BTW3 pretalk 2: Symplectic Geometry

UNIVERSITY OF OXFORD
Mathematical Institute

Guillem Cazassus, January 11 2023

Def: (M,ω) is a symplectic manifold if

- M: smooth mfd,
- $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

$\hookrightarrow d\omega = 0$

$\hookrightarrow \forall x, \ \ker \omega_x = \{0\} \subset T_x M$
$(\omega_x : T_x M \to T_x M^*)$

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<u>Rks</u>: * $\dim M$ even

* M is oriented (by ω^n)

* if M compact, then

$\forall i = 0, \dots, n$, $0 \neq H^{2i}(M;\mathbb{R}) \ni [\omega]^i \neq 0$

(since $[\omega]^n = [\omega]^i \wedge [\omega]^{n-i} \neq 0$)

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canonical 1-form:

$\lambda_{(q,p)} \cdot v = p(d\pi \cdot v)$

with $\pi : T^*Q \to Q$

in coord:
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$$\chi(\Sigma, G) = \mathrm{Hom}(\pi_1 \Sigma, G) / G^{ad}$$ (where smooth)

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G: semi-simple Lie group; $\mathfrak{g}^*$: dual Lie algebra

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* $\mathbb{CP}^n$, smooth projective varieties / $\mathbb{C}$.

* ...

<u>Def</u>: A submfd $S \overset{\iota}{\hookrightarrow} (M,\omega)$ is:

* <u>isotropic</u> if $\iota^*\omega = 0$ i.e. $TS \subset TS^{\perp\omega} = \{ v \in TM \,/\, \omega(v,\xi)=0 \ \forall \xi \in TS \}$

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are the important submfd to look at, and will form the objects of $\mathrm{Fuk}(M)$ (with extra structure)

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* Conormal bundles: $S \subset Q$ smooth submfd

$$N_S^*Q = \left\{ (q,p) \;/\; \begin{array}{l} \cdot\, q \in S \\ \cdot\, p|_{T_qS} = 0 \end{array} \right\} \subset T^*Q$$

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↳ $S = Q \to$ zero section

↳ $S = \{q_0\} \to T^*_{q_0}Q$ cotangent fibre.

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* $f : (M, \omega) \longrightarrow (M', \omega')$ smooth map

$\rightsquigarrow \Gamma(f) = \{(x, f(x))\} \subset \text{"}M^- \times M'\text{"} = (M \times M', -\omega \oplus \omega')$

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* Y^3 compact or. 3-mfd, $\Sigma \overset{\sim}{\underset{\iota}{\longrightarrow}} \partial Y$

$$\chi(Y; G) \xrightarrow{\iota^*} \chi(\Sigma, G)$$

(singular, immersed) Lagrangian.

* Hamiltonian group actions

"moment map"

Def: $G \circlearrowright (M,\omega)$ is <u>Hamiltonian</u> if $\exists\ \mu : M \longrightarrow \mathfrak{g}^*$ G-equivariant s.t. $\forall \xi \in \mathfrak{g}$,

$$X_\xi(m) = \nabla^\omega \langle \mu, \xi \rangle$$

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infinitesimal action:

$$X_\xi(m) = \frac{d}{dt}\Big|_{t=0} \left(e^{t\xi} m\right)$$

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"Weinstein Lagrangian":

$$\Lambda_G(M) := \left\{ \left((g,p), m, m' \right) \,\middle|\, \begin{array}{l} m \in M \\ m' = g m \\ R_{g^{-1}}^* p = \mu(m) \end{array} \right\}$$

$$\subset T^*G \times M^- \times M$$

* ...

Def: An almost-complex structure (acs) on a $\mathbb{R}$-v.b. $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is $J \in \operatorname{End} E \simeq \Gamma\left(\begin{array}{c} E^* \otimes E \\ \downarrow \\ B \end{array} \right)$

s.t. $J^2 = -\mathrm{Id}$.

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- <u>ω-compatible</u> if ω-tame + $\forall v,w$, $\omega(Jv,Jw) = \omega(v,w)$

$\Leftrightarrow g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ J-invariant Riem. metric.

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Th: (Gromov) (M,ω) symplectic. The spaces $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ of ω-tame and ω- compat. acs are nonempty and contractible.

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<u>Rk:</u> * Only depends on $u_*[\Sigma] \in H_2(M;\mathbb{Z})$ (say Σ closed)

(since $d\omega = 0$, Stokes formula ...)

$$E(u) = \|du\|^2_{L^2_{g_J}} \qquad A(u) = \int_\Sigma u^*\omega$$

<u>Proposition</u>: $E(u) = A(u) + \int_{\Sigma} |\bar\partial_J u|^2_{g_J} \, d\mathrm{vol}_{\Sigma}$,

with $\bar\partial_J$ the <u>Cauchy-Riemann</u> operator:

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$\Leftrightarrow \bar\partial_J u = 0$

$\Leftrightarrow du$ is $\mathbb{C}$-linear: $du \circ j = J \circ du$

$$\begin{tikzcd} T_z\Sigma \arrow[r, "du"] \arrow[d, "j"'] & T_{u(z)} M \arrow[d, "J"] \\ T_z\Sigma \arrow[r, "du"] & T_{u(z)} M \end{tikzcd}$$

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* if $\partial\Sigma_1 \neq \emptyset$, usually ask <u>Lagrangian boundary condition</u>

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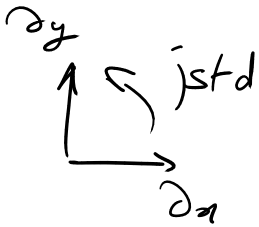
* if $\Sigma_1 \subset (\mathbb{C}, j_{std})$



$$\bar{\partial}_J u = 0 \iff 0 = \frac{\partial u}{\partial x} + J \cdot \frac{\partial u}{\partial y} \quad \left(= \bar{\partial}_J u \cdot \frac{\partial}{\partial x} \right)$$

Th: [Gromov compactness] (More details in Talk 3.1).

