Syllabus: M722 Topics in Topology: Symplectic Topology and Fukaya Categories, Spring 2020

Outline:

The goal of this class is to introduce some standard techniques in Floer theory, and focus on Fukaya categories of symplectic manifolds.

We will start by reviewing some basic definitions, results and open questions from contact and symplectic topology. Then in preparation for Floer theory we will review the construction of Morse homology, and the Morse category, a nice and simple toy model for the Fukaya category.

We will then follow Seidel's book (chapters 1 and 2) and introduce the compact Fukaya category in the exact setting.

If time permits, we will introduce the wrapped Fukaya category, corresponding to symplectic manifolds and Lagrangians that are "conical at infinity".

In order to pass, you will have to give a presentation on a topic of your choice, related to this class. A list of suggestions is given below, but you are free to choose other topics. You can work in groups on a same topic, and are encouraged to.

Bibliography:

- McDuff-Salamon, *Introduction to symplectic topology*, 3rd edition
- Hutchings, *Lecture notes on Morse homology (with an eye towards Floer theory and pseudo-holomorphic curves)*
- Fukaya, *Morse homotopy, A_∞-category, and Floer homologies*
- Seidel, *Fukaya categories and Picard-Lefschetz theory*
- Abouzaid, *On the wrapped Fukaya category and based loops*
- Abouzaid, *A geometric criterion for generating the Fukaya category*
- Auroux, *A beginner's introduction to Fukaya categories*
- Smith, *A symplectic prolegomenon*

Suggested topics:

- Biran, Cornea, *Lagrangian Cobordism and Fukaya categories*
- Abouzaid, *On the Fukaya Categories of Higher Genus Surfaces*
- Abouzaid, Kragh *Simple homotopy equivalence of nearby Lagrangians*
- More general approaches, bounding cochains, immersed Floer theory (Akaho-Joyce)
- Joyce, *Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow*
- Contact analogues: dga of a Legrendrian, augmentation categories, augmentation of a Lagrangian filling...
- Something about Homological Mirror Symmetry
- Relations with Instanton homology: the Atiyah-Floer conjecture and the Donaldson category
- Hedden, Herald, Hogancamp, Kirk *The Fukaya category of the pillowcase, traceless character varieties, and Khovanov Cohomology*

- Wehrheim-Woodward's theory: quilts, the geometric composition theorem, A-infinity functors, The symplectic category...
- Auroux, *Fukaya categories of symmetric products and bordered Heegaard-Floer homology*
- Pascaleff, *Poisson geometry, monoidal Fukaya categories, and commutative Floer cohomology rings* (needs quilts)
- Haydys, *Fukaya-Seidel category and gauge theory*

Plan of the course

sympl. of "complex"

I - Intro to ~~sympl~~ / contact topol

I.1 - Physics origins (→ see McDuff - Sal, Chap 1)

I.2 - Sympl. manifolds, Hamiltonian flows

Def: (M, ω) sympl: M: smooth mfld (C^∞), $\omega \in \Omega^2(M)$ s.t.

(i) $d\omega = 0$

(ii) ω non deg: $\forall v \neq 0, \exists w$ s.t. $\omega(v, w) \neq 0$ ⟺ ~~[illegible]~~ $\omega^\#: T_xM \to T_x^*M$ iso ⟺ ω^n volume form

dim $M = 2n$, and

Rk: • dim M even

• in a basis of T_xM, $\omega_x = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \end{bmatrix}$

Ex: * $M = \mathbb{C}^n \simeq \mathbb{R}^{2n}$ $\quad z_j = x_j + i y_j$

$\omega = \sum_j dx_j \wedge dy_j = \sum_j \frac{1}{2i} dz_j \wedge d\bar{z}_j = \mathrm{Im}$ (std. Hermitian inner prod.)

* oriented surfaces with vol. forms

* $M = T^*Q$, q_i: coord' Q, p_i: dual coord. $\quad \omega = \sum_i dq_i \wedge dp_i = -d\underbrace{\left(\sum_i p_i \, dq_i\right)}_{= \lambda_{can}}$

$\pi: T^*Q \to Q$

$\lambda_{can\,(q,p)} \cdot v = p(d\pi \cdot v)$

→ **Def**: M exact if ω exact.

* coadjoint orbits G: Lie group, ⟨,⟩: bi-inv^t scal. prod / g = Lie G

$G \circlearrowright g^* \rightsquigarrow O \subset g^*$: orbit,

$$T_\eta O = \{ ad(\xi)^* \eta \mid \xi \in g \}, \text{ with } ad(\xi) : g \to g, \quad \xi' \mapsto [\xi, \xi']$$

$$\omega_\eta \left(ad(\xi)^* \eta , ad(\xi')^* \eta \right) = \langle \eta, [\xi, \xi'] \rangle$$ symplectic

* Character varieties Σ: closed surf, G + bi-inv^t ⟨,⟩ on g

$M = Hom(\pi_1 \Sigma, G)/_{ad G}$ → $T_{[\rho]} M \simeq H^1(\Sigma, \rho)$ ω: cup prod + ⟨,⟩

$(= \chi(\Sigma, G))$ → $F \hookrightarrow E \downarrow \Sigma$

Def: $\phi : (M_0, \omega_0) \to (M_1, \omega_1)$ symplectomorph:

diffeo + $\phi^* \omega_1 = \omega_0$ ($\Rightarrow \dim M_0 = \dim M_1$)

→ Symp(M, ω): group of symp $M \circlearrowright$

Lie alg: $\mathfrak{X}(M, \omega) \subset \mathfrak{X}(M)$: "symplectic vf": X st. $\iota_X \omega$ closed

$$\Leftrightarrow \mathcal{L}_X \omega = 0$$
$$\underset{\iota_X d\omega + d\iota_X \omega}{\shortparallel}$$

Hamiltonian flows

$H(=H_t) : M \times I \to \mathbb{R}$ smooth fct

→ symplectic gradient: $X_H = \nabla^{\omega} H$ is the dual of dH via $\mathcal{X}(M) \xrightarrow{\sim} \Omega^1(M)$

$$X \longmapsto \iota_X \omega$$

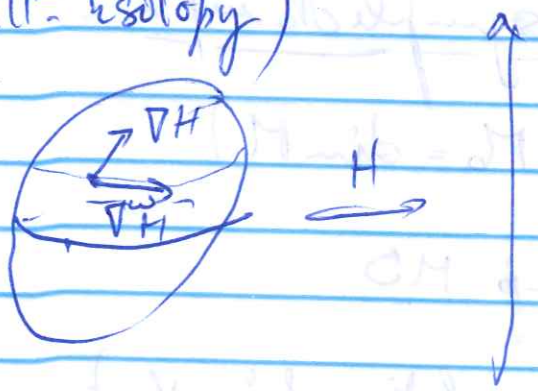
i.e. $\iota_{X_H} \omega = dH$

if X_H is complete (ex. M closed) ⇝ $\phi_H^t \in \mathrm{Diff}\, M$

STOP

"Hamilt. flow" (or Hamilt. isotopy): $\phi_H^0 = id$, $\frac{d}{dt} \phi_H^t = X_H \circ \phi_H^t$

ex:



→ X_H tangent to the level sets.

Rk: $\iota_{X_H} \omega$ exact ⇒ closed ⇒ $X_H \in \mathcal{X}(M, \omega)$

⇒ ϕ_H^t symplecto.

Fact: $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$ normal subgroup.

Rk: need time-dep hamilt, otherwise not true

The Arnold conjecture → see Chap 11 for more ...

$\phi \in \mathrm{Ham}(M,\omega)$, $\mathrm{Fix}\,\phi = \{m \mid \phi(m) = m\}$ **M compact**

- $\#\,\mathrm{Fix}\,\phi \geqslant \min\limits_{f : M \to \mathbb{R}} \#\,\mathrm{Crit} f$
- if the fixed pts are all non deg,

$\#\,\mathrm{Fix}\,\phi \geqslant \min\limits_{f\ \text{morse}} \#\,\mathrm{Crit} f$ $\quad \hookrightarrow \mathrm{Ker}\{D_m\phi - \mathrm{Id}_{T_mM}\} = \{\vec{0}\}$

Compact symplectic mfds

ω^n vol $\Rightarrow [\omega^n] \neq 0$ in $H^{2n}_{dR}(M)$

$\Rightarrow \forall k,\ [\omega]^k \neq 0$ in $H^{2k}_{dR}(M)$

$\Rightarrow \forall k,\ b_{2k} M \neq 0$

$\Rightarrow S^1 \times S^3$ not symplectic (for ex...) (but complex...)

S^n not symp...

$\Rightarrow$ exact symplectic mfds are non compact.

Ex: assume M connected, prove $\mathrm{Ham} \circlearrowright M$ is k-transitive, $\forall k \geq 1$

Ex: (Q, g) Riemannian mfd $\to T^*Q \simeq TQ$

$\rightsquigarrow \underbrace{H(q,p) = \frac{1}{2}|p|^2}_{\text{Kinetic en.}} \; \underbrace{(+ V(q))}_{\text{potential en.}}$

$\rightsquigarrow \phi_H^t$: geodesic flow.

Symplectic volume: $(M^{2n}, \omega) \rightsquigarrow \dfrac{\omega^n}{n!}$ volume form

ex: $M = \mathbb{R}^{2n} \to$ std. vol.

$U \subset M \rightsquigarrow \mathrm{Vol}_\omega U = \int_U \dfrac{\omega^n}{n!}$: ~~invariant by symplecto-morph~~

$\rightsquigarrow$ Symplecto are volume-preserving.

Th. [Darboux] Every (M^{2n}, ω) is loc. symplecto to $(\mathbb{R}^{2n}, \omega_{std})$, i.e. $\forall m \in M, \exists U_m$ nbd, $\varepsilon > 0$ s.t. $(U_m, \omega) \simeq B_\varepsilon \mathbb{R}^{2n}, \omega_{std}$

Proof: "Moser's trick"

Rk: ε matters... $\to$ largest $\varepsilon \simeq$ symplectic width of M: (indep/m) (Gromov)

$w_G(M) = \sup\{\pi r^2 \mid B_r(\mathbb{R}^{2n}) \hookrightarrow M \text{ sympl.}\}$ (see chap 12 McD-S.)

$\to$ "non squeezing..."

1.3 - Lagrangian submfds

$V \subset (E^{2n}, \omega)$ lin. subsp. $V^{\perp} = \{ w \in E \mid \omega(v,w) = 0 \ \forall v \in V \}$

V isotropic : $V \subset V^{\perp}$ $\to \dim V \leq n$
V coiso : $V^{\perp} \subset V$ $\dim V \geq n$
V Lagr : $V = V^{\perp} \Leftrightarrow$ ~~max~~ iso of max dim $\Leftrightarrow$ coiso min dim
$\rightarrow \dim V = n$

Def: $L \subset (M, \omega)$ Lag if $\forall m \in L$, $T_m L \subset (T_m M, \omega_m)$ Lag.

Weinstein's creed: "Everything is Lagrangian"

Ex: • $\mathbb{R}^n \subset \mathbb{R}^{2n}$, $i\mathbb{R}^n$, ...

• $0_Q \subset T^*Q$ zero sec, $T_q^*Q \subset T^*Q$ fiber

• $\alpha \in \Omega^1(Q) \rightsquigarrow \Gamma(\alpha) \subset T^*Q$ graph, $\Gamma(\alpha)$ Lag $\Leftrightarrow d\alpha = 0$

[• $C \subset Q$ submfd, $N_C^*Q = \{ (q,p) \in T^*Q \mid q \in C,\ p|_{T_qC} \equiv 0 \}$
"conormal bdle"

• $f : (M_0, \omega_0) \longrightarrow (M_1, \omega_1)$ smooth diffeo $\rightsquigarrow \Gamma(f) \subset M_0^- \times M_1$
(in partic. $\Delta_M \subset M^- \times M$ ...) "$(M_0 \times M_1, -\omega_0 \oplus \omega_1)$"

$\Gamma(f)$ Lagr $\Leftrightarrow$ f symplecto $\qquad \chi(\Sigma, g)$

- $\partial Y^3 = \Sigma$

$$\rightsquigarrow L(Y) \subset M(\Sigma) \text{ (singular, immersed) Lagr.}$$

G + $\langle , \rangle$ on $\mathfrak{g}$

$$\begin{tikzcd} \Big\{ \rho: \pi_1(\Sigma) \arrow[r] & G \Big\} \\ \pi_1(Y) \arrow[u] \arrow[ur] & \end{tikzcd}$$

- Hamiltonian group actions $\quad G$: Lie group

Def: $G \circlearrowright M$ Hamilt. if $\exists \mu: M \to \mathfrak{g}^*$ G-eq map ("moment map")

s.t.: • if $\xi \in \mathfrak{g} \rightsquigarrow X_\xi \in \mathfrak{X}(M)$: ~~$X_\xi(m) =$~~

$$X_\xi(m) = \frac{d}{dt}\Big|_{t=0} \left(e^{t\xi} \cdot m\right)$$

$$\rightarrow \boxed{X_\xi(m) = \nabla^\omega \langle \mu, \xi \rangle}$$

$$\Leftrightarrow \iota_{X_\xi} \omega = d\langle \mu, \xi \rangle$$

→ encoded by the action Lagrangian (Weinstein)

$$L_G(M) \subset T^*G \times M^- \times M$$

$$\| \quad \{ ((g,p), m, m') \mid m' = gm, \; R^*_{g^{-1}} p = \mu(m) \}$$

④

(ex: $G \circlearrowright Q \rightsquigarrow G \circlearrowright T^*Q$, and $\mu : T^*Q \to \mathfrak{g}^*$ def by

$$\langle \mu(q,p), \xi \rangle = \textcircled{\pm}\, p(X_\xi, q)$$)

↳ conventions... probably ⊕

Symplectic quotient (Marsden-Weinstein)

$G \circlearrowright M \xrightarrow{\mu} \mathfrak{g}^* \rightarrow M/_G$ not sympl. in G^{al} (ex dim G odd...)

Assume $G \circlearrowright M$ is "regular": $\cdot\ 0 \in \operatorname{Reg}\mu$

$\cdot\ G \circlearrowright \mu^{-1}(0)$ free

$\rightarrow M/\!\!/_G = \mu^{-1}(0)/_G$ symplectic ($\omega|_{\mu^{-1}(0)}$ descends to a sympl. form)

$\rightsquigarrow \mu^{-1}(0) \xhookrightarrow{i \times \pi} \bar{M} \times M/\!\!/_G$ is Lagr. "Canonical Lagr",

Ex: $\mathbb{CP}^N = \mathbb{C}^{N+1}/\!\!/_{S^1}$, get ω_{FS} this way.

Def: [Weinstein's "Category"] "Symp"

- ob: (M, ω)
- mph: "Lag corr" $L : M \to M'$: $L \subset M^- \times M'$

compo:
$$\begin{tikzcd} M_0 \arrow[r, "L_{01}"] \arrow[rr, bend right] & M_1 \arrow[r, "L_{12}"] & M_2 \end{tikzcd}$$

$$L_{01} \circ L_{12} = L_{02} = \pi_{02}(L_{01} \times M_2 \cap M_0 \times L_{12})$$
$$= \left\{ (m_0, m_2) \,/\, \exists\, m_1 \in M_1 : \begin{array}{l} (m_0, m_1) \in L_{01} \\ (m_1, m_2) \in L_{12} \end{array} \right\}^{\overline{\ell}}$$

Th: (Guillemin–Sternberg) if $\phi \circ \pi_{02}|_{\Delta}$ emb. then L_{02} is Lagr.

Rk: (Wehrheim - Woodward + Mau + Bottman + ...) Symp can be completed to a cat. and should be seen as an "$(A_\infty, 2)$-category", with $\hom'(M, M') = \mathrm{Fuk}(M^- \times M')$ (→ Bottman + collab.)

in partic, $L : M \to M' \rightsquigarrow \Phi_L : \mathrm{Fuk}\, M \to \mathrm{Fuk}\, M'$ A_∞-functor (Mau - W - W ...)

Th: [Weinstein's neighborhood th]

$L \subset M$ compact lag. Then $\exists\, \mathcal{U}L \subset M$ nbd st

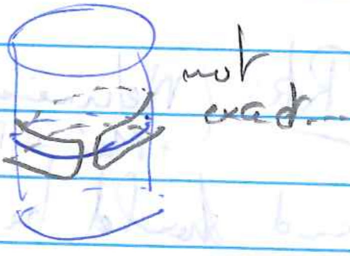
$$\mathcal{U}L \xrightarrow[\phi]{\sim} D^*_\varepsilon L \subset T^*L, \text{ with } \phi|_L = \mathrm{id}.$$

(symplecto) nbd of 0-sec

→ what are the nearby lag. of a given $L \subset M$?

Conj: [Nearby Lag. conj]

Q: closed smooth
$L \subset T^*Q$ closed exact $\Big) \Rightarrow L = \phi^1_H(0_Q)$



partial result:

Th [Abouzaid-Kragh]

——— $\Big) \Rightarrow L \to Q$ homotopy equiv.

Proof involves Fukaya categories... → idea for a topic...

1-4 – Almost complex structures, pseudo-hol. curves

→ Gromov, 85' "Pseudo-hol curves in sympl. mfds"

Def: • Almost-cplex mfd: M + structure of a $\mathbb{C}$-v.b on TM

($\Rightarrow$ dim M even)

ex: M complex $\Rightarrow$ M almost-cpx : "integrable a.c.m"

Def: • Almost complex str on M (or any v.b/M...) is $J : TM \circlearrowleft$ & $J^2 = -\mathrm{Id}$.

• (M,ω) sympl, an a.c.s J is ω-~~compat~~ tame if $\forall v \neq 0$, $\omega(v, Jv) > 0$

• J is ω-compat if ω-tame + $\omega(Jv, Jw) = \omega(v,w)$

→ $\mathcal{J}(M)$: set of a.cs.

$\mathcal{J}_\tau(M,\omega)$: ___ ω-tame a.cs.

$\mathcal{J}(M,\omega)$: ___ ω-compat a.cs.

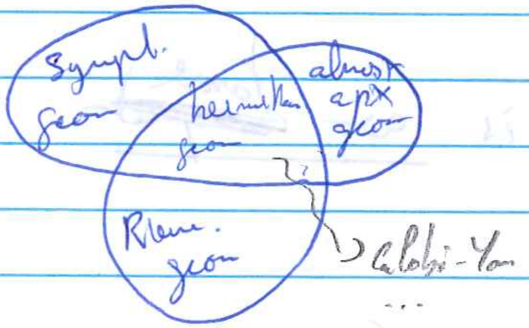
Rk: J a.c.s., J ω-~~compat~~ tame $\Leftrightarrow$ $g_J = \omega(\cdot, J\cdot)$ ~~compat~~ Riem. metric

J ω-compat $\Leftrightarrow$ g_J J-inv^t Riem met.

