1}(M,f)$$
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prop: $\partial^2 = 0$

$\Rightarrow HM_*(M,f) := \dfrac{\ker \partial}{\mathrm{Im}\, \partial}$

$\simeq H_*^{cell}(M, \{\mathcal{U}_x\})$ if f self-indexing

$\simeq H_*(M; \mathbb{Z})$ in general.