Let $u_n : (\Sigma, j_n) \longrightarrow (K, J)$, $K \subset (M, \omega)$ compact

$\partial^i \Sigma \longrightarrow L_i$ $\quad J \in \mathcal{J}(M, \omega)$

$L_1, L_2, \dots \subset M$ Lagrangians.

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then $\{u_n$ converges* to a "stable map" in the Gromov topology$\}$

* up to a subsequence

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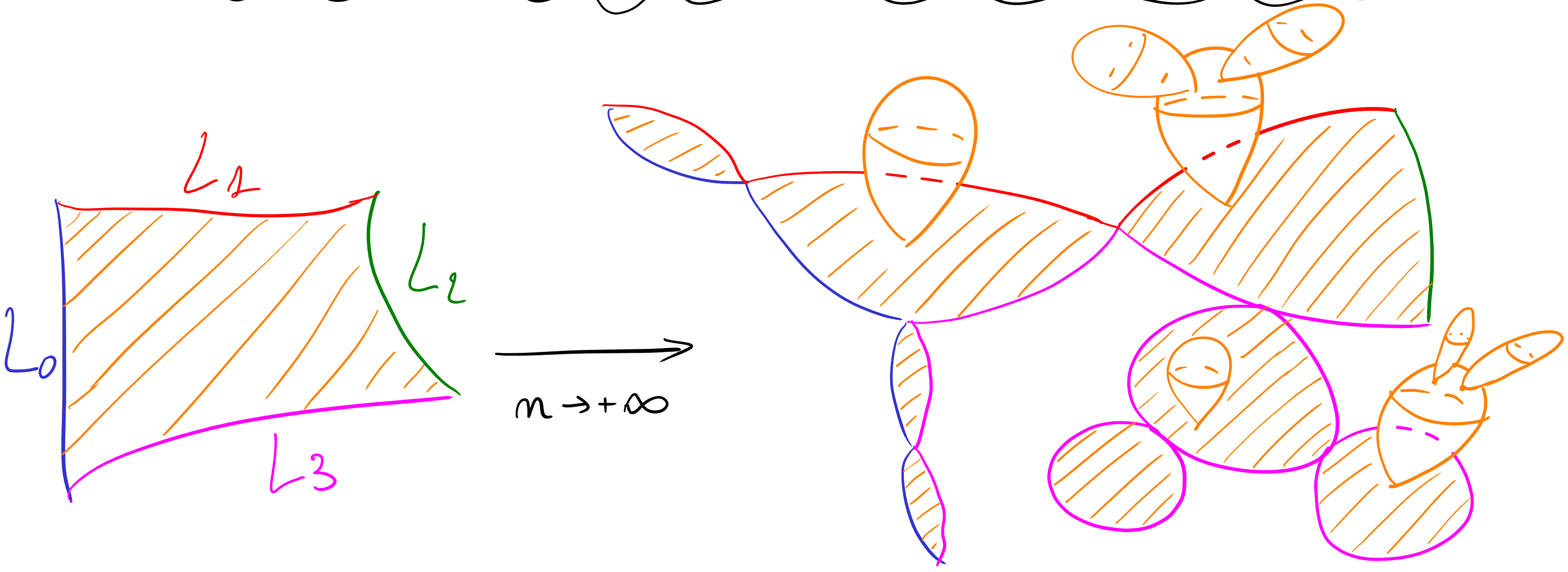
then $\{u_n\}$ converges to a "stable map" in the Gromov topology:



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$\partial^i \Sigma \longrightarrow L_i$ $\qquad J \in \mathcal{J}(M, \omega)$

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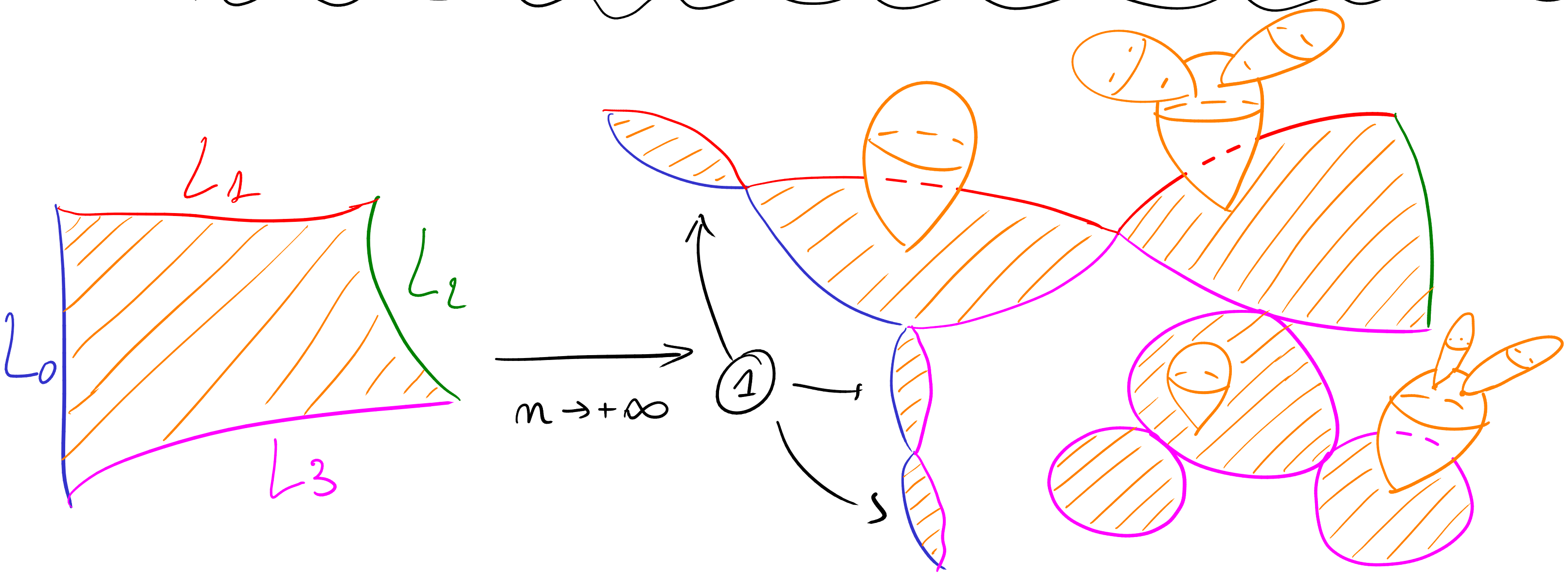