Rk: $\omega, J \rightsquigarrow g = \omega(\cdot, J\cdot)$

$g, J \rightsquigarrow \omega = g(J\cdot, \cdot)$

$\omega, g \rightsquigarrow J =$ see prop 2.5.6 McD-S



in $GL(2n, \mathbb{R})$:

$GL(n,\mathbb{C}) \cap O(2n) = GL(n,\mathbb{C}) \cap Sp(2n)$
$= Sp(2n) \cap O(2n) = U(n)$

Ex: * $M = \mathbb{C}^n$, $J = i \times$.

* $M = \mathbb{C}$, fix $m = 0 \in M$

$J_m \overset{1:1}{\longleftrightarrow} J_m \cdot \frac{\partial}{\partial x}$ (since $J^2 = -Id$ ...)

"$\mathcal{J}_m$" $\xrightarrow{\sim} \{z \mid \mathrm{Im}\, z \neq 0\}$

$\mathcal{J}(M,\omega)_m \simeq \mathcal{J}(M,\omega)_m \xrightarrow{\sim} \{z \mid \mathrm{Im}\, z > 0\}$

In G^{al}:

Th: $\boxed{\text{Gromov}}$ ω non-deg 2-form (need not be closed)

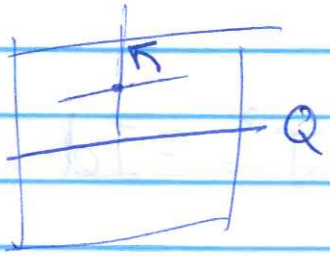
$\mathcal{J}(M,\omega)$ and $\mathcal{J}_\tau(M,\omega)$ are non-empty and contractible

(will be very important for later) → topic?

Conseq: $c_1(M,\omega) = c_1(TM,J) \in H^2(M,\mathbb{Z})$, is well-def (= indep. on $J \in \mathcal{J}(M,\omega)$)

Other examples $\ast$ $Q + g$: Riem. metric $\rightsquigarrow T^*Q \simeq TQ$

Levi-Civita conx° $\Rightarrow$ $T_{(q,v)} \simeq \underbrace{T_q Q}_{\text{horiz}} \oplus \underbrace{T_q Q}_{\text{vertic.}}$



$\hookrightarrow$ take $J = \left(\begin{array}{c|c} 0 & -Id \\ \hline Id & 0 \end{array}\right)$

$\ast$ Σ closed surf. $\rightsquigarrow \chi(\Sigma, G) \simeq \{\text{flat } G\text{-bdles}\} / \mathcal{G}_{gauge} = \mathcal{M}(\Sigma)$

g: metric$/\Sigma \rightsquigarrow$ a.c.s$/ \mathcal{M}(\Sigma)$ (Hodge star operator)

(only depends on the conformal class)

Symplectic vs Riemannian gradient $\quad J$: ω-compat

$$H \rightsquigarrow \nabla^\omega H : dH = \omega(\nabla^\omega_H, \cdot)$$

$$\rightsquigarrow \nabla^{g_J} H : dH = g_J(\nabla^{g_J} H, \cdot) = \omega(\nabla^{g_J} H, J \cdot)$$

$$= \omega(-J \nabla^{g_J} H, \cdot)$$

$$\Rightarrow \boxed{\nabla^{g_J} H = J \cdot \nabla^\omega_H}$$

cf McDuff-Sal: J-hol. curves and sympl. topol.

Pseudo-hol. curves: (Σ, j) ~~surface~~ Riemann surface

(in $\dim_{\mathbb{C}} = 1$, complex $\Leftrightarrow$ almost complex)

Def: (M, J) almost-cpx, (Σ, j) Riem. surf.

a pseudo-hol (or (j,J)-holom) curve $u : (\Sigma, j) \to (M, J)$ is a map such that du is $\mathbb{C}$-linear

$$\Leftrightarrow \quad \begin{tikzcd} T_z\Sigma \arrow[r, "j_z"] \arrow[d, "du_z"'] \arrow[dr, phantom, "\circlearrowright"] & T_z\Sigma \arrow[d, "du_z"] \\ T_{u(z)}M \arrow[r, "J_{u(z)}"'] & T_{u(z)}M \end{tikzcd} \qquad du \circ j = J \circ du$$

$\Leftrightarrow \bar\partial_J u = 0$, with $\bar\partial_J(u) = \frac{1}{2}\left(du + J \circ du \circ j\right)$

$\bar\partial_J$ (or just $\bar\partial$): Cauchy-Riemann operator.

Rk: u smooth $\rightsquigarrow A_\omega(u) = \int_\Sigma u^*\omega$: "symplectic area"

$(\Sigma, j, dvol_\Sigma)$ Riem. surf. $\rightsquigarrow E_{g_J}(u) = \frac{1}{2}\int_\Sigma |du|^2_{g_J} \, dvol_\Sigma$ "Energy"

$\left(|L|_{g_J} = |\xi|^{-1}\sqrt{|L(\xi)|^2 + |L(j\xi)|^2} \right)$ = square of L^2-norm

indep^t on $\xi \neq 0 \in T_z\Sigma$

prop: $E_{g_J}(u) = A_\omega(u) + \int_\Sigma \left|\bar{\partial}_J u\right|^2_{g_J} \, d\mathrm{vol}_{\Sigma}$

Rk:

→ J-hol curves are the curves of minimal energy (= Area).

→ very important tools in ST

→ appear in finite dimension

→ under bounded energy assumption, have some compactness properties (Gromov compactness Th) → later.

Th: [Gromov, non-squeezing Th] aka symplectic camel

$Z(r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n / |z_1| < r\}$

$B(R) \xhookrightarrow[\text{sympl.}]{} Z(r) \to R \leq r$ i.e. $w_G(Z(r)) = \pi r^2$

Proof: → topic?
→ ask vinicius

Rk: explains the uncertainty principle in QM → Topic?

Rk: If Σ has boundary, usually impose l.b.c

$$(\Sigma, \partial\Sigma) \longrightarrow (M, L)$$

(Wehrheim-Woodward)

Quilts: Def: $\underline{S}$: quilted surface:

S : Riem. surf + $\mathcal{S} = \{ s \subset S$ real analytic curves$\}$ "seams"

ex:

$\rightsquigarrow \mathcal{P}$ = closures of connected comp. of $S \setminus$ seams "patches"

· decoration of $\underline{S}$: $(\underline{M}, \underline{L})$: $P \in \mathcal{P} \rightsquigarrow M_P$

$\sigma \in \mathcal{S} \rightsquigarrow L_\sigma \subset M_P^- \times M_{P'}$

if $\frac{P}{P'}\Big| \sigma$

· pseudo-hol quilt: $\underline{u} : \underline{S} \longrightarrow (\underline{M}, \underline{L})$ = coll. of $u_P : P \to M_P$

pseudo-hol + seam cond: if $x \in \sigma$

$(u_P(x), u_{P'}(x)) \in L_\sigma$

1.5 - Contact mfds

can find α globally def...

Def: A (co-oriented) contact structure on M^{2n+1} is a "maximally non-integrable" hyperplane field $\xi \subset TM$, i.e. $\xi = \ker \alpha$, with $\alpha \wedge (d\alpha)^n$ volume form/M

α: contact form

but volume depends on α...

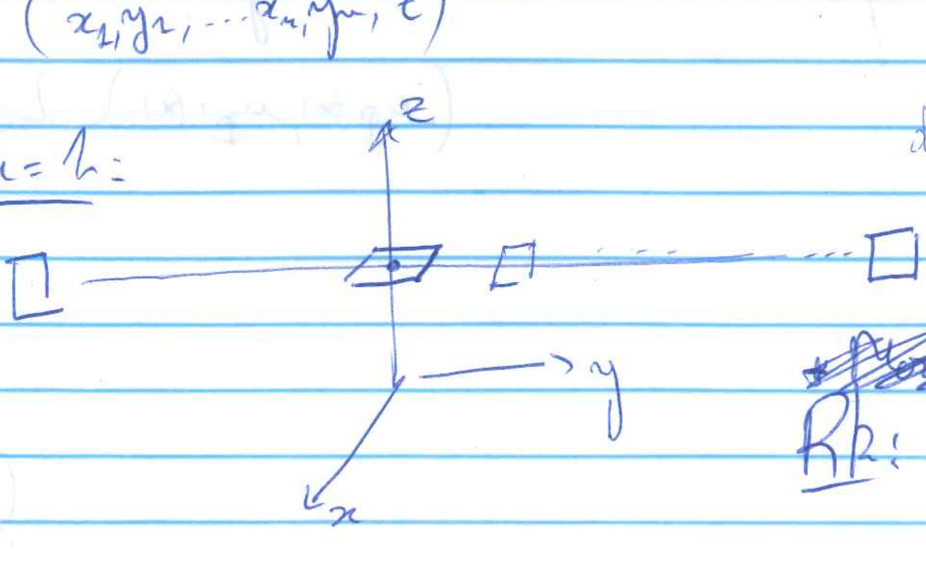
Rk: * $\alpha \wedge (d\alpha)^n$ volume form $\Leftrightarrow$ $d\alpha|_{\xi}$ symplectic

* ξ foliation $\Leftrightarrow$ $d\alpha|_{\xi} = 0$ ...

* α contact form + f nevervanishing $\rightsquigarrow$ $f \cdot \alpha$ other contact form for ξ

Ex: $\left(\underset{(x_1, y_1, \dots x_n, y_n, z)}{\overset{\cup}{\mathbb{R}^{2n+1}}}, \xi_{std} = \ker \alpha_{std}\right)$ $\alpha_{std} = \sum y_i dx_i - dz$

$n=1$:



$dz = y\,dx$

~~* More globally~~

Rk: $\mathbb{R}^{2n+1} \simeq J^1(\mathbb{R}^n)$.

* 1-jet space $J^1 Q = \underset{(q,p),\ z}{T^*Q \times \mathbb{R}}$ ⇝ $\alpha = \lambda_{can} - dz$

$\cup$

* Q + g riem. metric → $M = (T^*Q) \subset T^*Q$ "unit sphere bdle"

$\overset{\shortparallel}{\{(q,p) / |p| = 1\}}$

$$\alpha = \lambda_{can}|_M$$

Def: Reeb v.f. : α : contact 1-form ⇝ $R_\alpha \in \mathfrak{X}(M)$ s.t.

- $\alpha(R_\alpha) = 1$
- $\iota_{R_\alpha} d\alpha = 0$ (contact cond ⇒ $\ker d\alpha$ is 1-dim)

Rk: • depends on α, not just $\xi = \ker\alpha$

* in $\mathbb{R}^{2n+1}$ and J^1M, with std. α, $R_\alpha = \frac{\partial}{\partial z}$

* $\mathcal{L}_{R_\alpha} \alpha = 0$ ⇒ Reeb flow preserves α (and therefore ξ)

Def: $f : (M, \underset{\ker\alpha}{\xi}) \to (N, \underset{\ker\alpha'}{\xi'})$ contactomorph. if $f_*\xi = \xi'$

⟺ $\exists\, g$ non-vanishing / M : $f^*\alpha' = g \cdot \alpha$

Darboux th: every (M,ξ) loc. contacto to $(\mathbb{R}^{2n+1}_{std})$.

Def: $L \subset (M,\xi)$ integral submfd if $TL \subset \xi$

$\Rightarrow \dim L \leq n$.

L is Legendrian if furthermore $\dim L = n$.

Ex: * $f : Q \to \mathbb{R} \rightsquigarrow j^1 f = \{(q, d_q f, f(q)) \; q \in Q\} \subset j^1 Q$

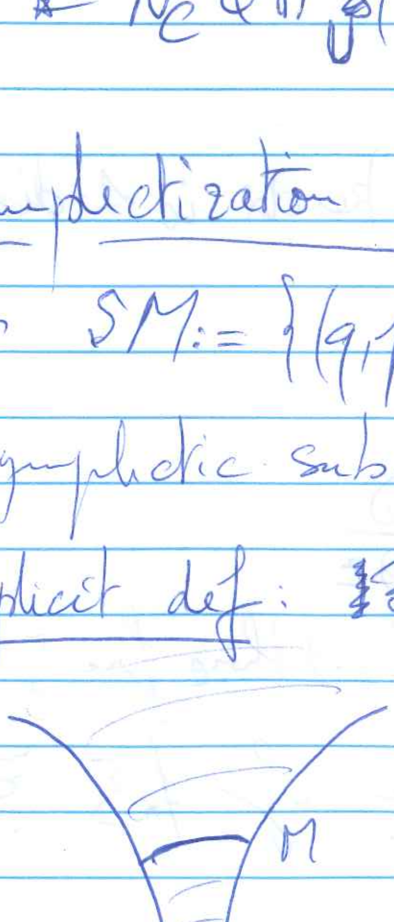
* $N^*_C Q \cap S(T^*Q)$ unit conormal.

Symplectization $(M, \xi = \ker\alpha)$ contact

$\rightsquigarrow SM := \{(q,p) \,/\, q \in M, \; p|_{\xi_q} = 0, \; \alpha(v) > 0 \Rightarrow p(v) > 0\}$

symplectic submfd of T^*M

explicit def: ~~SM~~ fix α: $SM \simeq (M \times \underset{\substack{\in \\ t}}{\mathbb{R}}, -d(e^t \alpha))$



Rk $\underbrace{\mathbb{R}_- \times M}_{Vol < \infty} \neq \underbrace{\mathbb{R}_+ \times M}_{Vol = \infty}$

Fact: * $T^*Q \setminus O_Q \simeq S(U(T^*Q))$

* $L \subset (M,\xi)$ Leg $\rightsquigarrow \mathbb{R}\times L \in SM$ Lagrangian.

⟹ State Courte's result symplecto symplectizat...

Liouville v.f, contact hypersurf

Def: • (M,ω) sympl $X \in \mathfrak{X}(M)$ is a Liouville vf if

$\mathcal{L}_X \omega = \omega$ (⟹ flow satisf $\phi^{t*}\omega = e^t\omega$)

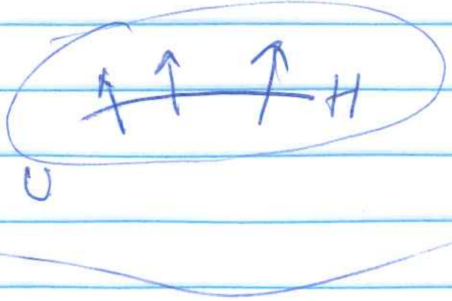
ex: • $SM = \mathbb{R}\times M$, ~~X~~ $X = \frac{\partial}{\partial t}$ Liouville (corresp. to X_{can}...)

• $M = T^*Q$, $X_{can}(q,p) = p$

• $H \subset (M,\omega)$ compact hypersurf is of contact type if

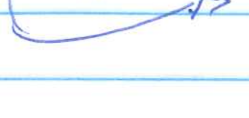
(i) ∃ α ctact form/H st $-d\alpha = \omega|_H$

⟺ (ii) ∃ $U \supset H$ nbd, $X \in \mathfrak{X}(U)$ Liouville vf $\pitchfork H$



• (M,ω) with ∂, M has cvx ∂ if

∃X Liouville near ∂M & outward pointing

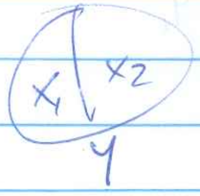


- (M, ω, X) is a Liouville domain if X: Liouville v.f. outward pointing..

1.6 - From Instanton Theory to Donaldson's cat

→ Topic

- X^4 closed smooth or. ⇝ $\mathcal{M}_{ASD}(X,g)$ "anti-self dual conn's"
  + g Riem.
  ⇝ $D_X \in \mathrm{Sym}\, H^2(X,\mathbb{Z})$ "Donaldson pol."

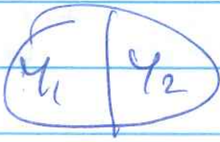
- 

  Y^3 ⇝ $I_*(Y,g)$ "Instanton homol."
  + g
  X_i ⇝ $D_{X_i} \in \mathrm{Sym}\, H^2(X_i;\mathbb{Z}) \otimes I_*(Y)$

  } ≃ TQFT $\mathrm{Cob}_{3+1} \to \mathrm{Vect}$

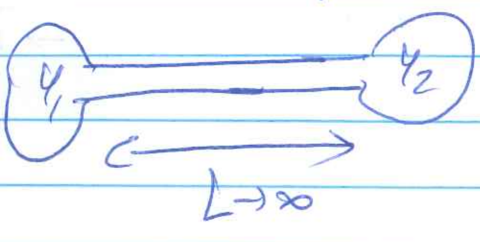
  Relative Don. Polyn.

Atiyah-Floer conjecture : $Y =$



Heegaard splitting

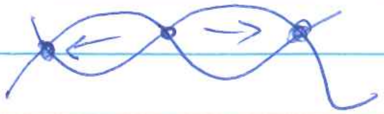
"neck stretch"



$\mathcal{M}(\Sigma)$: SU(2) - char. var. (or flat conn's)
∪
L_i : extend flatly to Y_i

$\Rightarrow I_*(Y) \simeq HF(L_0, L_1)$ "conj" :

- chain cpx gen by $L_1 \cap L_2$
- ∂ counts J-hol discs :

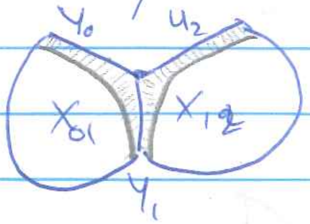


"Lagrangian Floer homol"

Relative TQFT: Σ surf, let Cob^{Σ}_{3+1}:

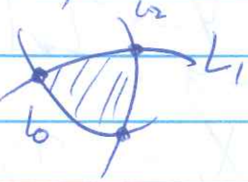
- ob: Y^3, $\partial Y = \Sigma$
- $X : Y_0 \to Y_1$:

- Compo:



⇝ expect conjecturally $Cob^{\Sigma}_{3+1} \to Don(\mathcal{M}(\Sigma))$

- ob: $L \subset M$
- $Hom(L_0, L_1) = HF(L_0, L_1)$
- compo counts



Fukaya: work at chain level ⇝ get an A_∞-cat instead ...

Ref: · Braam-Donaldson: Floer's work on instanton homol, knots & surgery (⇝ including RK)

· Seidel, A LES for symplectic Floer cohomol.
→ Topic?

Chap 2 - Morse homol. & the Morse categ.

2.1 - Morse fcts

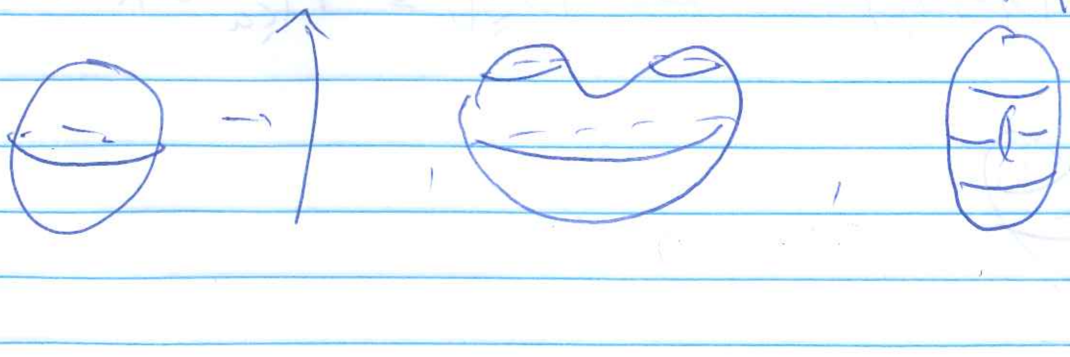
Def: $f: M \to \mathbb{R}$ is Morse if all its c.p. are nondeg, i.e. $\text{Hess}_x f$ nondeg. $k = \text{ind}\, x = \#\{\text{neg. eigenval. of } \text{Hess}_x f\}$

"Morse index"

Lemma: [Morse lemma] near a c.p, ∃ a chart in which in loc. coord. $f(x_1, \dots x_n) = -x_1^2 - \dots x_k^2 + x_{k+1}^2 + \dots + x_n^2 + f(x_0)$.

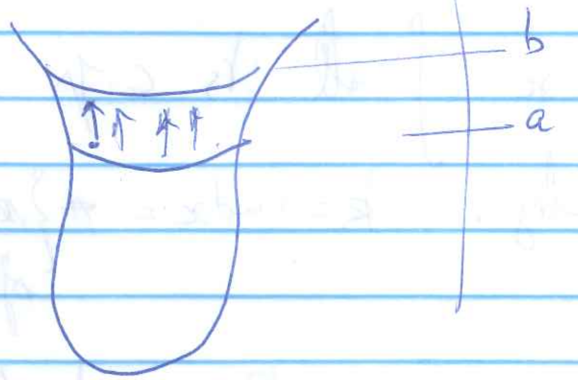
Idea: study M by looking at $M_{\leq a} = f^{-1}((-\infty, a])$

ex:



prop: if $[a,b]$ contains no critical values, then
$M_{\leq a} \underset{diff}{\cong} M_{\leq b}$

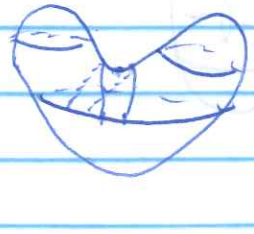
proof:



fix Riem. metric,
take $X = \frac{\nabla f}{|\nabla f|^2}$ ~~$\cdot \varepsilon$~~
on $f^{-1}([a,b])$, extend smoothly...

Ex: (Reeb's thm) assume ~~f has 2~~ cr. $f: M \to \mathbb{R}$
has only 2 crit pts, M compact, prove M homeo to a
sphere.

prop: if $f^{-1}([a,b])$ contains exactly 1 crit of index k
(outside $f^{-1}(\{a,b\})$), then $M_{\leq b} \simeq M_{\leq a} \cup k$-handle:



• Every smooth mfld has the htpy type of a cw-cpx

Some applic: • Morse ineq: c_i = #crit pts of index i
b_i : betti numbers

$$c_i - c_{i-1} + c_{i-2} + \dots \geq b_i - b_{i-1} + \dots$$

(in partic. $c_i \geq b_i$)

• The h-cob thm ($\Rightarrow$ Poincaré conj $n \geq 5$ ...)

Def: [Pseudo-gradient] $f : M \to \mathbb{R}$ Morse, $X \in \mathcal{X}(M)$

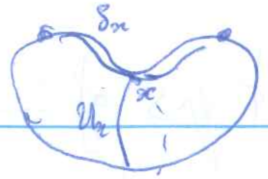
is a pseudo-grad. for f if $\begin{cases} \bullet\ \forall x \text{ not crit, } d_x f \cdot X < 0 \\ \bullet\ X = -x_1 \partial x_1 \dots - x_k \partial x_k + \dots \text{ near a crit pt, in Morse coord.} \end{cases}$

Ex: $X = -\nabla^g f$, g Riem. metr.

Def: stable / unstable mflds ϕ^t : flow of ps. grad X, $x \in \mathrm{Crit} f$

$$S_x = \{ y \mid \lim_{t \to +\infty} \phi^t(y) = x \}$$

$$U_x = \{ y \mid \lim_{t \to -\infty} \phi^t(y) = x \}$$

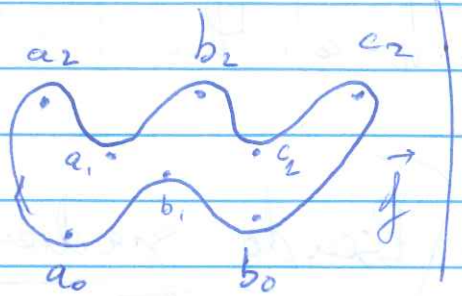
Ex: 

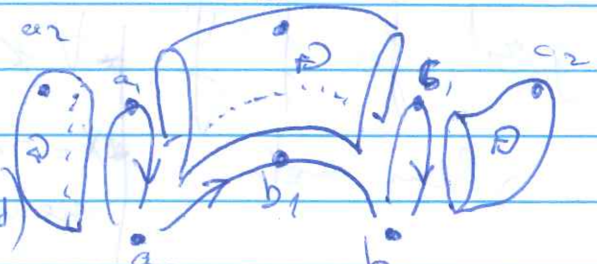
Prop: • $U_x \simeq \mathbb{R}^k$, $k = \mathrm{ind}(x)$

• $S_x \simeq \mathbb{R}^{(n-k)}$, &

• $M = \coprod_{x \in \mathrm{Crit} f} S_x = \coprod_{x \in \mathrm{Crit}} U_x$

~~Ex:~~ 

$M =$ 

⇒ gives a decomposit°: 

(Rk: works because f is self-indexed)

⇒ Can compute H_*^{cell}: ① orient the U_x

② $C_0 = \mathbb{Z}a_0 \oplus \mathbb{Z}b_0$ $C_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}c_1$
$C_2 = \mathbb{Z}a_2 \oplus \mathbb{Z}b_2$

③ $\partial a_2 = a_1$, $\partial b_2 = c_1 - \cancel{b_1} - a_1 + \cancel{b_1}$, $\partial c_2 = -c_1$

$\partial a_1 = a_0 - a_0 = 0$, $\partial b_1 = b_0 - a_0$, $\partial c_1 = b_0 - b_0 = 0.$

Obs: • C_* gen by Crit f, ∂ counts flow lines...

2.2 - Morse homology

Def: * $\tilde{\mathcal{M}}(x,y) = \left\{ \gamma : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \lim_{-\infty} \gamma = x \\ \lim_{+\infty} \gamma = y \\ \dot{\gamma}(t) = X(\gamma(t)) \end{array} \right\} \circlearrowleft \mathbb{R}$ (reparam)

* $\mathcal{M}(x,y) = \tilde{\mathcal{M}}(x,y) / \mathbb{R}$

Rk: $\tilde{\mathcal{M}}(x,y) \simeq \mathcal{U}_x \cap \mathcal{S}_y$, $\quad \mathcal{M}(x,y) \simeq \mathcal{U}_x \cap \mathcal{S}_y \cap f^{-1}(c)$

$\gamma \longmapsto \gamma(0)$ $\qquad$ for $c \in (f(y), f(x))$

Def: (f, X) is Morse-Smale if $\forall p, q$, $\mathcal{S}_p \pitchfork \mathcal{U}_q$.

Th: * Morse-Smale pairs (f, X) are "generic". (def. later)

* Assume (f, X) Morse-Smale, and $\mathcal{U}_p$ is oriented for each p, then $\tilde{\mathcal{M}}(x,y)$ and $\mathcal{M}(x,y)$ are smooth mfd, oriented in a canonical way, and of dimension:

$\dim \tilde{\mathcal{M}}(x,y) = \operatorname{ind}(x) - \operatorname{ind}(y)$

$\dim \mathcal{M}(x,y) = \operatorname{ind}(x) - \operatorname{ind}(y) - 1$ ← except if $\operatorname{ind} x = \operatorname{ind} y$:

$\tilde{\mathcal{M}}(x,y) = \begin{cases} \emptyset & \text{if } x \neq y \\ \{pt\} & \text{if } x = y \end{cases}$

- If ind x = ind $y+1$: $\mathcal{M}(x,y)$ compact, zero-dim.

- If ind x = ind $y+2$, $\mathcal{M}(x,y)$ can be compactified to a cpct 1-mfd with ∂; $\overline{\mathcal{M}}(x,y) = \mathcal{M}(x,y) \cup \partial\overline{\mathcal{M}}$ with

$$\partial\overline{\mathcal{M}}(x,y) = \coprod_{\substack{z \\ \text{ind } z = \text{ind } y+1}} \mathcal{M}(x,z) \times \mathcal{M}(z,y).$$

Rk: contains 4 key features in Floer/Morse theory:

- Transversality (Morse Smale = generic)
- Orientations $\mathcal{M}(x,y)$ or.
- Compactness $\partial\overline{\mathcal{M}} \subset \coprod_z \mathcal{M}\times\mathcal{M}$
- Glueing $\coprod_z \mathcal{M}\times\mathcal{M} \subset \partial\overline{\mathcal{M}}$

Def: Morse complex

- $C_i^{\text{Morse}}(f,X) = \bigoplus_{x \in \text{Crit}_i f} \mathbb{Z} \cdot x$ $\quad \text{Crit}_i f = \left\{ \begin{matrix}\text{crit pts of} \\ \text{ind} = i\end{matrix} \right\}$

$\partial : C_* \to C_{*-1}$ def by:

$$\partial x = \sum_y{}' \, n(x,y) \cdot y, \quad \text{with } n(x,y) = \underset{\substack{\uparrow \\ \text{signed count}}}{\#} \mathcal{M}(x,y)$$

$\rightsquigarrow HM_* = \dfrac{\ker \partial}{\operatorname{Im} \partial}$ ...

Co-homology: $C^* = \mathrm{Hom}(C_*, \mathbb{Z})$,

$d : C^* \to C^{*+1}$ def by

$$(d\varphi)x = (-1)^{\deg x + 1}\varphi\,\partial x$$

Prop: $\partial^2 = 0$;

Proof:
$$\partial^2 x = \sum_y n(x,y)\,\partial y$$

$$= \sum_{y,z}' n(x,y)\,n(y,z)\,z$$

$$= \sum_z' \left(\# \left(\underbrace{\coprod_y \mathcal{M}(x,y)\cdot\mathcal{M}(y,z)}_{= \partial \mathcal{M}(x,z) \text{ by prev. Thm.}}\right)\right) z$$

$$= 0.$$

2.3- Orientations

Def: A submfd $V \subset M$ is co-oriented if $TM_{|V}/TV \to V$ is oriented.

$\Rightarrow \forall v \in V$, any complement K st $T_vM = T_vV \oplus K$ is canonically oriented, since $K \xrightarrow{\sim} T_vM/_{T_vV}$.

Rks If U_x is oriented, then S_x is co-oriented (since $T_xM = T_xU_x \oplus T_xS_x$)
and S_x contractible

→ Orientation of $\widetilde{\mathcal{M}}(x,y) \simeq \mathcal{U}_x \cap S_y$

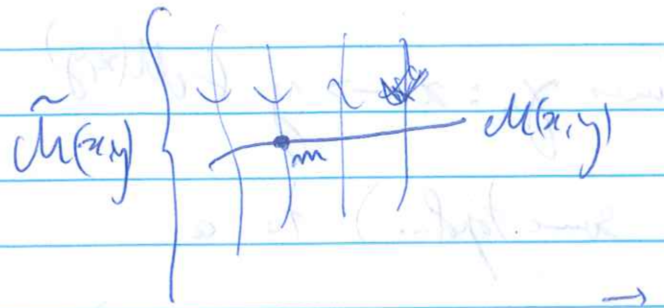
T_mS_y

(f,X) Morse-smale $\Rightarrow$ $T_mM = \underbrace{T_m(\mathcal{U}_x \cap S_y) \oplus K_m}_{T_m\mathcal{U}_x} \oplus L_m$

- $\mathcal{U}_x$ oriented $\Rightarrow T_m(\mathcal{U}_x \cap S_y) \oplus K_m$ or.
- S_y co-or $\Rightarrow K_m$ or

orient $T_m(\mathcal{U}_x \cap S_y)$ so that the two o. agree.

→ Orientat° of $\mathcal{M}(x,y)$ view $\mathcal{M}(x,y) \subset \tilde{\mathcal{M}}(x,y)$ by fixing a level set



$$T_m \hat{\mathcal{M}} = T_m \mathcal{M} \oplus \underbrace{\mathbb{R}\cdot X_m}_{\text{oriented by } X}$$

← outward pointing first...

→ orient $T_m \mathcal{M}$ so that the orient. of $T_m \tilde{\mathcal{M}}$ agrees with the $\oplus$ of or-

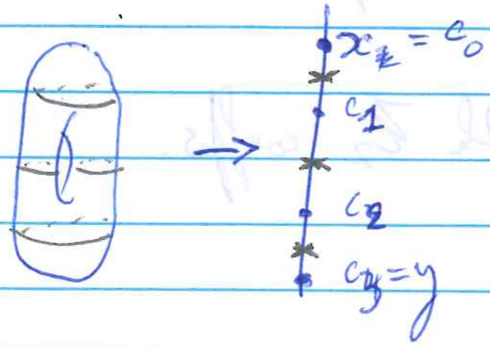
Rk: * Doesn't generalize to Floer theory in such a simple way (S_x and U_x will be ∞ dim) ⇒ will have to define orientation by orienting a Fredholm operator...

* Can avoid orientations by working with $\mathbb{Z}_2$ coeffs.

2.4 - Compactness

Lemma: Any sequence of flow lines $\gamma_n : x \to y \in \mathcal{M}(x,y)$ has a subseq. that converges (in some topol...) to a broken flow line $\gamma_\infty = \left(x \xrightarrow[\gamma_\infty^1]{} x_1 \xrightarrow[\gamma_\infty^2]{} x_2 \xrightarrow[\gamma_\infty^3]{} \cdots \xrightarrow[\gamma_\infty^k]{} x_k = y\right)$

Proof: let $f(x) > c_1 > c_2 > \cdots c_N > f(y)$ be all the crit. values of f in $]f(y), f(x)[$, and take regular values $r_0, \dots r_N$ in between:



On each (compact) level set $f^{-1}(r_i)$, the seq $\{m_n^i\} = \gamma_n \cap f^{-1}(r_i)$ converges up to subseq., and defines pieces of the broken flow line...

(needs to check defines a broken flow line ... exercise.)

$\square$

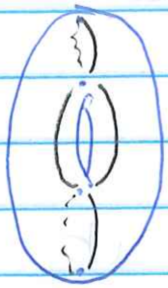
Conseq: $\partial \overline{\mathcal{M}}(x,y) \subset \coprod_{(x_1,\dots,x_{k-1})} \mathcal{M}(x,x_1) \times \dots \mathcal{M}(x_{k-1},y)$

(f,X) Morse-Smale $\Rightarrow$ $\mathcal{M}(x,y) = \emptyset$ if $\begin{cases} \text{ind } x < \text{ind } y \\ \text{ind } x = \text{ind } y \text{ and } y \neq x \end{cases}$

$\rightarrow$ if $\text{ind } x = \text{ind } y + 1$, $\partial \overline{\mathcal{M}} = \emptyset$

$\rightarrow$ $\text{ind } x = \text{ind } y + 2$, $\partial \overline{\mathcal{M}}(x,y) = \coprod_z \mathcal{M}(x,z) \times \mathcal{M}(z,y)$.

Rk:

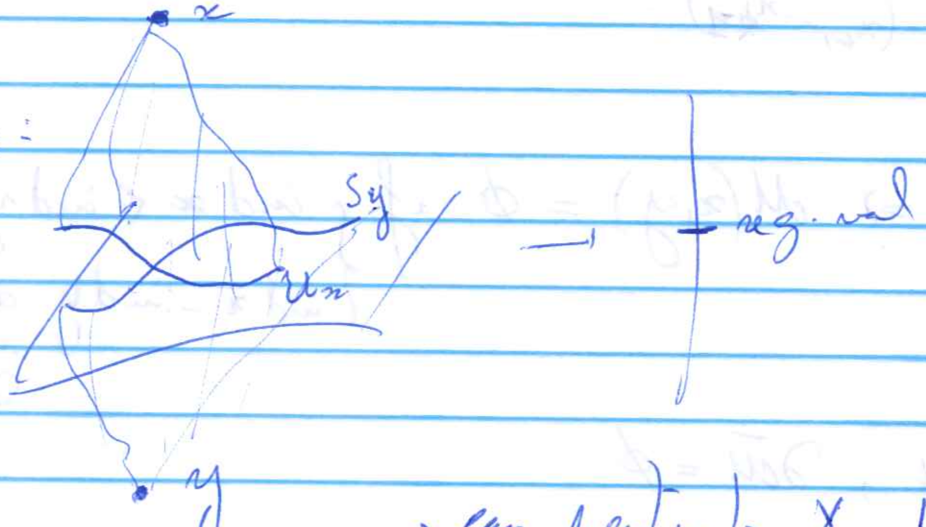


$\rightarrow$ Not Morse-Smale

$\Rightarrow$ perturb:

2.5 – Transversality

idea:



→ can perturb X above the lev. set: U_x can be perturbed in an arbitrary way while leaving S_y unchanged...

Def: P: top. space, $U \subset P$ is "generic" if it contains a countable $\cap$ of open dense subsets. A property $P(p)$ is generic (in $p \in P$) if it holds true on a generic set.

Rk: ① P complete metric space, Baire: generic $\Rightarrow$ dense.