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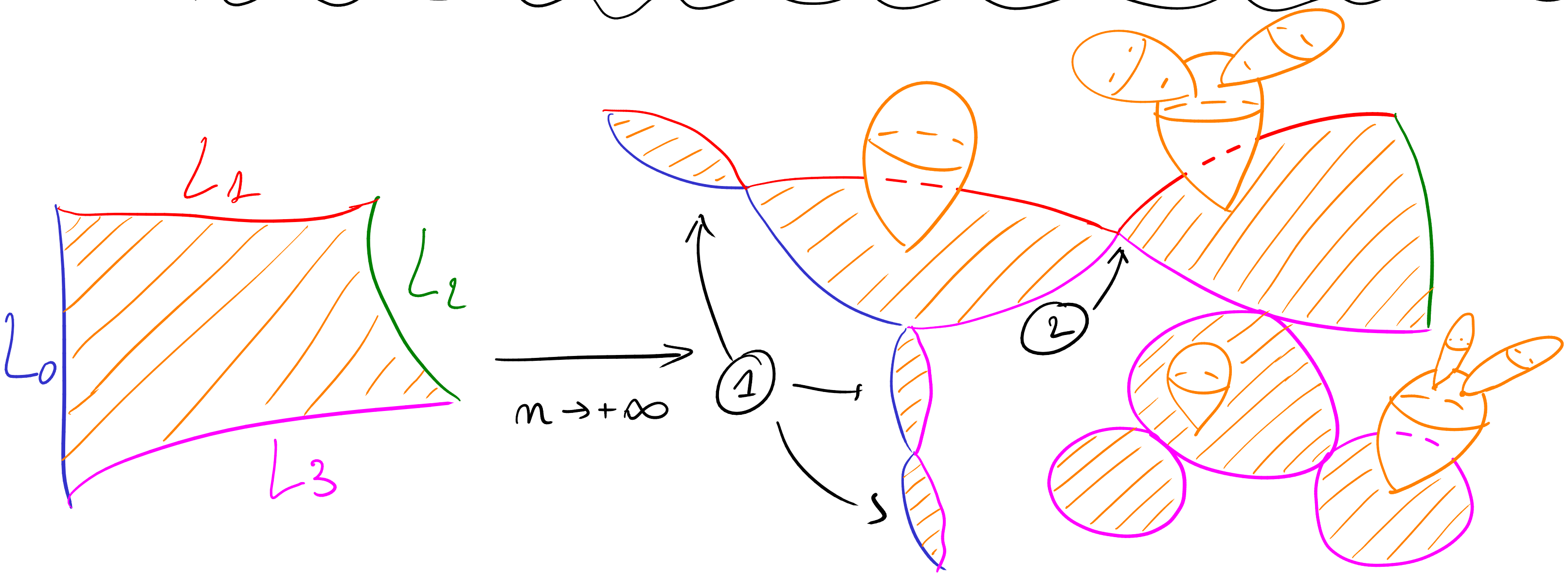

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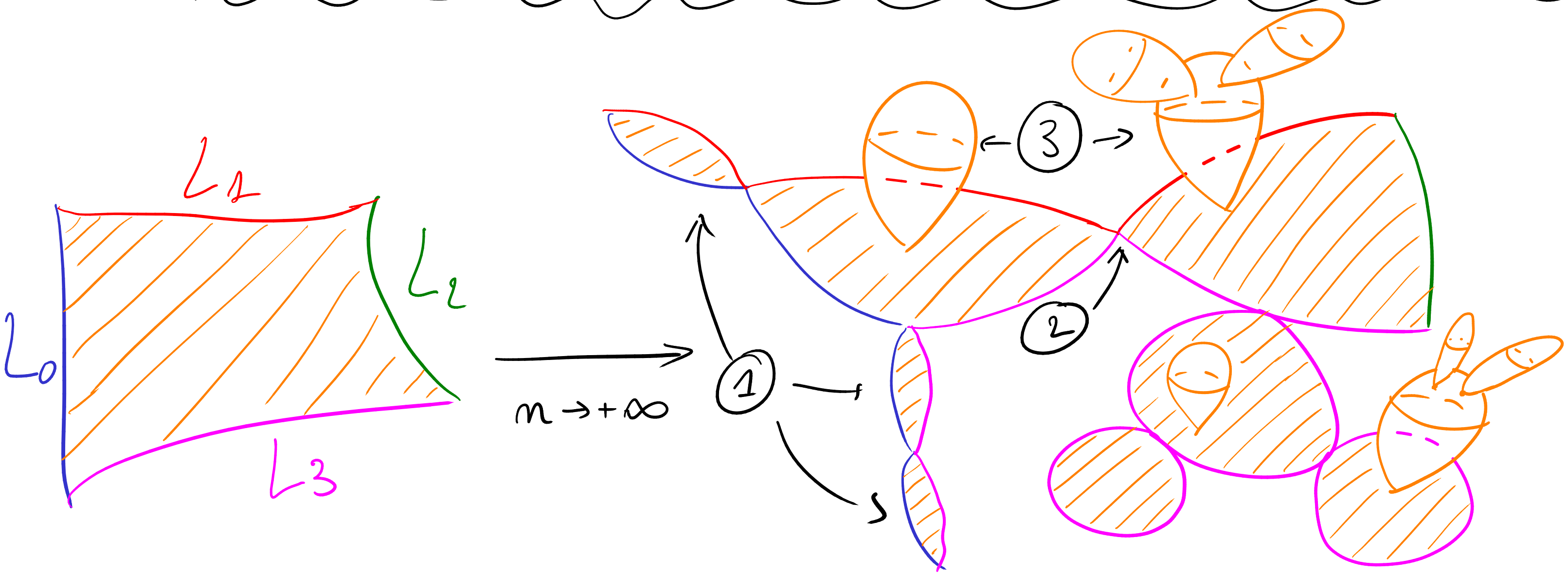