② U, V generic $\Rightarrow U \cap V$ generic (important, not true if generic $\rightsquigarrow$ dense)

Def: V, W : Banach spaces, $F: V \to W$ bounded lin. op. is Fredholm if
- $F(V) \subset W$ closed
- $\ker F$, $\operatorname{coker} F$ have finite dim.

$\operatorname{ind} F := \dim \operatorname{Ker} F - \dim \operatorname{Coker} F$

prop:
- F_0 Fredholm, $\exists \varepsilon > 0 : \|F - F_0\| < \varepsilon \Rightarrow F$ Fredholm, and $\operatorname{ind} F = \operatorname{ind} F_0$
- F_0 Fredholm, K compact $\Rightarrow F_0 + K$ Fredholm, $\operatorname{ind} F_0 + K = \operatorname{ind} F_0$

Th [Sard-Smale] X, Y : separable Banach mflds, $f: X \to Y$ C^k map, s.t. $df_x : T_x X \to T_{f(x)} Y$ is Fredholm, of ind l, Assume $k \geq \max\{1, l+1\}$

Then $Y^{reg} \subset Y$ the set of reg. val is generic.

→ how to use? Recall: $\tilde{\mathcal{M}}(x,y) = \mathcal{F}_X^{-1}(0)$, with

$$\mathcal{F}(\gamma) = \dot{\gamma} - X(\gamma), \quad \gamma: \mathbb{R} \to M$$
$$-\infty \to x$$
$$+\infty \to y$$

General strategy to prove transversality

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \end{tikzcd}$$
: Banach bundle, $f_p : \mathcal{B} \to \mathcal{E}$ smooth section

parametrized by $p \in \mathcal{P}$ parameter

Want to show $\mathcal{M}_p = f_p^{-1}(0)$ smooth for generic p

Natural approach: show $d_\gamma f_p$ surj on $f^{-1}(0\text{-sec})$ ...

Strategy: consider $\mathcal{F} : \mathcal{P} \times \mathcal{B} \to \mathcal{E}$

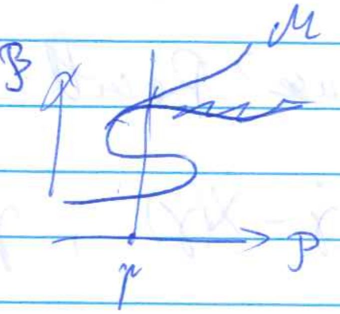
$$(p, \gamma) \mapsto f_p(\gamma)$$

$$\mathcal{M} := \mathcal{F}^{-1}(0\text{-sec}) = \bigcup_p \mathcal{M}_p \quad \to \mathcal{M}_p = \mathcal{M} \cap \{p\} \times \mathcal{B}$$

① show $\mathcal{M}$ smooth by proving $\frac{\partial \mathcal{F}}{\partial p}$ surjective on it ($\Rightarrow d\mathcal{F}$ surj)

② Apply Sard-Smale's thm to $\pi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$, with

$\pi : \mathcal{P} \times \mathcal{B} \to \mathcal{P}$ the proj.



Def: p is regular if it is a reg. val. of $\pi|_{\mathcal{M}}$ ($\Leftrightarrow df_p|_{\mathcal{M}_p}$ surj)

→ Applic. to Morse homol.

$$\begin{tikzcd} \mathscr{E} \arrow[d] \\ \mathcal{B} \arrow[u, bend left, "f_P"'] \end{tikzcd}$$

- $\mathcal{P}' = \{ X \mid \text{pseudo-}\nabla \}$
- $f_X : C^\infty(\mathbb{R}, M; x, y) \overset{\mathcal{B}'}{=} \longrightarrow \mathscr{E}$

$$\gamma \longmapsto \dot{\gamma} - X(\gamma)$$

$$\mathscr{E}'_\gamma = C^\infty(\gamma^* TM)$$

<u>Pb</u>: $\mathcal{P}', \mathcal{B}', \mathscr{E}'$ not complete

→ take their completion, $p > 1$

<u>Recall</u>: $W^{k,p}(\mathbb{R}^m, \mathbb{R}^n) = \{ \text{distrib } \mathbb{R}^m \to \mathbb{R}^n \text{ whose } k \text{ first deriv. are in } L^p \}$

$$\|u\|_{k,p} = \Big(\int \sum_{|\alpha| \leq k} |\partial^\alpha u|^p dx \Big)^{1/p}$$

· $\mathcal{B}$: $W^{1,p}$-complet° of $\mathcal{B}'$ (as a Banach mfd, fix Riem metric, and use exponential maps to construct atlas (→ see Audin–Damian)

$\mathscr{E} = L^p$-complet° of $\mathscr{E}' = L^p(\gamma^* TM)$

$\mathcal{P}$: $W^{1,p}$-complet°.

$$\frac{\partial F}{\partial X} \cdot Y = Y \circ \gamma \quad : \quad \frac{\partial F}{\partial X} \text{ surjective.}$$

Rk1: * in G^{al}, $\mathcal{M}_p = f_p^{-1}(0) / G_p \leftarrow$ symmetries

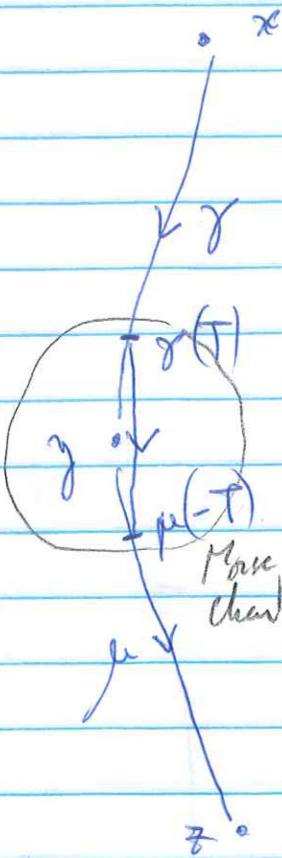
→ can give rise to complications

* the reparametrization $\mathbb{R} \times C^k(S^1, \mathbb{R}) \to C^k(S^1, \mathbb{R})$

is not smooth: only "sc-smooth" in the sense of Polyfold theory (→ HWZ), i.e. $\mathbb{R} \times C^k \to C^{k-1}$

2.6 - Gluing

<u>Rk</u>: See Audin - Damian for the intersect° picture.



Say ind x = ind y + 2, and Morse - Smale.

some nbhd of (γ_0,μ_0)

Goal: construct $g_y : [T_0,\infty) \times \mathcal{M}(x,y) \times \mathcal{M}(y,z) \to \mathcal{M}(x,z)$

so that $\mathcal{M}(x,z) \setminus \bigcup_y \operatorname{im} g_y$ is compact.

(in general: can have several param. for several breaking pts.)

<u>Step 1</u>: construct a "pre-gluing" map:

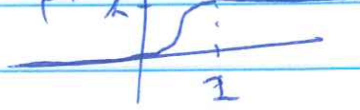
$$\tilde g_y : [T_0,\infty) \times \tilde{\mathcal{M}}(x,y) \times \tilde{\mathcal{M}}(y,z) \longrightarrow C^\infty(\mathbb{R}, M, x, z)$$

take T_0 large enough so that $\gamma([T_0,\infty))$ and $\mu((-\infty,-T_0]))$ are contained in a Morse chart,

define then for $T > T_0$:

$$\tilde g_y(T,\gamma,\mu) : t \longmapsto \begin{cases} \gamma(t+T) & t \le 0 \\ \mu(t-T-1) & t \ge 1 \\ (1-\varphi(t))\gamma(t+T) + \varphi(t)\mu(t-T-1) \end{cases}$$

with smooth φ:



Step 2: Work... (ex. Newton-Picard Method) to get flow traj from $\tilde{g}_\gamma$.

Step 3: * check $\lim_{T\to\infty}$ = broken traj.

* check g_γ embedding.

* $\gamma_n \to (\gamma, v) \Rightarrow$ contained in image for large n.

~~2.7 - Invariance, continuation maps~~

~~Prop: (f_0, X_0) and (f_1, X_1) Morse-Smale on M.~~

~~Then $\underset{\substack{\shortparallel \\ C_0}}{C_*(f_0,X_0)}$ and $\underset{\substack{\shortparallel \\ C_1}}{C_*(f_1,X_1)}$ are homotopic~~

~~i.e $\exists$~~

$$\begin{tikzcd} C_0 \arrow[r, bend left, "\phi_{01}"] \arrow[loop below, "\partial"', "H_0"] & C_1 \arrow[l, bend left, "\phi_{10}"] \arrow[loop below, "\partial"', "H_1"] \end{tikzcd}$$

~~s.t.~~

$$\phi_{01} \circ \phi_{10} = Id_1 + H_1 \partial + \partial H_1$$

$$\phi_{10} \circ \phi_{01} = Id_0 + H_0 \partial + \partial H_0$$

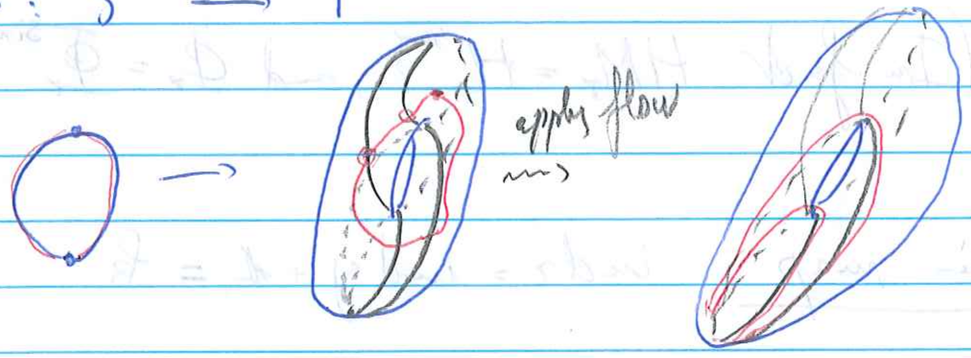
~~in particular, ϕ_{01} and ϕ_{10} induce inverse isom on H_*~~

[redacted] 2.7 - Functoriality, invariance

Goal: $\underset{\substack{(f,X)\\ \text{Morse-Smale}}}{\Phi : M} \longrightarrow \underset{\substack{(g,Y)\\ \text{Morse Smale}}}{N}$ smooth map (or even correspondence)

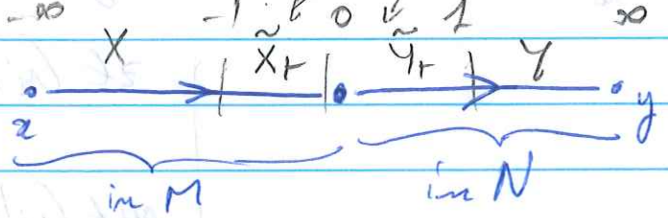
$\Rightarrow$ can def $C\Phi_* : CM_*(M,f,X) \longrightarrow CM_*(N,g,Y)$? (chain map

ex: $\Phi : S^1 \longrightarrow T^2$



$\Rightarrow$ Define $\langle C\Phi_* x, y\rangle$ by counting "grafted lines":

ind $x \neq$ ind y



allow to perturb so to have transversality

say $\tilde{X}_t \equiv X$ for $t \leq -1$
$\tilde{Y}_t \equiv Y$ for $t \geq 1$

$$\Leftarrow \mathcal{M}(\phi, x, y) = \left\{ (\gamma_-, \gamma_+) \;\middle|\; \begin{array}{ll} \gamma_- : \mathbb{R}_- \to M & \gamma_+ : \mathbb{R}_+ \to M \\ \quad -\infty \mapsto x & \quad \infty \mapsto y \\ \dot{\gamma}_-(t) = \tilde{X}_t(\gamma_-(t)) & \dot{\gamma}_+(t) = \tilde{Y}_t(\gamma_+(t)) \\ \multicolumn{2}{c}{\Phi(\gamma_-(0)) = \gamma_+(0)} \end{array} \right\}$$

prop: for generic $\tilde X_t, \tilde Y_t$, $\mathcal{M}(\Phi, x, y)$ compact, 0-dim, oriented. $C\Phi_*$ is a chain map: $C\Phi_*\partial = \partial\Phi_*$, and different choices of $\tilde X_t, \tilde Y_t$ yield homotopic maps

$(C\phi' - C\phi = H\partial + \partial H)$

In partic, $\Phi_* : HM_* \to HM_*$ is well-def and indep^t of $\tilde X_t^{Z}\ \tilde Y_t$. (In fact $HM_* = H_*^{sing}$, and $\Phi_* = \Phi_*^{sing}$ ...)

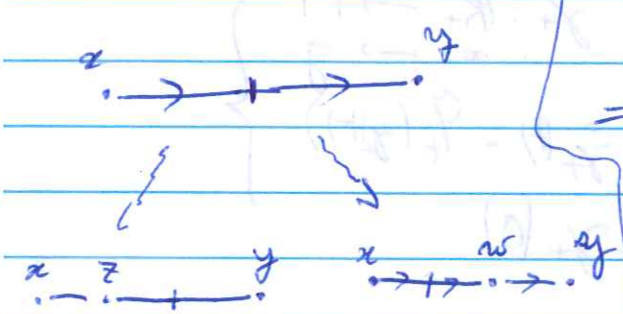
Proof: • Chain map $\operatorname{ind} x = \operatorname{ind} y + 1 = k$

$$\langle C\phi_*\partial - \partial C\phi_*\ x, y\rangle = \sum_z{}' \#\mathcal{M}(x, z)\,\#\mathcal{M}(\Phi, z, y) - \sum_w{}' \#\mathcal{M}(\Phi, x, w)\,\#\mathcal{M}(w, y)$$

$z \in \operatorname{Crit}_{k-1}(f)$

$w \in \operatorname{Crit}_k(g)$

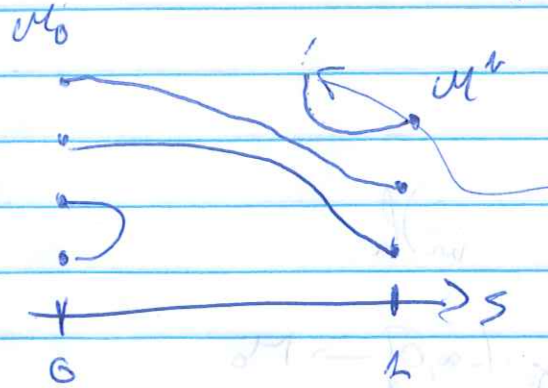
Compact. of $\overline{\mathcal{M}}(\phi, x, y)$:



$= \#\,\partial\overline{\mathcal{M}}(\phi, x, y) = 0$

• Different $\tilde X_t$ $\tilde Y_t$: Let $(\tilde X_t^0, \tilde Y_t^0)$ and $(\tilde X_t^1, \tilde Y_t^1)$ be ≠ choices, chose $(\tilde X_t^s, \tilde Y_t^s)$ $s \in [0,1]$ that interpolates, and define for $k = \operatorname{ind} x - \operatorname{ind} y = \begin{cases} 0 \\ -1 \end{cases}$

$$\mathcal{M}^{par}(\phi, x, y) = \bigcup_{0 \leq s \leq 1} \{s\} \times \mathcal{M}(\phi, x, y)_s \quad : \text{ of } \dim = k+1$$



breaking: $\overline{\mathcal{M}^{par}} = \mathcal{M}^{par} \cup \coprod_z \mathcal{M}^{par} \times \mathcal{M} \cup \coprod_w \mathcal{M} \times \mathcal{M}^{par}$

if

⇒ • if $k = 0$, $\mathcal{M}^{par}(\phi, x, y)$ defines the htpy H.

⇒ • if $k = 1$ $\overline{\mathcal{M}^{par}}$ can be used to prove that $C\phi_*^0 - C\phi_*^1 = \partial H + H \partial$.

prop: $\underset{(f_0,X_0)}{M_0} \xrightarrow{\Phi_{01}} \underset{(f_1,X_1)}{M_1} \xrightarrow{\Phi_{12}} \underset{(f_2,X_2)}{M_2}$, then ~~$\Phi_{12}$~~

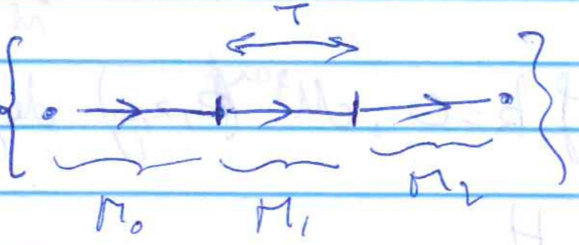
$$C(\Phi_{12}\circ\Phi_{01})_* = C\Phi_{12*}\circ C\Phi_{01*} + \partial H + H\partial$$

in partic $(\Phi_{12}\circ\Phi_{01})_* = \Phi_{12*}\circ\Phi_{01*}$ in homol, and the Morse cplex for ≠ choices of (f,X) are htpy equiv.

(apply to $\Phi = id$)

proof: consider $\mathcal{M} = \bigcup_{0\le T<\infty} \{T\}\times\mathcal{M}_T$, with

$\mathcal{M}_T =$



i.e. $\begin{cases}\gamma_0 : (-\infty,0] \to M_0\\ \gamma_1 : [0,T] \to M_1\\ \gamma_2 : [T,\infty) \to M_2\end{cases}$

grafted by Φ_{01} & Φ_{12}

$$\begin{tikzcd} T=0 \arrow[d] & T\to\infty \arrow[d] & \text{breaking} \arrow[d] \\ C(\Phi_{12}\circ\Phi_{01})_* & C\Phi_{12*}\circ C\Phi_{01*} & \partial H + H\partial \end{tikzcd}$$

□

Assume coeffs in $\mathbb{Z}_2$

Conseq: $HM_*(f,X) \simeq HM_*(-f,-X) \simeq HM^{n-*}(f,X)$

(Poincaré Duality). $\overset{x}{\cdot}\xrightarrow[X]{}\cdot^{y} \leftrightarrow \overset{y}{\cdot}\xleftarrow[-X]{}\cdot^{x}$

Rk: signs issues in general, but works if M is oriented...

Th: $HM_* \simeq H_*$

Proof 1: use invariance to reduce to a self-indexed Morse fct, and identify with H_*^{cell}

Proof 2: construct directly a htpy eq. to H_*^{sing} (→ see Hutchings's notes)

Proof 3: (Witten) g metric ⇝ $H^*_{dR} \simeq Harm^* = \{\alpha \,/\, d\alpha = d^*\alpha = 0\}$ $\Leftrightarrow \Delta\alpha = 0$

use f to deform the eq $d(e^{tf}\alpha) = d^*(e^{-tf}\alpha) = 0$ and $t \to \infty$

→ Topic.

2.8 - Product structure, the Morse category

Assume either M oriented or $\mathbb{Z}_2$/coeffs.

cup prod: $H^p(M)\otimes H^q(M)\to H^{p+q}(M)$

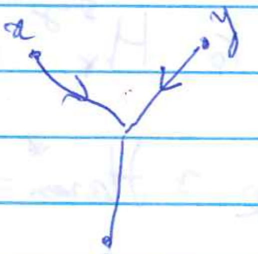
is defined dually by intersecting cycles.

Rk: can't do it if (f, X) is fixed

$$\to C_*\underset{X_{12}}{(f_2-f_1)}\otimes C_*\underset{X_{01}}{(f_1-f_0)}\to C_*\underset{X_{02}}{(f_2-f_0)}$$

$$x\otimes y\longmapsto {\sum_z}' \#\mathcal{M}(x,y;z)\,z$$

where $\mathcal{M}(x,y;z)=\Big\{$



$\Big\}$

$$=\left\{(\gamma_{12},\gamma_{01},\gamma_{02})\;\middle|\;\begin{array}{l}\gamma_{12},\gamma_{01}:\mathbb{R}_-\longrightarrow M\\ \gamma_{02}:\mathbb{R}_+\longrightarrow M\\ \lim_{-\infty}\gamma_{12}=x\\ \lim_{-\infty}\gamma_{01}=y\\ \lim_{+\infty}\gamma_{02}=z\end{array}\quad\begin{array}{l}\gamma_{ij}\text{ flow line of }X_{ij}\\ \text{+ coincide at }0\end{array}\right\}$$

prop: • $\mathcal{M}(x,y;z) \simeq U_x \cap U_y \cap S_z$, generically smooth, oriented, of dim ind x + ind y − ind z − n

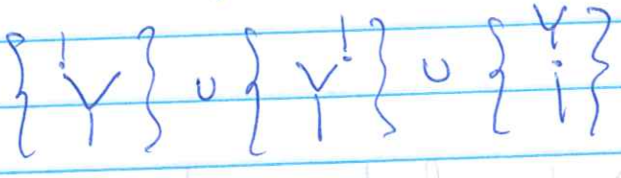
* • is a chain map ($C_* \otimes C_* \to C_*[n]$)

* • is assoc. up to htpy.

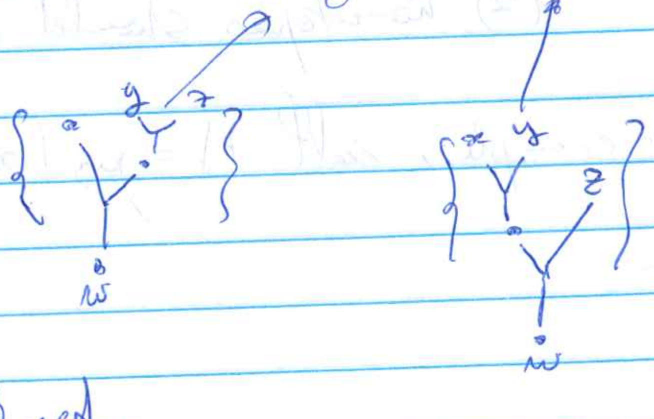
proof: orientat' U_x or + M or ⇒ S_z or ⇒ ∩ are oriented...

* chain map: if dim $\mathcal{M}(x,y;z) = 1$,

$\partial\overline{\mathcal{M}}(x,y;z) =$

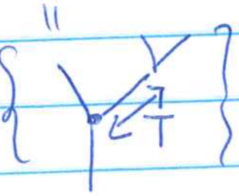


• assoc up to htpy: wants to show $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ + htpy:

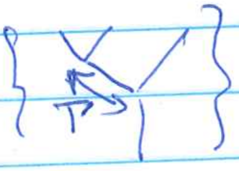


→ define $\mathcal{A} = \bigcup_{T\geq 0} \{T\} \times \mathcal{A}_T$

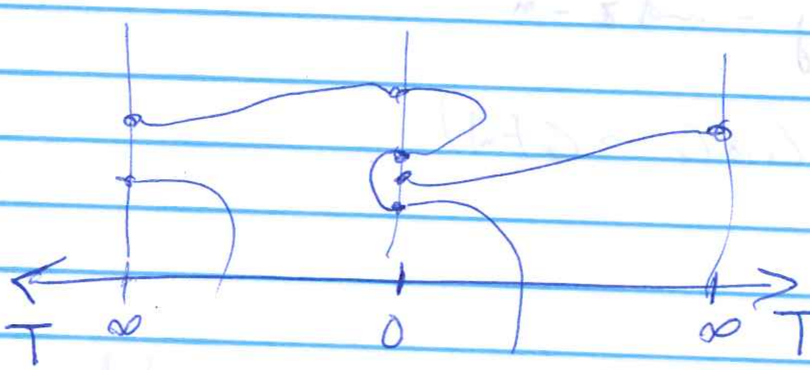
||



$\mathcal{B} = \bigcup_{T\geq 0} \{T\} \times \mathcal{B}_T =$



$$\mathcal{A}_0 = \mathcal{B}_0 \Rightarrow \mathcal{M}(x,y,z;w) = \mathcal{A} \underset{\mathcal{A}_0=\mathcal{B}_0}{\cup} \mathcal{B}$$



... use 0-dim part to define htpy, 1-dim to prove formula.

□

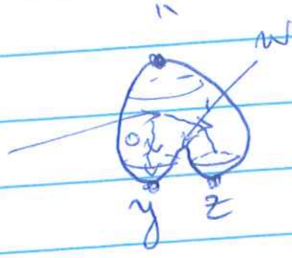
<u>Two principles</u> ① homol = bad, chain cpx = good

② homotopies should be part of the str

↪ in prev example, call $H = \mu_3(x,y,z)$ ...

Rk: example where htpy is nontrivial: $\Phi : M \to N$

$M = \mathrm{pt} \ni x$, $N =$

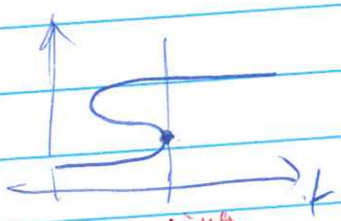


for some pseudo-grad X^0, X^1, $C\Phi^0_* x = y$

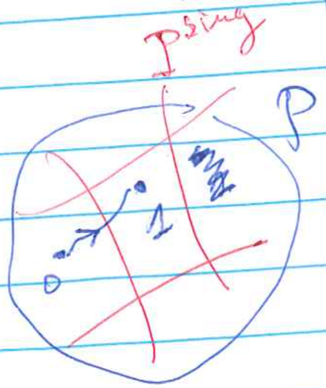
$C\Phi^1_* x = z$

→ for any family $\{X^t, t \in [0,1]\}$, $\exists\, t$ s.t. $\Phi(x) \to w$

such traj are not regular in the non-param sense, but are reg. in a param sense:



$$F : [0,1] \times \mathcal{P} \times \mathcal{B} \to \mathcal{E}$$
$$(t, p, \gamma)$$



$\frac{\partial F}{\partial p}$ can have cokernel of dim 1,

compensated by $\frac{\partial F}{\partial t}$

→ Now switch to <u>cochain</u> complexes

⟺ grade the cpxes by $n - \mathrm{ind}(x) = \dim S_x$

⟺ flow upwards : $\partial =$  , $(\cdot\circ\cdot) =$ 

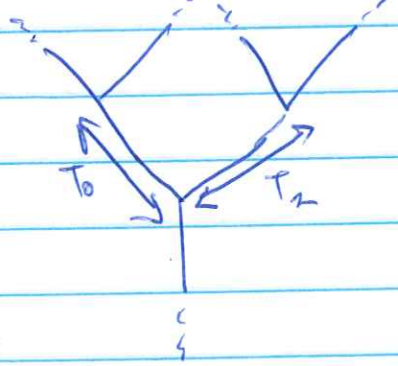
... so to have nicer degrees ...

$$\mathcal{M}(x,y;z) = S_x \cap S_y \cap U_z$$

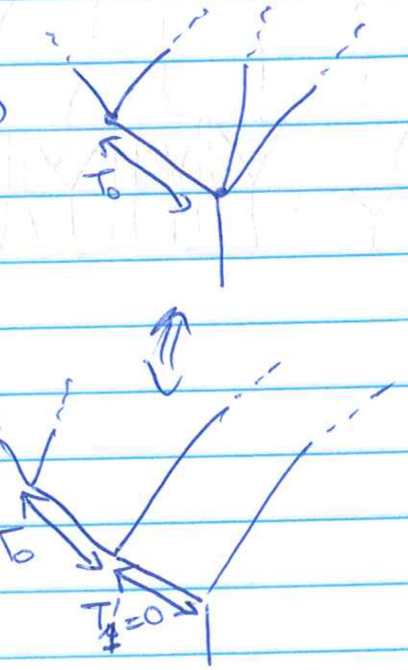
$$\dim = \mathrm{ind}\, z - \mathrm{ind}\, x - \mathrm{ind}\, y \Rightarrow C^p(f_2 - f_1) \otimes C^q(f_1 - f_0) \to C^{p+q}(f_2 - f_0).$$

<u>Def</u>: $d \geq 1$ T_d : moduli space of (rooted ribbon) trees with d incident semi-infinite edges, one root, and intermediate edges of length $\in [0,\infty)$.

<u>d=4</u>:

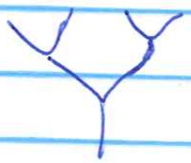


$T_1 = 0$ ⟺



Rk: $T \in T_d \iff$ bracketing of $\{1, \dots d\}$

$+ \ d-2$ parameters in $[0, \infty)$
(or less)

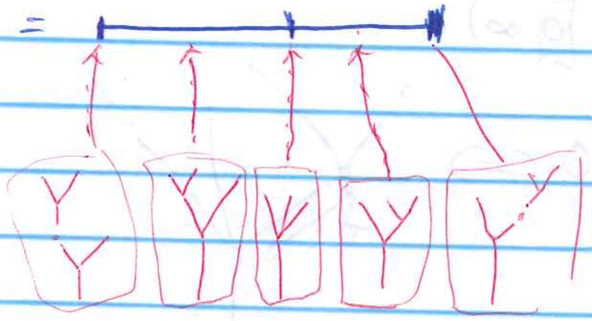
 $\iff ((1\ 2)(3\ 4)) + (T_0, T_1)$

 $\iff ((1,2)\ 3\ 4) + T_0$

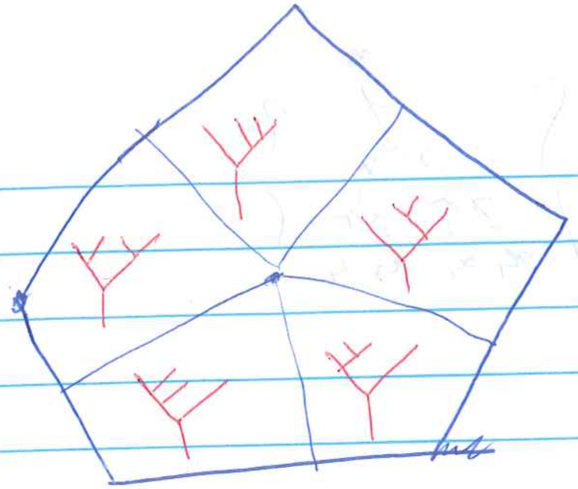
$\overline{T_d}$: compactifications $\iff$ broken trees
$\iff$ parameters can be $= \infty$

ex; $T_1 = \overline{T_1} = T_2 = \overline{T_2} = pt$

$\overline{T_3} =$



$\overline{\mathcal{T}}_4 =$



ex: draw $\overline{\mathcal{T}}_5$ ...

prop: • $\overline{\mathcal{T}}_d$ polytope of dim $d-2$

$$\partial^1 \overline{\mathcal{T}}_d = \coprod_{2\le k\le d-1} \{1, \dots k\} \times \overline{\mathcal{T}}_k \times \mathcal{T}_{d-k+1}$$

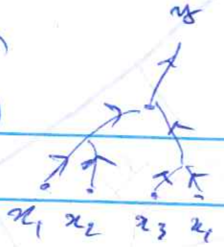
$d=4$
$k=3$

$(\partial^k \overline{\mathcal{T}}_d = k$ breakings ...$)$

Def: $f_0, \dots f_d$, $x_{01}, x_{12}, \dots$ pseudo-grad

a (broken) flow tree is a pair $(T \in \mathcal{T}_d$ (or $\overline{\mathcal{T}}_d$), $\gamma : T \to M)$
with γ flow line on each interval (flowing upwards, coherent ...)

$$\rightsquigarrow \mathcal{M}(x_1, \dots, x_d; y) = \Big\{$$



$$\Big\}$$

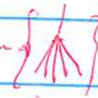
$$\downarrow$$

$$T_d$$

Generically,

prop: • smooth, oriented, of $\dim = \underbrace{\operatorname{ind} y - \operatorname{ind} x_1 - \dots \operatorname{ind} x_d}_{}+\underbrace{(d-2)}_{= \dim T_d}$

$= \dim \{$

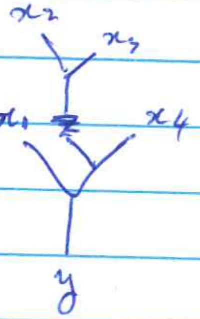


$\}$

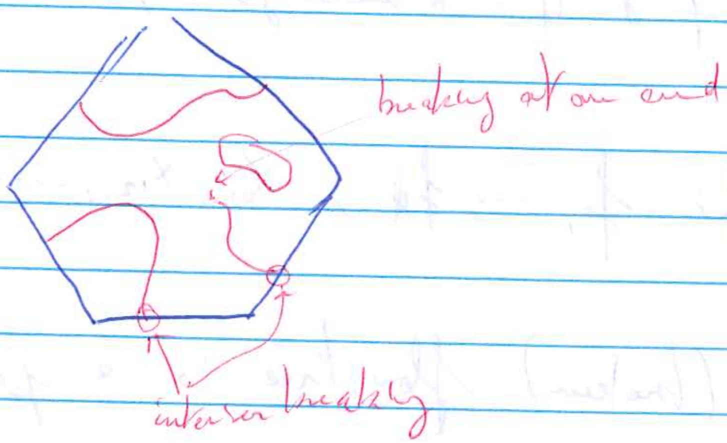
$= \dim (S_{x_1} \cap \dots \cap S_{x_d} \cap U_y)$

• compact if $\dim = 0$

• if $\dim = 1$ $\partial \overline{\mathcal{M}}(x_1, \dots, x_d; y) = \displaystyle\coprod_{\substack{1 \leq k \leq d \\ 1 \leq i \leq k \\ z}} \mathcal{M}(x_1, \dots x_{i-1}, z, x_{i+d-k+1}, \dots; y) \times \mathcal{M}(x_i, \dots \dots x_{i+d-k}; z)$



$d = 4$
$k = 3$
$i = 2$



$\Rightarrow$ defines maps $\mu^d : C^*(f_d - f_{d-1}) \otimes \dots C^*(f_1 - f_0) \to C^*(f_d - f_0)$ of degree $2-d$ satisfying the A_∞-relations:

$\forall d \geq 2$:

$$\sum_{m,n} (-1)^{\maltese_n} \mu^{d-m+1}\left(a_d, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1\right)$$

$= 0$

with $\maltese_n = |a_n| + \dots + |a_1| - n$

2.9 - Some A_∞-algebra

Def: An A_∞-algebra $(A, (\mu^n)_{n \geq 1})$ is a graded module $A = \bigoplus_n A_n$ with maps $\mu^n : A^{\otimes n} \to A$ satisfying

Rk: $d=2$; $\mu^1 \circ \mu^1 = 0$: μ^1 differential

$d=3$: (up to sign) $\mu^2(1 \otimes \mu^1 + \mu^1 \otimes 1) + \mu^1 \circ \mu^2 = 0$

μ^2 chain map

$d=4$: assoc up to htpy given by μ^3 ...

Rk: • more Grally, can define curved A_∞-alg: $(A, \{\mu^n\}_{n\geq 0})$ satisfying similar relations (can happen in G^{al} symplectic mfds...)

→ • $\mu^1(\mu^0) = 0$

• $\mu^1\circ\mu^1 \pm \mu^2(\mu^0, \cdot) \pm \mu^2(\cdot, \mu^0) = 0$) μ^1 might not be a chain map...

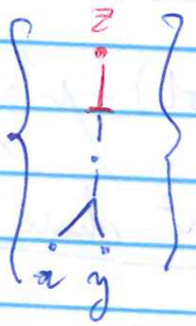
etc...

→ <u>A_∞-morphisms</u> vague def: $F : A \longrightarrow B$ s.t. $F(a.b) = F(a).F(b)$ up to htpy ...

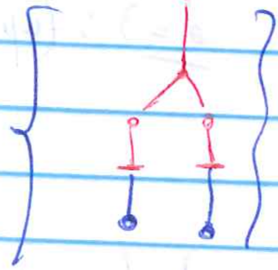
<u>Ex:</u> $\Phi : M \to N$ smooth

⇝ $C\Phi^* : C^*N \to C^*M$ def by counting

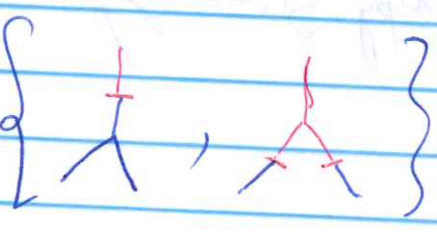
$C\Phi^*(x, y)$ counts



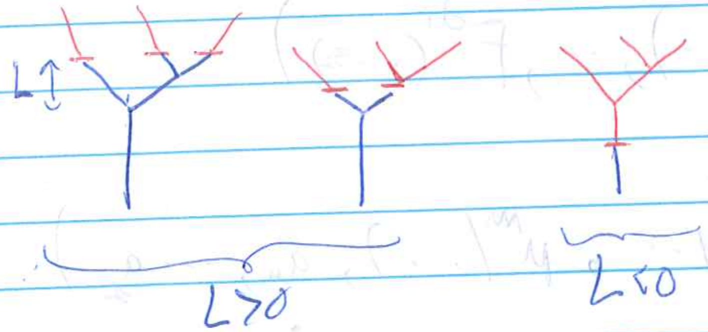
, $(C\phi^* x).(C\phi^* y)$ counts



⇒ htpy has to count

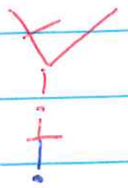


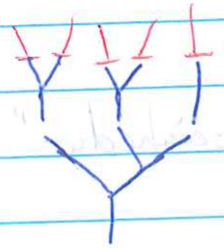
Def: A 1-grafted tree is a tree T + graftings at distance $L \in \mathbb{R}$ from the root:



$\rightsquigarrow T_{d,1} \simeq T_d \times \mathbb{R}$ (⚠ compactification is not $\overline{T_d} \times \overline{\mathbb{R}}$ ...)

$$\partial^1 \overline{T_{d,1}} = \underbrace{T_d \times T_{1,1}}_{L \to -\infty} \cup \underbrace{\partial^1 T_d \times \mathbb{R}}_{L \to \text{finite}} \cup \bigsqcup_{d = d_1 + \cdots d_k} T_k \times T_{d_1,1} \times \cdots T_{d_k,1}$$





Rk: grafting parameters drop codimension...

ex:
$$\begin{tikzcd} \overline{T_{3,1}} \arrow[d] \\ T_3 \end{tikzcd}$$

...