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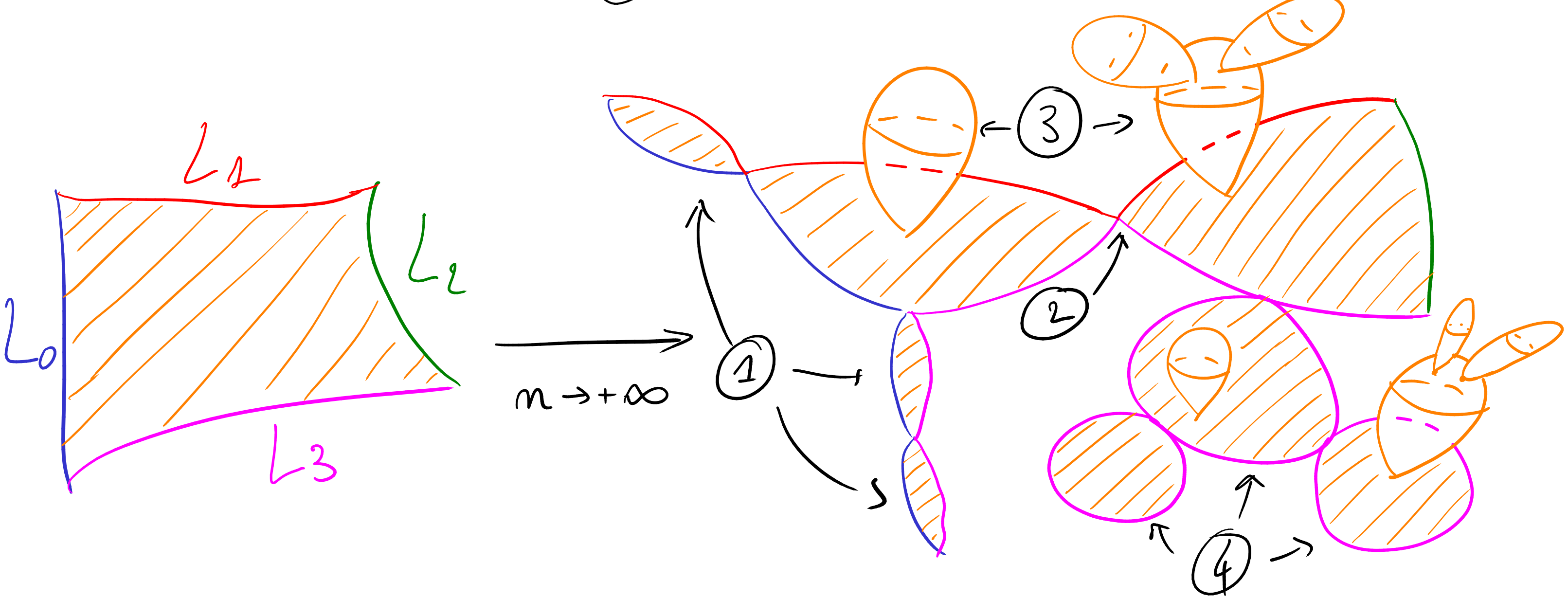

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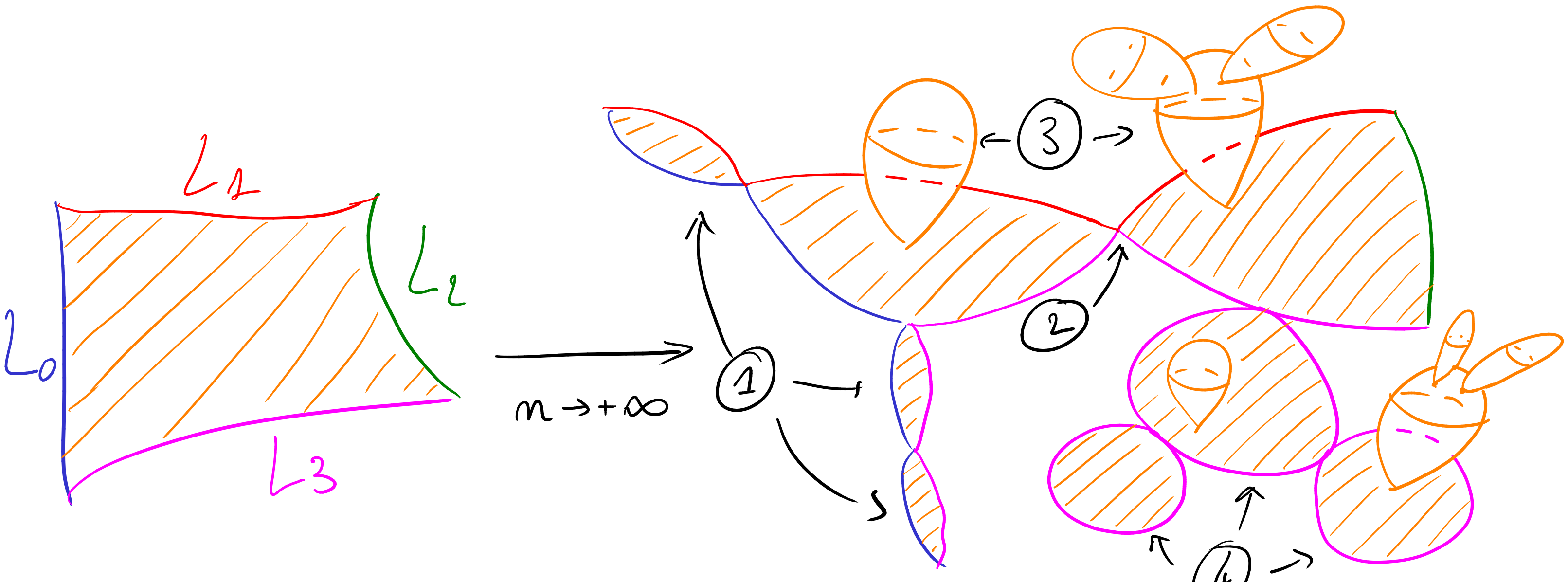

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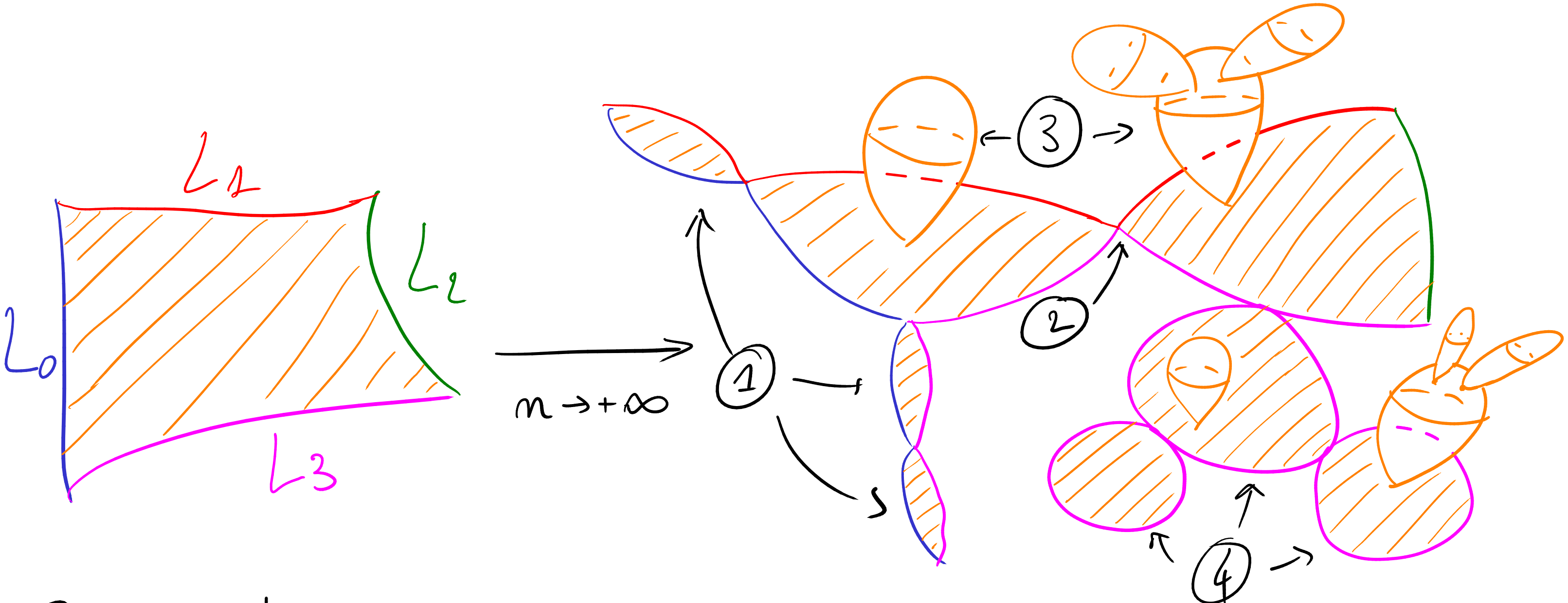
<u>Remarks:</u> * K compact & $E(u_n) \leq C$ are important assumptions



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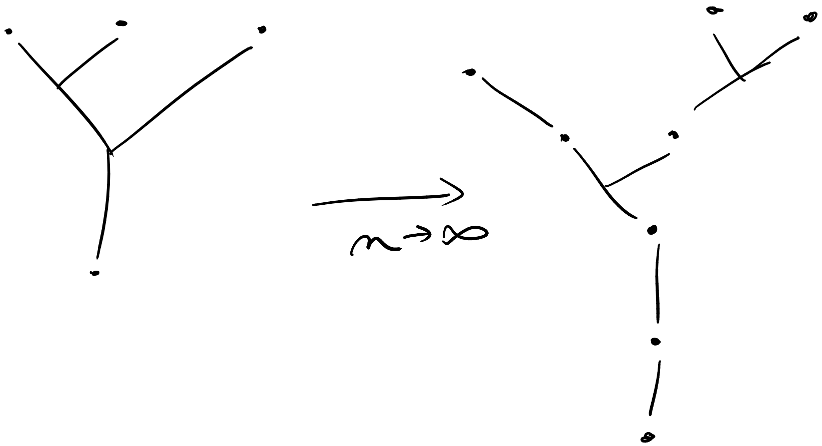
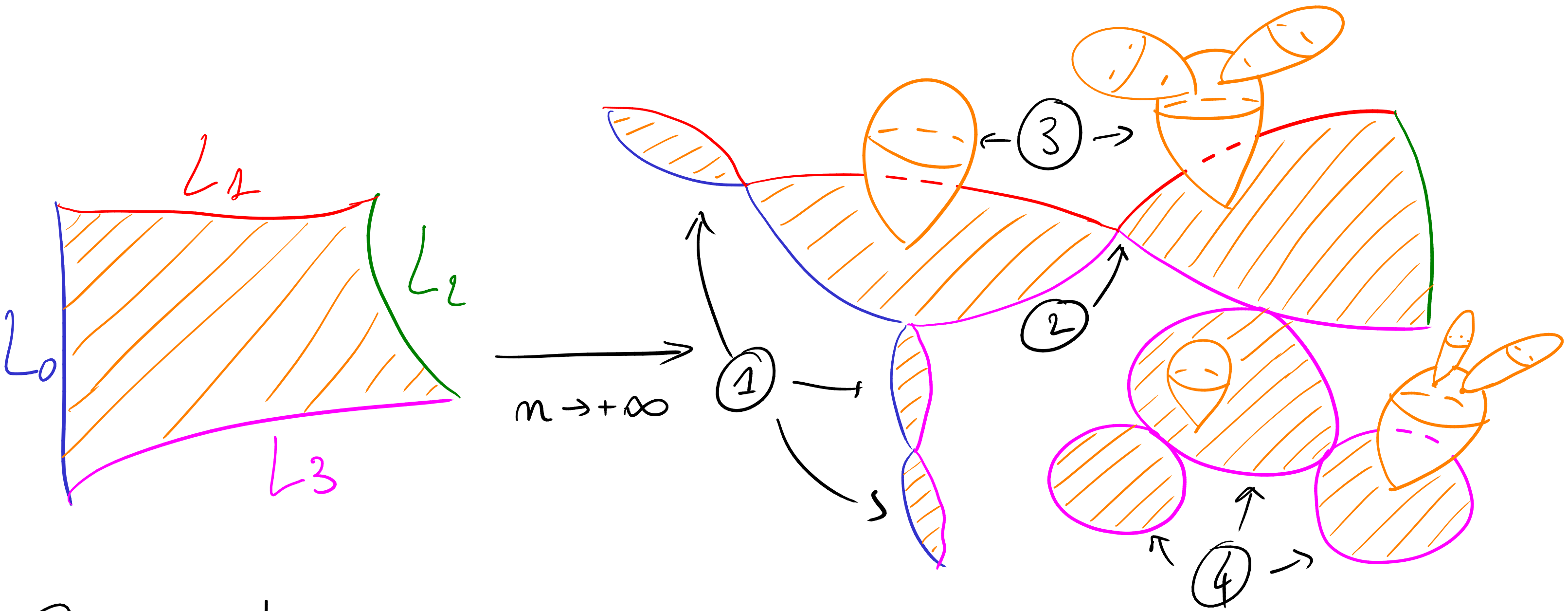
* K compact & $E(u_n) \leq C$ are important assumptions
* degenerations 1 and 2 already happen in Morse theory (see talk 2.5) but 3 and 4 are new features of Floer theory.