⇒ Def: A_n (non-unital) A_∞-mph $F : (A, \{\mu^n_A\}) \longrightarrow (B, \{\mu^n_B\})$

is a coll. $F^1, F^2, \dots$ with $F^k : A^{\otimes k} \longrightarrow B[1-d]$ s.t.:

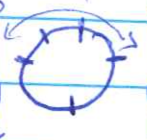
$$\sum_{\substack{r \\ d = d_1 + \dots d_k}}' \mu^d\left(F^{d_k}(a_d \dots), F^{d_{k-1}}(\dots), \dots, F^{d_1}(\dots a_1)\right)$$

$$= \sum_{m,n}' (-1)^{\maltese_n} F^{d-m+1}\left(a_d \dots, \mu^m(\dots), \underbrace{a_n, \dots a_1}_{n}\right)$$

<u>Composition</u> $A \xrightarrow{F} B \xrightarrow{G} C$

$$(G \circ F)^d = \sum_{\substack{r \\ d = d_1 + \dots d_r}}' G^r\left(F^{d_r}(\dots), \dots, F^{d_1}(\dots)\right)$$

→ strictly assoc.

<u>Rk</u>: $\overline{T}_d$ = "associahedron" ≃ $\Big\{$  $\Big\}$

$\overline{T}_{d,1}$: "multiplihedron" ≃ $\Big\{$  $\Big\}$ (→ Ma'u - Wehrheim - Woodward)

→ are part of a more G^al str, the "2-associahedra", which should give symp the str of an "$(A_\infty, 2)$ category"

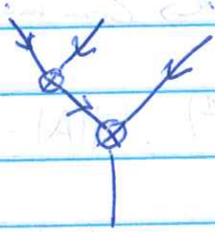
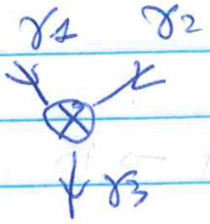
→ cf Bottman (→ topic)

→ A_∞ - modules

Vague def: $A \circlearrowright M$ with $(a_1 a_2) m = a_1 (a_2 m)$
up to htpy
(≈ associativity)

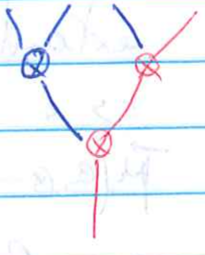
Prototype: G: Lie group, $G \circlearrowright M$ smooth mfd

$\Rightarrow C_*(G) \circlearrowright C_*(M)$ (chains, not cochains)

A_∞-str on $C_*(G)$: counts  , where 

means $\gamma_1(0)\gamma_2(0) = \gamma_3(0)$

(right dim, no shift needed...)

A_∞-module str: count 

Def: An A_∞-module M over (A, μ_A^k) is

$M = \bigoplus_n M^n$, $\mu_M^d : A^{\otimes d-1} \otimes M \to M[2-d]$

satisfying the A_∞-rel:

$$\sum (\pm 1)^{*} \mu_M^{d-m+1}\Big(\ldots, \underset{\uparrow}{\mu^m}(\ldots), \underbrace{\ldots}_{n}\Big) = 0$$

either μ_M or μ_A depending on the last input.

- Homology $(A, \mu^k) \to H(A) = \dfrac{\ker \mu^1}{\operatorname{Im} \mu^1}$

  ⟳ commutative algebra

- $F : A \to B$ mph $\leadsto H(F^1) : H(A) \to H(B)$

- M A-module $\leadsto H(M)$ $H(A)$-module ...

Rk: $A \leadsto T^c A = \bigoplus_n A^{\otimes n}$ co-algebra

$$\begin{tikzcd} T^cA \arrow[r, "\{\mu^k\}"] \arrow[dr, dashed, "\vec{\mu}"'] & A \\ & T^cA \arrow[u] \end{tikzcd}$$

extends in a unique way to T^cA ⟲ as a "co-derivation" $\{ \vec{\mu}(a \otimes b) = \pm \vec{\mu}(a) \otimes b \pm a \otimes \vec{\mu}(b)$

$\vec{\mu}(a_1 \otimes \cdots a_n) = \sum_{n,m} \pm (a_1 \otimes \cdots) \mu^m(\cdots \otimes \cdots$

A_∞ rel $\Leftrightarrow (\vec{\mu})^2 = 0$.

∘ Categories (→ Seidel...)

Def: A (non-unital) A_∞-categ: $\mathcal{A}$ is

* a set of objects $\mathrm{Ob}\,\mathcal{A}$
* a graded vector space (or module) $\mathcal{A}(X_0, X_1) = \hom_{\mathcal{A}}(X_0, X_1)$

for each pair of obj

* $\forall d \geq 1$, $\mu^d_{\mathcal{A}} : \mathcal{A}(X_{d-1}, X_d) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \longrightarrow \mathcal{A}(X_0, X_d)[2-d]$

s.t. A_∞ - rel holds... (same rel.)

Def: Cohomol. categ $H(\mathcal{A})$: same obj, but

$$H(\mathcal{A})(X_0, X_1) = H(\mathcal{A}(X_0, X_1), \mu^1), \text{ and compo:}$$

$$[a_2] \circ [a_1] = (-1)^{|a_1|} [\mu^2(a_2, a_1)]$$

* Opposite cat $\mathcal{A}^{opp}$: same ob, $\mathcal{A}^{opp}(X_0, X_1) = \mathcal{A}(X_1, X_0)$,

$$\mu^d_{\mathcal{A}^{opp}}(a_d, \dots, a_1) = (-1)^{\maltese_d} \mu^d_{\mathcal{A}}(a_1, \dots, a_d)$$

Def: A (non-unital) A_∞-functor $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ consist of

- a map $\mathcal{F}: \mathrm{Ob}\mathcal{A} \to \mathrm{Ob}\mathcal{B}$
- multilin maps; $\forall d \geq 1$: $\mathcal{F}^d : \mathcal{A}(X_{d-1}, X_d) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)$

$$\to \mathcal{B}(\mathcal{F}(X_0), \mathcal{F}(X_d))[1-d]$$

satisfying same rel as A_∞-mphs.

(induce $H(\mathcal{F}): H(\mathcal{A}) \to H(\mathcal{B})$).

→ compose the same way as A_∞-mph.

Modules

Def: ~~the~~ A (non-unital) right module $\mathcal{M}$ over $\mathcal{A}$ is:

- $\forall X \in \mathrm{Ob}\mathcal{A}$, $\mathcal{M}(X)$ graded v.sp.
- $\forall d \geq 1$, $\mu^d_{\mathcal{M}} : \mathcal{M}(X_{d-1}) \otimes \mathcal{A}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)$

$$\to \mathcal{M}(X_0)[2-d]$$

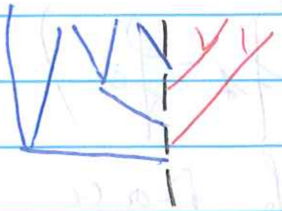
satisfying anal. rel

Fact nu-mod$(\mathcal{A})$ = ~~F~~ nu-fun$(\mathcal{A}^{opp}, Ch_*)$

Rk: · $H(\mathcal{M})$ is a $H(\mathcal{A})$-mod

· left modules: nu-fun$(\mathcal{A}, Ch_*)$

* $(\mathcal{A}, \mathcal{B})$-bimodules: $k, l \geq 0$, $\mu^{k,l} : \mathcal{A}^{\otimes k} \otimes \mathcal{M} \otimes \mathcal{B}^{\otimes l} \to \mathcal{M}$



2.10 – From Morse to Floer (really from Fukaya to Fukaya~)

Def: $Morse(M)$: * Ob = $f : M \to \mathbb{R}$ smooth

(naive def)

* $hom(f_0, f_1) = C^*(f_1 - f_0)$

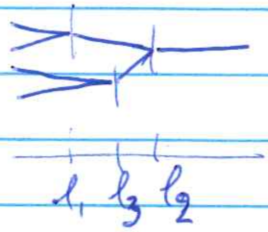
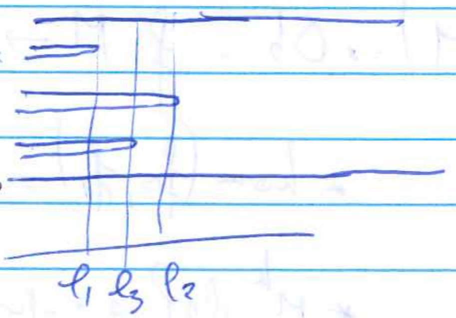
* μ^k def as previously

is <u>almost</u> an A_∞-categ.

$\hookrightarrow$ issues:
- * $f_1 - f_0$ needs to be Morse ($\to hom(f_0, f_0) = ?$)
- * μ^k depends on pseudo-grad..

$\Bigg\}$ $\Rightarrow$ need to incorporate perturbations in the str ($\to$ later)

Dictionnary:

Morse theory	Floer theory
M	T^*M (sympl)
$f: M \to \mathbb{R}$	$\Gamma(df) \subset T^*M$ (Lagr) (exact...)
$\mathrm{Crit}(f_1 - f_0)$	$\Gamma(df_1) \cap \Gamma(df_0)$
$f_1 - f_0$ Morse	— $\pitchfork$ —
$\gamma : \mathbb{R} \to M$ flow line (take $X = \nabla^g(f_1 - f_0)$ for some metric g)	$u : \mathbb{R} \times [0,1] \to T^*M$ $(s,t) \mapsto (\underbrace{\gamma(s)}_{q}, \underbrace{df_0 + t(df_1 - df_0)}_{p})$ J_g (almost) J-hol strip.
$(T, \gamma : T \to M)$ flowtree	$u : \tilde{T} \to T^*M$ polygon
T: 	$\tilde{T}$: 
Rk: in pratice, $T_d \simeq \mathbb{R}^{d-1}/\mathbb{R}$	idea: $L_i^{\varepsilon} = \Gamma(\varepsilon df_i)$ and $\varepsilon \to 0$ strips become vertical

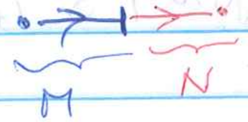
$\Phi : \overset{f}{M} \longrightarrow \overset{g}{N}$ smooth

$L_\Phi \subset T^*M \times T^*N \simeq (T^*M)^- \times (T^*N)$

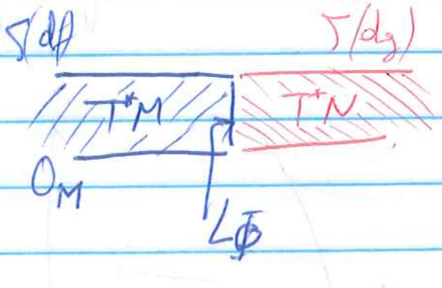
$(q,p) \to (q,-p)$

$\overset{\shortparallel}{N_{\Gamma(\Phi)}}$

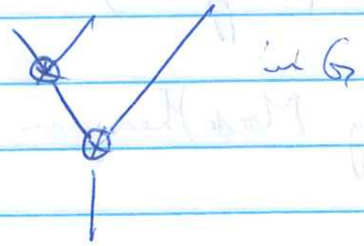
• grafted line

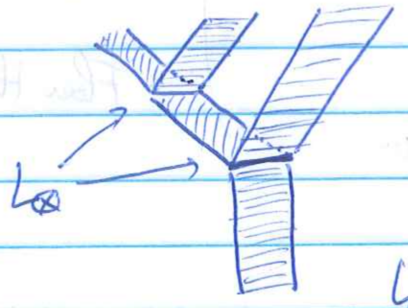


quilt



•





$m : G \times G \to G$

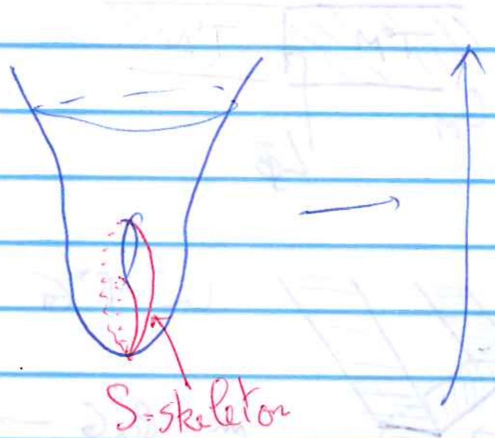
$L_\otimes = N_{\Gamma(m)} \subset (T^*G)^3$

Ref: Fukaya-Oh: Zero loop open strings in the cotangent bdle and Morse homotopy: identif moduli spaces of trees & polygons in T^*M.

Other appearances of trees: • Ekholm: Morse flow trees and Legendrian contact homology.

• Ongoing work (Nadler, Starkston, Eliashberg, Gavela, ...)

Weinstein domain: Liouville domain (W,ω,X) + Morse function f loc. ct on ∂W and s.t. X is gradient-like for f



idea: $W \approx T^*S$ (+ structure (S...)

Q: Can understand (wrapped?) Floer theory on W by Morse theory on S?...

Chap 3 - The Fukaya category

Naive def: (M,ω) symplectic,

- $Ob(\mathrm{Fuk}\,M)$: $L \subset M$ Lagrangians
- $hom(L_0, L_1) = CF(L_0, L_1) = \bigoplus_{x \in L_0 \pitchfork L_1} \mathbb{Z}\cdot x$
- μ^k counts J-hol polygons

... so that $D\pi(M) = H(\mathrm{Fuk}\,M)$

Issues: * $\overset{new}{}$ compactness phenomenon: "bubbling"

→ restrictions on M, L_i so that no bubbling (Seidel)

→ More G^{al} approach: FOOO

* no Morse index ⇒ $CF(L_0, L_1)$ not graded a priori

⇒ grading structure on L (Seidel)

* Orientations: Pin structures ...

* Perturbations, transversality ...

⇒ objects: "Lagrangian branes".

3.1 – Compactness (→ McDuff–Salamon, chap 5 for precise statements)

→ Bubbling

Ex: $C_\varepsilon = \{xy = \varepsilon\} \subset \mathbb{CP}^2$ (in the chart $[x:y:1]$ ...)

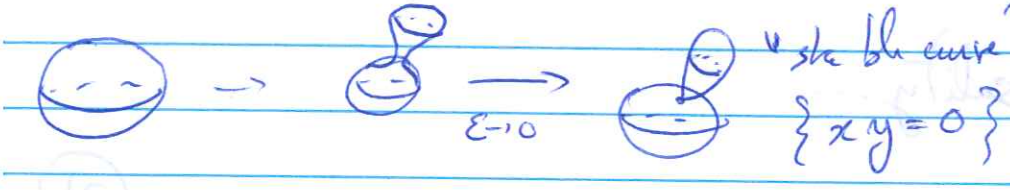
. $u_\varepsilon : \mathbb{CP}^1 \to \mathbb{CP}^2$
$z \longmapsto (z, \frac{\varepsilon}{z})$ $\xrightarrow[\text{simple conv. outside } 0]{}$ $\{ z \mapsto (z, 0) \}$

. $v_\varepsilon : \mathbb{CP}^1 \to \mathbb{CP}^2$
$z \longmapsto (\frac{\varepsilon}{z}, z)$ $\longrightarrow$ $\{ z \mapsto (0, z) \}$

What happens? when $\varepsilon \to 0$ $\|du_\varepsilon(0)\|_\infty \to \infty$

energy concentrates at zero

$\Rightarrow$ rescale: $z \longmapsto u_\varepsilon\left(\frac{z}{\|du_\varepsilon(0)\|_\infty}\right) \approx v_\varepsilon$ ... and see the bubble...



Thm [Gromov compactness]. M: compact (possib. with ∂)

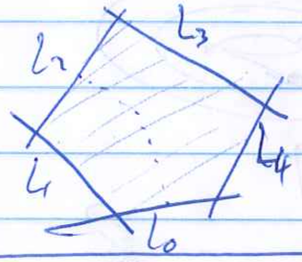
Σ: Riem. surf. with ∂ and corners ($\to$ polygon)

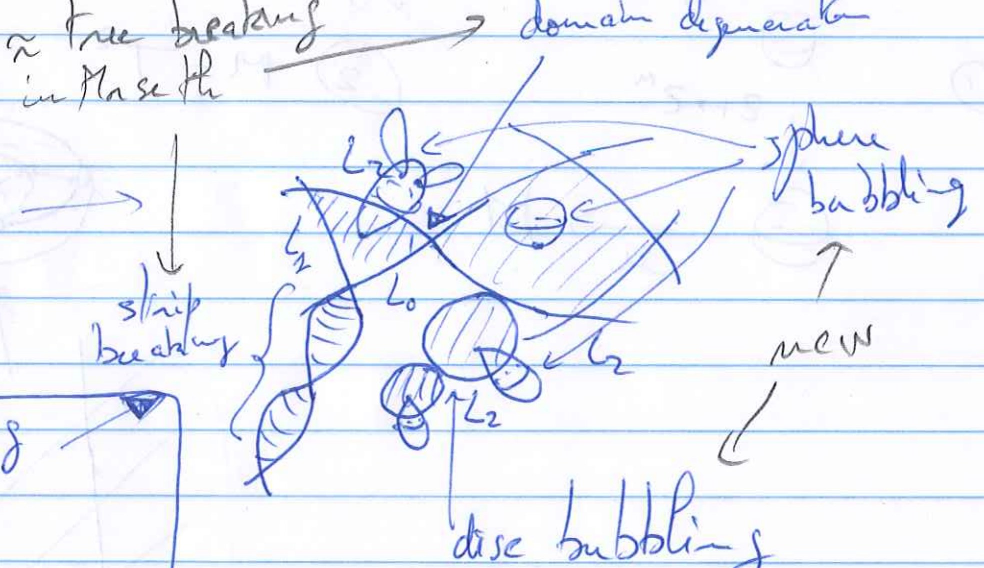
$u_n : (\Sigma, \partial^i \Sigma) \longrightarrow (M, L_i)$ seq. of J_n-hol curves, with bounded energy

$J_n \xrightarrow{n\to\infty} J \ (\in \mathcal{J}_c(M))$ $\quad \sup_n E(u_n) < \infty$

Then u_n converges to a stable curve in the Gromov topol.

ex:





Rk: picture is wrong

more realistic:

$\Rightarrow$ Assumption: (M,ω) exact: $\omega = d\lambda$

$\Rightarrow$ no sphere bubbles: if $u : \mathbb{C}P^1 \to M$,

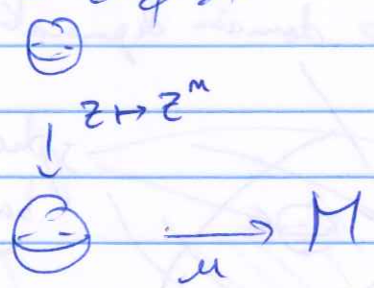
Then $E(u) = \int_{\mathbb{C}P^1} u^*(d\lambda) \underset{\text{Stokes}}{=} 0 \Rightarrow u$ constant.