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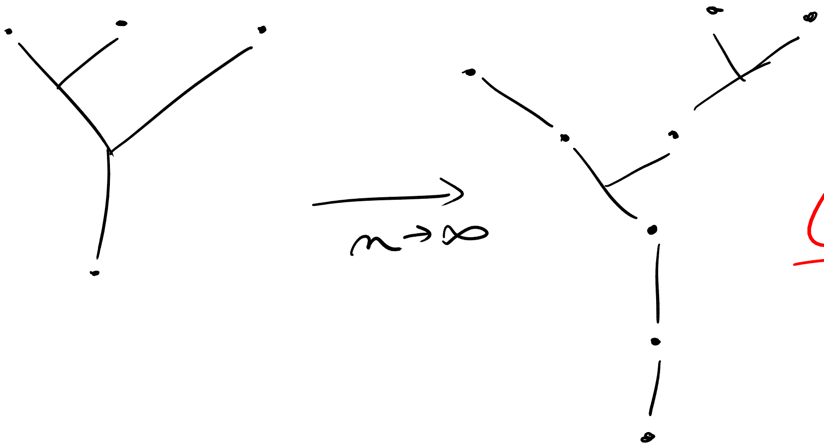

Goal: Put assumptions on M, L, J that:

→ ensure stay in K compact + bounded E

→ rule out 3 and 4

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<u>Ex:</u> $M = T^*Q$, $L = \Gamma(df)$, $L = N_S^* Q$
(graph of exact 1-form) (conormal bundle)

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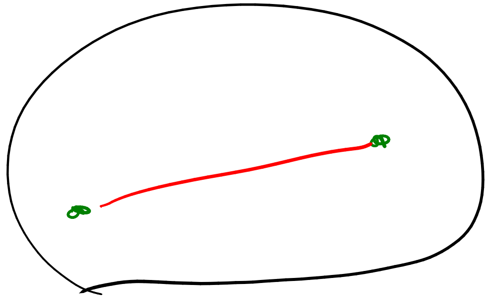
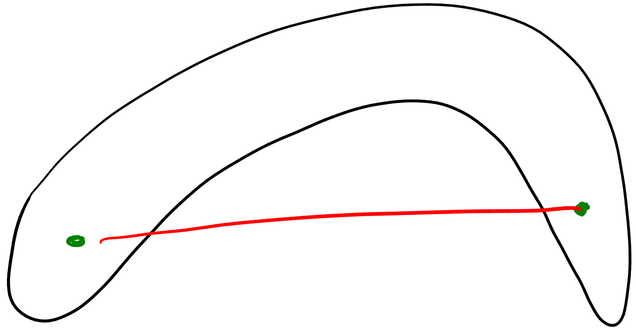
Bad idea: any exact M must be noncompact (otherwise $[\omega] \neq 0 \in H^2(M)$) ⇒ use convexity instead.

Convexity:

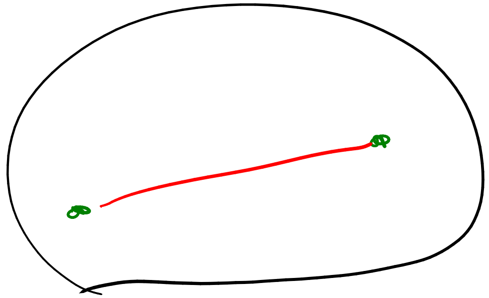


convex

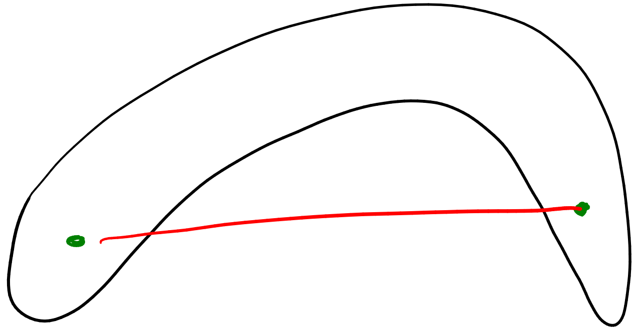


not convex

Convexity:



convex



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⚠ Really depends on which J...

Def $K \subset (M, \omega, J)$ is <u>convex</u> if $\forall u : \Sigma \longrightarrow M$ (compact with boundary) J-holomorphic and s.t. $u(\partial \Sigma) \subset K$, one has $u(\Sigma) \subset K$.

__Def:__ $(X^{2n+1}, \xi = \ker\alpha)$ is a (co-oriented) __contact manifold__ if $\alpha \in \Omega^1(X)$ is s.t. $\alpha \wedge (d\alpha)^n$ is a volume form ($\Leftrightarrow d\alpha|_\xi$ is symplectic)

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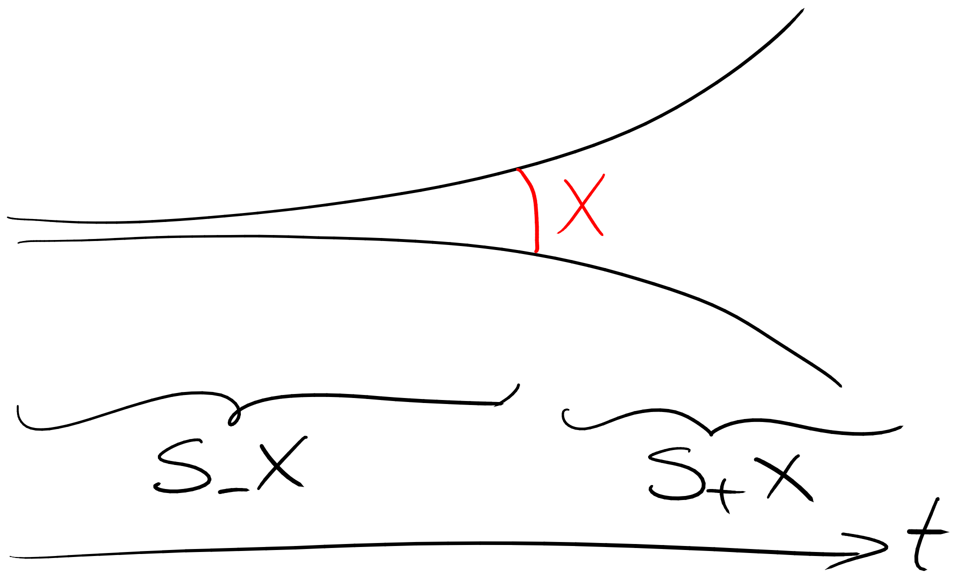

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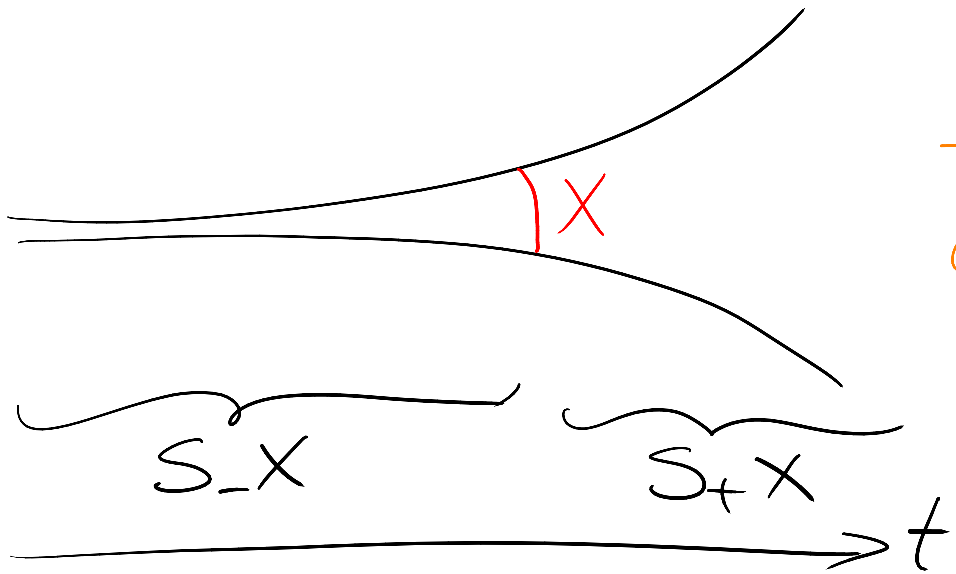

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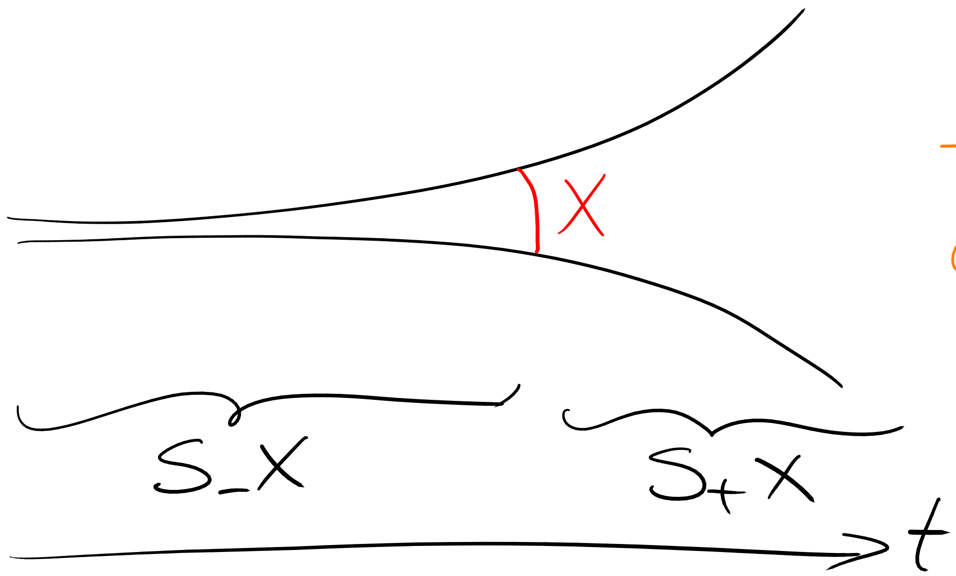

$\rightarrow$ We will model the ends of M on S_+X.