$L \subset (M, \omega = d\lambda)$ exact: $\lambda|_L = df$

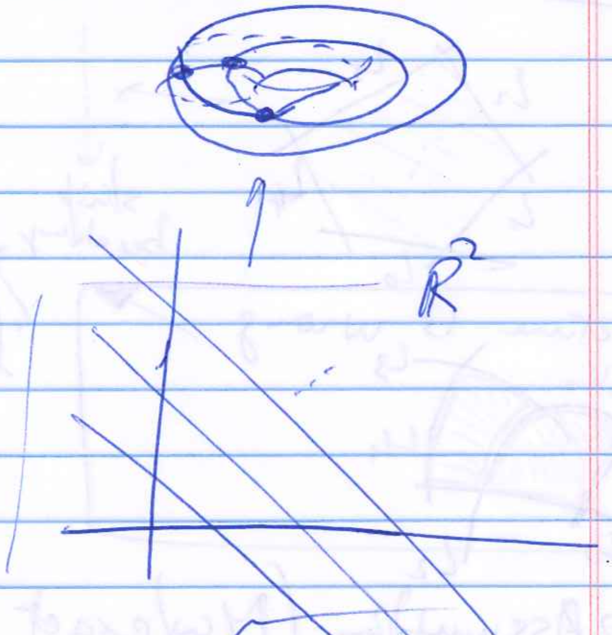
$\Rightarrow$ no disc bubbling:

$u : \mathbb{D} \to M$, $E(u) = \int_{\mathbb{D}} u^* d\lambda \underset{\text{stokes}}{=} \int_{\partial D} u^* \lambda = \int_{S^1} u^* df \underset{\text{stokes}}{=} 0$

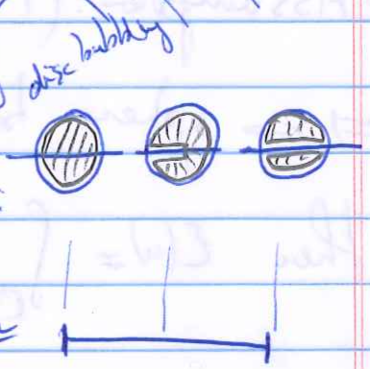
Rk: ~~[illegible]~~ Energy unbounded.

① 

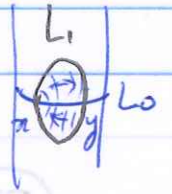
② $M = T^2$



Rk: can be fixed by using "Novikov rings"



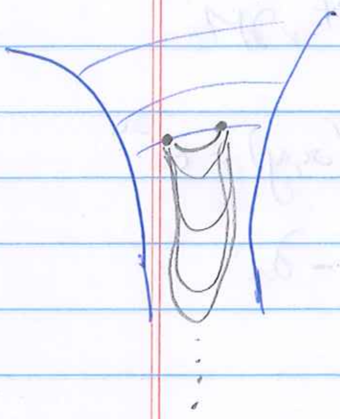
Rk: Disk bubbling can obstruct $\partial^2 = 0$:



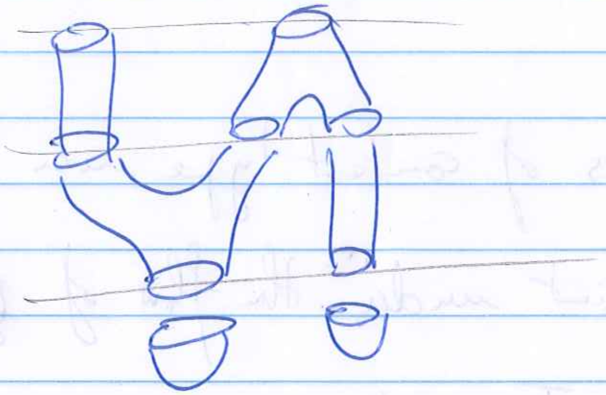
$\partial x = y$
$\partial y = x$

reason: $\overline{\mathcal{M}}^1(x,y) \simeq$

Rk: non-compactness: M = symplect. cob'



→ "SFT compactness" ...

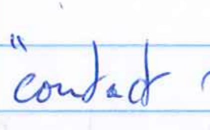


→ "contact type boundary" (≃ Liouville dom)

$(M, \phi_M, \theta_M, I_M)$

with ∂ mfd — sympl — $d\theta = \phi$ — ϕ-compat acs. $\left(\Rightarrow \partial M + \theta_{|\partial M} \text{ contact}\right)$

X_θ : negative Liouville v.f. : $\phi(\cdot, X_\theta) = \theta$



~~Def~~

$h_M : U\partial M \to [0,1]$

def by $\begin{cases} h_M|_{\partial M} = 1, \text{ and} \\ X_\theta \cdot h_M = -h_M \end{cases}$

complet:

$(M, \phi, \theta, \ ) \to (\hat{M}, \hat{\phi}, \hat{\theta}) = M \cup$ posit. symplectiz $\mathbb{R}^+ \times \partial M$

$h_{\hat{M}}(r,y) = e^r$

$\rightsquigarrow X = -\partial_r$

Def: J is of contact type near ∂M if

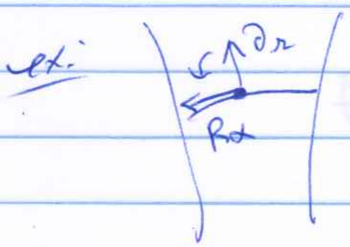
(1) invariant under the flow of X_{θ_M}

(2) $dh_M \circ J = -\theta_M$

Rk: (2) $\Rightarrow$ J restricts to the contact structure $\xi = \ker \theta|_{h^{-1}(c)}$:

$v \in \xi : \begin{cases} \theta(v) = 0 \\ dh.v = 0 \end{cases} \rightsquigarrow \begin{array}{l} \theta(Jv) = dh.v = 0 \\ dh.(Jv) = -\theta(v) = 0 \end{array}$

ex:



on the end:

Reeb

$$T\hat{M} = \xi \oplus \mathbb{R}.\partial_r \oplus \mathbb{R} R_\theta$$

$$\rightsquigarrow J = J_\xi \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Rk: J contact type $\rightsquigarrow \hat{J}$ on $\hat{M}$ ...

Lemma: $\hat{J}$ as before, $u : S \to [1,\infty) \times \partial M \subset \hat{M}$ $\hat{J}$-hol,

then $\rho = h \circ u : S \to \mathbb{R}$ has no local max.
(unless loc. constant...)

$\Rightarrow$ if $u(\partial S) \subset \mathring{M}$, then $u(S) \subset \mathring{M}$.
(for $u : S \to \hat{M}$)

Proof: max principle:

$$\Delta \rho = -d(d\rho \circ j) \quad (j : \text{cpx str} / S)$$

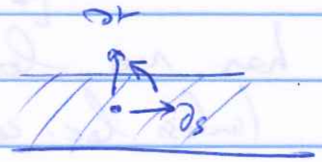
$$= -d(dh \circ J \circ du) \quad (u \ J\text{-hol})$$

$$= d(\theta \circ du) = u^*\phi \geq 0 \quad \square$$

$\to$ can then apply Gromov compactness...

3.2 – Transversality

$S = \mathbb{R} \times [0,1]$ with $j \partial_s = \partial_t$



(= Z in Seidel's book) (s,t)

$u : (S, j) \to (M, J)$ pseudo-hol $\Leftrightarrow du(j\partial_s) = J\, du(\partial_s)$

$\Leftrightarrow \partial_t u = J \partial_s u$

$\Leftrightarrow \partial_s u + J \partial_t u = 0$

C.R. eq.

↳ Perturb by varying $J \in \mathcal{J}(M,\omega)$?

pb1: if $L_0 \cap L_1$ not transverse, will never be enough

pb2: unlike gradient flow lines, J-hol curves might not be injective (ex: multiple covers...)

pb1: perturb $L_1 \rightsquigarrow \varphi_H^1(L_1)$ by Hamiltonian isotopy (of which HF is invariant)

$\Leftrightarrow$ perturb the equation:

$$\begin{cases} \partial_s v + J\partial_t v = 0 \\ v(s,0) \in L_0,\ v(s,1) \in \varphi_H^1(L_1) \end{cases}$$

$$\Updownarrow \xleftarrow{\quad} v(s,t) = \varphi_H^t(u(s,t))$$

$$\begin{cases} \partial_s u + J(\partial_t u - X_H) = 0 \quad : \text{"}\underline{\text{Floer equation}}\text{"} \\ u(s,0) \in L_0,\ u(s,1) \in L_1 \end{cases}$$

<u>pb 2</u>: use domain-dependent a.c.s:

$J \in \mathcal{J}(M,\omega) \rightsquigarrow J \in C^\infty(S, \mathcal{J}(M,\omega))$

$\rightsquigarrow \partial_s u + J(u).(\partial_t u - X_H) = 0$

$\Rightarrow$ Pb 2': want $\underset{\mathrm{Aut}(S)}{G} \circlearrowright S \rightsquigarrow G \circlearrowright M$

- $\underset{(s,t)}{S = \mathbb{R} \times [0,1]}$, $G = \mathbb{R}$-transl $\Rightarrow J \in C^\infty([0,1], \mathcal{J}(M,\omega))$ ✓

- $S =$  : $G \circlearrowright \mathring{S}$ transitive : <u>real issue</u>

(⇒ virtual perturbations, Kuranishi str...)

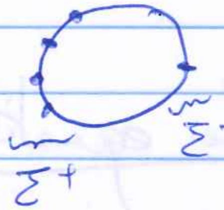
→ Seidel, sec 8

$\hat{S}$ = cpct Riem. surf. $\Sigma = \Sigma^- \cup \Sigma^+ \subset \partial \hat{S}$ finite

Σ^- ↗ incoming punctures

Σ^+ ↑ outgoing punctures

$S = \hat{S} \setminus \Sigma$



$\rightsquigarrow \mu^k : CF \otimes \cdots CF \longrightarrow CF$ (cohomology)

Lagrangian labels: $C \subset \partial S$ connected comp $\rightsquigarrow L_C$

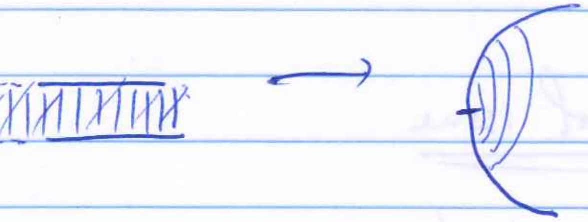
$\zeta \in \Sigma \to C_{\zeta,0}, C_{\zeta,1}$

Convention: Σ^-:

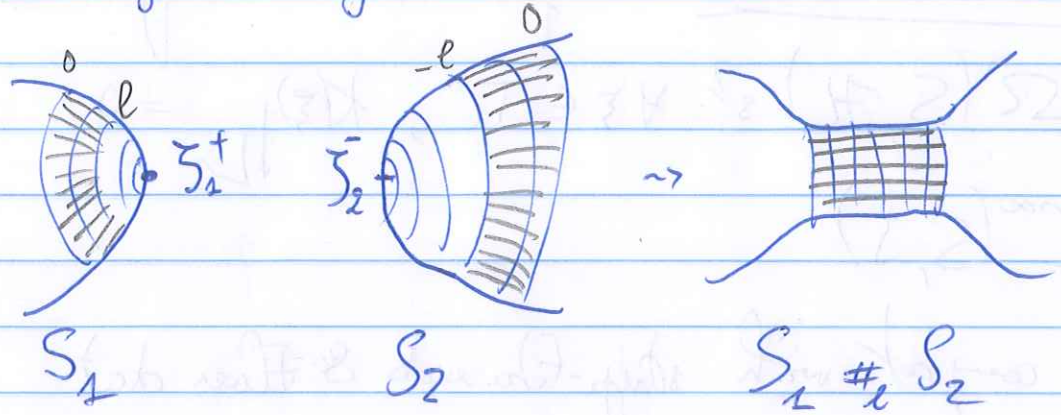
, Σ^+:

<u>Strip-like ends:</u> $Z^\pm = \mathbb{R}^\pm \times [0,1]$

$\zeta \in \Sigma^\pm \to \epsilon_\zeta : Z^\pm \longrightarrow S$ holomorphic, st $\begin{cases} \epsilon_\zeta^{-1}(\partial S) = \mathbb{R}^\pm \times \{0,1\} \\ \lim\limits_{s \to \pm\infty} \epsilon_\zeta(s,\cdot) = \zeta \end{cases}$



→ Can glue surfaces:



<u>Floer data</u> / <u>perturb. data</u>

I : of exact type near ∂M : <u>fixed</u>.

- $\mathcal{J} = \{ J \in \mathcal{J}(M, \phi) \mid \underset{\substack{\uparrow \\ \text{some nbd.}}}{J_{|\cup \partial M}} \equiv I \}$
- $C^\infty(T, \mathcal{J}) = \{ \{J(t)\}_{t \in T},\ J(t) \in \mathcal{J} \}$.
- $\mathcal{H} = C^\infty_c(\mathrm{int}(M), \mathbb{R}) \to C^\infty(T, \mathcal{H})$.

<u>Def</u>: * $L_0, L_1 \subset M$ exact lag. A <u>Floer datum</u> for (L_0, L_1)

$= (H, J) \in C^\infty([0,1], \mathcal{H}) \times C^\infty([0,1], \mathcal{J})$ s.t.

$\phi^1(L_0) \pitchfork L_1$

- $S + \underset{\text{Lag labels}}{\{L_C\}} + \underbrace{\epsilon_\xi}_{\text{strip-like ends}} + \underset{\text{Floer datum}}{(H_\xi, J_\xi)}$ $\forall \xi \in \Sigma$

A perturbation datum for S is a pair (K, J),

- $K \in \Omega^1(S, \mathcal{H})$ s.t. $\forall \xi \in TC$, $K(\xi)|_{L_C} = 0$
- $J \in C^\infty(S, \mathcal{J})$

that are compat. with strip-like ends & Floer dat.:

$$\epsilon_\zeta^* K = H_\zeta(t)\,dt \quad , \quad J(\epsilon_\zeta(s,t)) = J_\zeta(t)$$

If $S = Z$ strip, take on the whole Z, so to have $\frac{\partial}{\partial s}$-translation invariance.

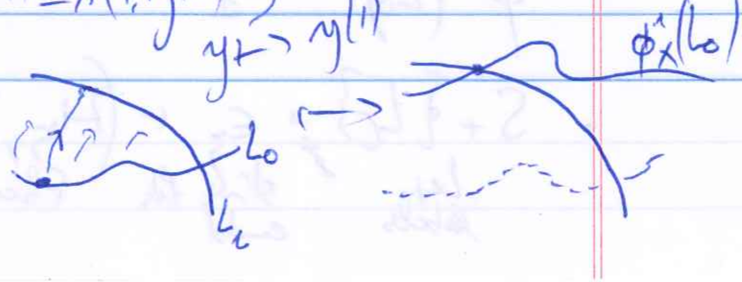
- Inhomog. pseudo-hol. maps

$L_0, L_1 \subset M$ exact compact Lag., (H, J) Floer datum
($\rightsquigarrow$ pert. datum $/ Z$)

Generators of chain cpx: $X = \nabla^\omega H$

$$\mathcal{C}(L_0, L_1) = \left\{ y : [0,1] \to M \;\middle|\; \begin{array}{l} y(0) \in L_0 \\ y(1) \in L_1 \\ y'(t) = X(t, y(t)) \end{array} \right\} \xleftrightarrow{1:1} \phi_X^1(L_0) \cap L_1$$

(chords of X)



Floer's eq: $u : Z \longrightarrow M$ smooth

$$(Fl) \begin{cases} \partial_s u + J(t,u)\left(\partial_t u - X(t,u)\right) = 0 \\ u(s,i) \in L_i \ , \ i = 0,1 \end{cases}$$

$$\rightarrow \mathcal{M}_Z(y_0, y_1) = \left\{ u \;\middle|\; (Fl) + u(s,\cdot) \begin{matrix} \xrightarrow{s\to-\infty} y_0 \\ \xrightarrow[s\to+\infty]{} y_1 \end{matrix} \right\} \circlearrowleft \mathbb{R}$$

s-shifts: ~~free~~

$\mathbb{R} \circlearrowleft \mathcal{M}_Z$ free, except when $y_0 = y_1$, in which case $\mathcal{M}_Z = \{\text{constant sol}^\circ\}$
$u(s,t) = y_0(t)$

$$\rightarrow \mathcal{M}_Z^*(y_0, y_1) = \begin{cases} \mathcal{M}_Z(y_0,y_1)/\mathbb{R} & y_0 \neq y_1 \\ \emptyset & \text{if } y_0 = y_1 \end{cases}$$

→ More G'd S (polygon ...)
+ decorat° $\{L_C\}$ + pert datum ...
(+ Floer dat/ends ...)

$$K \in \Omega^1(S, H) \rightarrow Y \in \Omega^1(S, C^\infty(TM))$$
$$Y(\xi) = \nabla^\omega K(\xi)$$

Inhomog. pseudo-hol map eq: $u : S \to M$ smooth:

$$\begin{cases} Du(z) + J(z,u) \circ Du(z) \circ I_S = Y(z,u) + J(z,u) \circ Y(z,u) \circ I_S \\ u(C) \subset L_C, \quad C \subset \partial S. \end{cases}$$

$$\Longleftrightarrow (Du - Y)^{0,1} = 0$$
$$\Leftrightarrow \bar{\partial}_J(u) = Y^{0,1}$$

→ solutions sets: $\mathcal{M}_S((y_\xi)_\xi)$

* Floer vs ∞^{te}-dim Morse

Rk: Lagrangian Floer homol $\simeq$ Morse homol. for the action functional

$$\mathcal{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$$
$$\overset{\shortparallel}{\left\{ y : [0,1] \to M \atop i \mapsto L_i \right\}}$$

L_0, L_1 exact: $\theta|_{L_k} = dh_k$, for some $h_k : L_k \to \mathbb{R}$

Let $\mathcal{A}_H : \mathcal{P}(L_0, L_1) \to \mathbb{R}$

$$y \mapsto \int_0^1 \left(-y^*\theta + H(t, y(t)) dt \right) + h_{L_1}(y(1)) - h_{L_0}(y(0))$$

Rk: * assume $u : \blacktriangledown \to M$ st
and $H = 0$ say...

stokes $\Rightarrow \mathcal{A}(\gamma) = A_\omega(u)$

* Crit $\mathcal{A}_H = \mathcal{C}(L_0, L_1)$

* $\gamma : \mathbb{R} \to \mathcal{P}(L_0, L_1) \leftrightarrow u : \mathbb{R} \times [0,1] \to M$

$\dot{\gamma} = -\nabla \mathcal{A}_H \iff$ Floer eq / u

$u : S \to M$ with l.b.c

(pert) energy: $E(u) = \int_S \frac{1}{2} \| Du - Y \|^2 \, d\mathrm{vol}_S$

(pert) sympl. area: $A(u) = \sum_{\xi \in \Sigma^\pm}' \mp A_{H_\xi}(\gamma_\xi) - \int_S R(z, u(z))$

$\to E(u) = A(u)$ iff u is perturb. J-hol...

curvature of K:

$R \in \Omega^2(S, \mathcal{H})$

$= (\partial_s K(\partial_t) - \partial_t K(\partial_s) - \{K(\partial_s), K(\partial_t)\}) \, ds \wedge dt$

Gromov Compactness still applies, eventual bubbles would be (unperturbed) J-hol, for domain-indep^t J...

$$\begin{tikzcd}(E, \phi_E) \arrow[d] \\ S\end{tikzcd}$$

<u>Cauchy-Riemann op.</u> $(E, \phi_E) \to S$ symplectic v.b

(typically, $E = u^*TM$ ...) + J_E acs/E

$F \subset E_{|\partial S}$: lag sub-bundle (u^*TL_C ...)