⚠ Still need to find good acs J to work with.

Def: An exact $(M, \omega = d\lambda)$ has convex ends if can find (X, ξ) + $\psi : S_+X \xhookrightarrow[\text{exact}]{} M$ such that $M \setminus \psi(\operatorname{int} S_+X)$ compact

sympl. embedding
$(\psi^*\lambda = e^t\alpha)$

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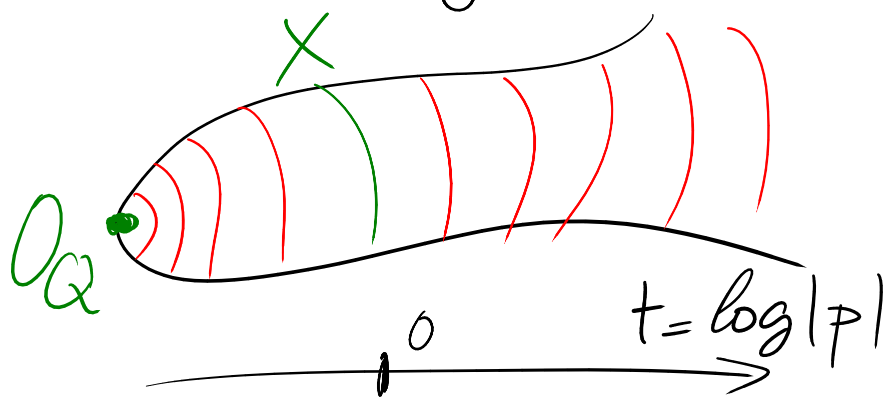
<u>Ex</u>: (Q, g) Riemannian

$\rightsquigarrow (M = T^*Q, \omega = d\lambda, J)$

$X := \{(q,p) \in T^*Q \mid |p| = 1\}$: unit cotangent bdle

$T^*Q \setminus 0_Q \simeq SX \Rightarrow T^*Q$ has convex ends

zero sect.



<u>Def:</u> $J \in \mathcal{J}(M,\omega)$ <u>of contact type</u> at the ends of M if:

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- $dh \circ J = -\lambda$, with $h : \psi(S_+ X) \longrightarrow \mathbb{R}$

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$\hat{\Sigma} := u^{-1}(E) \subset \Sigma$, $\rho = h \circ u$

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "u"] \arrow[dr, "\rho"'] & E \arrow[d, "h"] \\ & \mathbb{R} \end{tikzcd}$$

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Then ρ can't have loc. max, unless locally constant.

proof: $(\Delta\rho) \cdot dvol_{\Sigma'} = -d(d\rho \circ j)$

$= -d(dh \circ J \circ du)$ ⟵ u is J-hol.

$= d(\lambda \circ du)$ ⟵ J is of contact type

$= u^*\omega$ ⟵ $\omega = d\lambda$

≥ 0.

proof: $(\Delta\rho)\cdot dvol_{\Sigma_1} = -d(d\rho\circ j)$

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Maximum principle: local max $\Rightarrow$ locally constant $\square$

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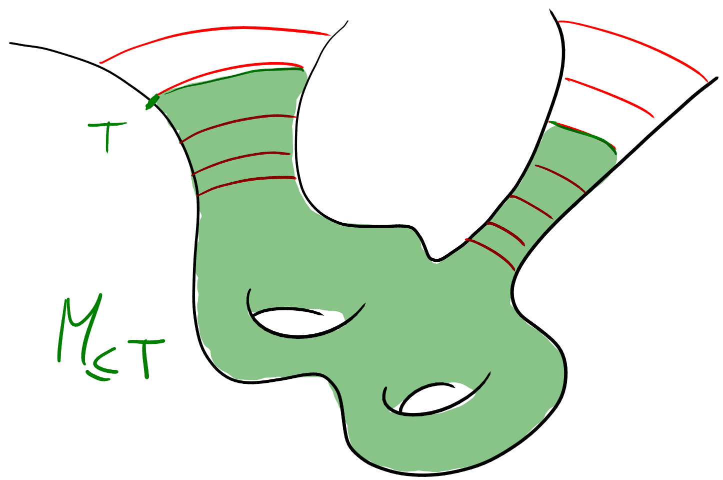
$= u^*\omega$ $\leftarrow$ $\omega = d\lambda$

≥ 0.

Maximum principle: local max $\Rightarrow$ locally constant $\square$

<u>Consequence</u>: The compact subsets $M_{\leq T} := M \setminus \{t > T\}$ are convex for all $T \geq 0$.



Conclusion: The working assumptions will be:

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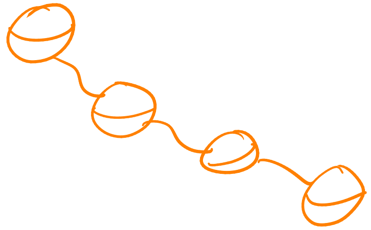
Remark: * can work with other assumptions

* failure of checking the four items doesn't mean it's impossible, but you usually need more sophisticated machinery (→ talk 4.3)

- Sphere bubbling ✗ → include "pearl trajectories" in your moduli spaces



- Disc bubbling ✗ → get curved A_∞-structures, use "bounding cochains" (talk 4.3)

→ failure of transversality
⇒ use "Kuranishi structures" (or "stabilizing divisors", or "polyfolds"...)

- Energy bound ✗ → keep track of the area with a Novikov ring (talk 4.3)

- Stay in K compact ✗ → use "SFT compactness"