Recall: $\underbrace{\Omega^1(S)\otimes\mathbb{C}}_{} = (\Omega^{1,0}\oplus\Omega^{0,1})$

if $s=z$ $\quad \mathbb{C}ds\oplus\mathbb{C}dt = \mathbb{C}dz \oplus \mathbb{C}d\bar{z}$

on vectors: $\partial_z = \partial_s + i\partial_t$

$\partial_{\bar{z}} = \partial_s - i\partial_t$ ↔ the opposite...

→ dual basis: $dz = \frac{1}{2}(ds - idt)$ ⟹ proj $\Omega^1 \to \Omega^{0,1}$

$d\bar{z} = \frac{1}{2}(ds + idt)$ $\quad \underset{(=Du\dots)}{\alpha} \mapsto \alpha(\partial\bar{z}).d\bar{z}$

∇ : connexion on E ≃ covariant derivative $\nabla : C^\infty(S,E,F) \to C^\infty(S, \Omega^1_S \otimes_{\mathbb{C}} E)$

⇝ Cauchy-Riemann operator $\bar{\partial}_\nabla = \nabla^{0,1} : C^\infty(S,E,F) \to C^\infty(S, \Omega^{0,1}_S \otimes_{\mathbb{C}} E)$

$u \mapsto \bar{\partial}u = Du(\partial\bar{z}).d\bar{z}$

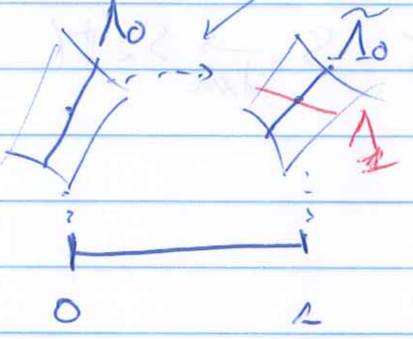
⇝ Fredholm op. $\bar{\partial}_\nabla : W^{1,p}_{loc}(S,E,F) \to L^p_{loc}(S, \Omega^{0,1}_S \otimes E)$ if S is <u>compact</u>

→ when strip like ends..

$$\zeta \in \Sigma \rightsquigarrow \underbrace{\begin{matrix} E_\zeta \\ \downarrow \\ [0,1] \end{matrix} + \left(\phi_{E_\zeta},\ J_{E_\zeta},\ \nabla_\zeta,\ \underset{\text{Lag},\ k=0,1}{\Lambda_{\zeta,k} \subset E_{\zeta,k}}\right)}_{\text{"limiting datum" at } \zeta}$$

↳ non-deg ⟺ $\tilde{\Lambda}_0 \pitchfork \Lambda_1$

transport along ∇_ζ



$\bar{\partial}_J$ is "admissible"

→ assume $\epsilon_\zeta^* E \simeq \pi^{\pm} E_\zeta$, $\pi^{\pm} : Z^{\pm} \to [0,1]$

+ everything matching asymptotically

↳ i.e. ~~modulo~~ modulo sth. decaying expon. fast.

→ Linearizat^n

$$\begin{tikzcd} \mathcal{E}_S \arrow[d] \\ \mathcal{B}_S \arrow[u, bend right, "\bar{\partial}-\nu"'] \end{tikzcd}$$

$\mathcal{B}_S \supset \mathcal{M}_S = (\bar{\partial}-\nu)^{-1}(0)$

- $\mathcal{B}_S = \left\{ u : S \to M \;\middle|\; \begin{matrix} \cdot\ u \in W^{1,p}_{loc} \\ \cdot\ u \xrightarrow[W^{1,p}]{} y_\zeta \end{matrix} \right\}$

$T_u \mathcal{B}_S = W^{1,p}(S, \underset{= u^*TM}{\underset{\uparrow}{E}}, \underset{u^*TL}{\underset{\nwarrow}{F}})$

a.c.s/Σ : $J_\Sigma(z) = J(z, u(z))$

- $(\mathcal{E}_S)_u = L^p(S, \Omega_S^{0,1} \otimes u^*TM)$

prop: $u \in \mathcal{M}_S$, $D_u(\bar\partial - \nu) : T_u\mathcal{B}_S \to (\mathcal{E}_S)_u$
is a $\bar\partial_\nabla$ as before, for some ∇

pf: - how to construct ∇:

· $\tilde u : S \xrightarrow[id \times u]{} S \times M$ → take $Z \in C^\infty(S \times M, T^*S \otimes TM)$

s.t. · $Z(z, u(z)) = du(z)$ (extend du to $S \times M$)

· $(Z - Y)^{01} = 0$

~~ξ~~ → $\nabla X := \tilde u^* [\underset{T^*S \otimes TM}{Z}, \underset{TM}{\tilde X}]$ for $\tilde X \in C^\infty(S \times M, TM)$ s.t. $X = \tilde u^* \tilde X$

why it works: take $\tilde\nabla^0$: torsion free conn on $\begin{matrix} TM \\ \downarrow \\ S \times M \end{matrix}$, $\nabla^0 = \tilde u^* \tilde\nabla^0$ on E.

↳ $[X,Y] = \tilde\nabla^0_X Y - \tilde\nabla^0_Y X$

$\xi \in \mathfrak{X}(S)$, $\nabla_\xi X = \left(\tilde u^* [Z, \tilde X]\right)(\xi)$

$$= \tilde u^*\left(\tilde\nabla^0_{\underset{Z(\xi)}{\underset{\shortparallel}{du(\xi)}}} \tilde X - \tilde\nabla^0_{\tilde X} \langle Z, \xi \rangle\right) = \nabla^0_\xi X - \tilde u^*\left(\tilde\nabla^0_X \langle Z, \xi \rangle\right)$$

$\to$ $z = s + it$, $\quad \to = \frac{1}{2}\left(\nabla_{\partial_s} X + J \nabla_{\partial_t} X\right)$

$$(\nabla X)^{0,1} = \frac{1}{2}\left(\nabla^0_{\partial_s} X + J \nabla^0_{\partial_t} X - \widehat{\nabla}^0_X \langle Z, \partial_s \rangle - J \widetilde{\nabla}^0_X Z(\partial_t)\right)$$

$$= \frac{1}{2}\left(\nabla^0_{\partial_s} X + J \nabla^0_{\partial_t} X + \tilde{u}^*(\widetilde{\nabla}^0_X J) du(\partial_t)\right)$$

$$- \frac{1}{2}\left\{\tilde{u}^* \nabla^0_X \left(\psi(\partial_s) + J \psi(\partial_t)\right)\right.$$

(Room for comments/questions)

From last time:

Def: $\Lambda_0, \Lambda_1 \in LGr(n)$ s.t. $\Lambda_0 \pitchfork \Lambda_1$

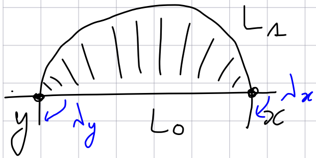
$\exists A \in Sp(2n)$ mapping $\begin{cases} \Lambda_0 \to \mathbb{R}^n \\ \Lambda_1 \to i\mathbb{R}^n \end{cases}$

Let $\Lambda_t := A^{-1}\left(e^{-\frac{i\pi t}{2}}\mathbb{R}\right)$, $0 \leq t \leq 1$

"Canonical short path" from Λ_0 to Λ_1: htpy class is indep^t on A.

• Maslov index of a strip

$u : \mathbb{R}\times[0,1] \longrightarrow (M, L_0, L_1)$



- λ_x : c.s.p $T_xL_0 \to T_xL_1$
- λ_y : c.s.p $T_yL_0 \to T_yL_1$
- ℓ_0, ℓ_1 corresp. to

$u^*_{\mathbb{R}\times\{i\}} TL_i$, going from $+\infty$ to $-\infty$

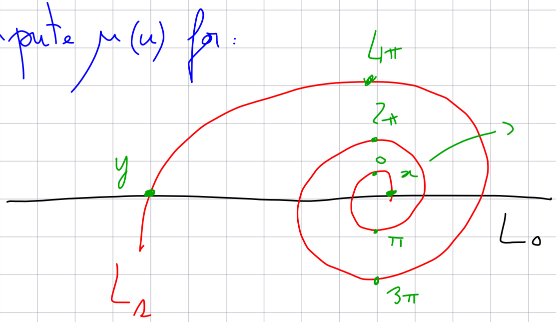
Def: $\mu(u) := \mu\left((-\ell_0)\#\lambda_x\#\ell_1\#(-\lambda_y)\right)$

ex: in picture above:

$$0 \xrightarrow[-\ell_0]{} 0 \xrightarrow[\lambda_x]{} -\pi/2 \xrightarrow[\ell_1]{} \frac{\pi}{2} \xrightarrow[-\lambda_y]{} \pi$$

$\Rightarrow \underline{\mu(u) = 1}$.

ex: compute $\mu(u)$ for:

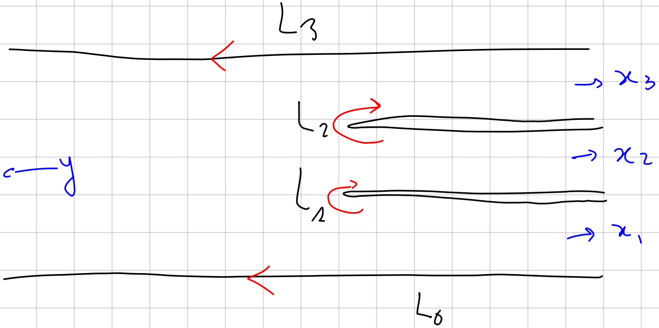


$$0 \xrightarrow[-\ell_0]{} 0 \xrightarrow[\lambda_x]{} -\frac{\pi}{2} \xrightarrow[\ell_1]{} 4\pi + \frac{\pi}{2} \xrightarrow[\lambda_y]{} 5\pi$$

$\rightarrow \mu(u) = 5.$

. . .

• (k+1)-gons



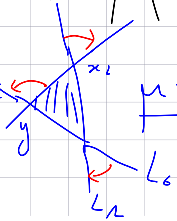
λ_{x_i} : c.s.p $T_{x_i} L_{i-1} \longrightarrow T_{x_i} L_i$

λ_y : c.s.p $T_y L_0 \longrightarrow T_y L_k$

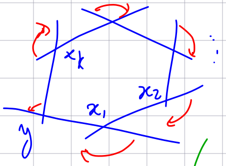
ℓ_i corresp. to L_i, oriented as in the picture

$\Rightarrow \mu(u) = \mu\left((-\ell_0) \# \lambda_{x_1} \# \ell_1 \# \cdots \# \lambda_{x_k} \# \ell_k \# (-\lambda_y) \right)$

Ex:



$\mu = 0$



$\rightarrow \mu(u) = 2 - k$
$= -\dim T_k$

($\Rightarrow$ dim of family = 0 ...)

- **Gradings** (recall: want $\mu(u) = \mathrm{ind}(y) - \mathrm{ind}(x)$)

Idea (Seidel): lift to the universal cover $\widetilde{LGr}(n) = \left\{ (\lambda, [\gamma]) \;\middle|\; \begin{array}{l} \lambda \in LGr(n) \\ [\gamma] : \lambda_0 \to \lambda \text{ htpy class of path } \gamma \end{array} \right\}$

λ_0: fixed

$$\begin{tikzcd} \mathbb{Z} = \pi_1(LGr_n) \arrow[r] & \widetilde{LGr}(n) \arrow[d] \\ & LGr(n) \end{tikzcd}$$

$$\begin{tikzcd} (M,\omega) \arrow[r, squiggly] & TM \arrow[d] \arrow[r, squiggly] & LGr(n) \arrow[r, hook] & LGr(TM) \arrow[d] \\ & M & & M \end{tikzcd}$$

(1) $\boxed{\text{Assume } 2c_1(TM) = 0}$ so to lift this bdle:

$\Leftrightarrow$ $(\det_{\mathbb{C}} T^*M)^{\otimes 2}$ is trivial

↳ Θ nowhere vanishing sec.

↳ phase function

$\varphi : TM \to \mathbb{R}/2\pi\mathbb{Z}$

$\varphi(\Lambda) = \arg(\Theta|_{\Lambda})$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \widetilde{LGr}(n) \arrow[r, hook] \arrow[d] & \widetilde{LGr}(TM) \arrow[d] & \\ & LGr(n) \arrow[r, hook] & LGr(TM) \arrow[r] \arrow[d] & \mathbb{R}/2\pi\mathbb{Z} \\ & & M & \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \widetilde{LGr(n)} \arrow[r, hook] \arrow[d] & \widetilde{LGr(TM)} \arrow[d] & \\
& LGr(n) \arrow[r, hook] & LGr(TM) \arrow[d] & \\
& & M & L \arrow[l, hook'] \arrow[ul, "s"'] \arrow[uul, dashed, "\tilde{s}"']
\end{tikzcd}$$

$L \xhookrightarrow{i} M$ Lagrangian

$\rightarrow$ TL defines a section s of $i^*LGr(TM)$

(2) • Say L is *gradable* if $\exists\, \tilde{s}$ lifting s to $\widetilde{LGr}(TM)$: $\tilde{s}$ is a *grading* of L, gradings differ by $\mathbb{Z}$ ....

• A *graded Lagrangian* is a pair $(L, \tilde{s})$

Rk: • L gradable $\iff$ $\mu_L = 0 \in H^1(L;\mathbb{Z}) \simeq \hom(\pi_1 L, \mathbb{Z})$

$\uparrow$

Maslov class: $\pi_1 L \longrightarrow \mathbb{Z}$

$[\gamma] \longmapsto \mu(T_{\gamma(\cdot)} L)$

* Phase lifts to $\mathbb{R}$:

$$\begin{tikzcd}
\mathbb{R} \arrow[d] & \widetilde{LGr(TM)} \arrow[l] \arrow[d] & \\
\mathbb{R}/2\pi\mathbb{Z} & LGr(TM) \arrow[l] \arrow[d] & \\
& M & L \arrow[l] \arrow[ul, "s"'] \arrow[uul, dashed, "\tilde{s}"'] \arrow[ull, "\varphi_L"] \arrow[uull, "\tilde{\varphi}_L"]
\end{tikzcd}$$

* Grading on $CF(L_0,L_1)$

Assume L_0, L_1 are graded,

let $x \in L_0 \pitchfork L_1$. $\Lambda_i := T_x L_i \subset T_x M$

$$\begin{tikzcd}
\widetilde{LGr}\ \{ & \widetilde{\Lambda}_0 \arrow[rr, bend left, "\gamma\ \text{unique}/\text{htpy}"] & {} \arrow[d, squiggly] & \widetilde{\Lambda}_1 \\
& & {[\gamma]} & \\
LGr\ \{ & \Lambda_0 \arrow[uu, dotted] \arrow[rr, bend left] \arrow[rr, bend right, "\text{canon. short path}"'] \arrow[loop right, "\alpha"] & & \Lambda_1 \arrow[uu, dotted]
\end{tikzcd}$$

$\rightarrow \boxed{\deg(x) := \mu(\alpha)}$

$(= \deg_{L_0,L_1}(x))$

Rk: Shifts: $\mathbb{Z} \simeq \pi_1(LGr(n)) \circlearrowright \widetilde{LGr}(n)$

→ acts on gradings $\tilde{s}$

$\tilde{L} = (L, \tilde{s}) \rightsquigarrow \tilde{L}[k] = (L, -k + \tilde{s})$ $(= S^k \tilde{L})$

→ $CF(\tilde{L}_0[k_0], \tilde{L}_1[k_1]) = CF(\tilde{L}_0, \tilde{L}_1)[k_1 - k_0]$

Rk: C cochain cpx → $C[k]$: $C[k]^i = C^{i+k}$

$C = (\cdots C^i \to C^{i+1} \to \cdots)$

Warning: degrees are shifted to the left:

	0	1	2	3	4
C	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$
$C[1]$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Rk: * If $2c_1(TM)$ is N-torsion

$$\begin{tikzcd} \cancel{\widetilde{LGr}} \arrow[r, squiggly] & \mathbb{Z}/N\mathbb{Z} \arrow[r, leftrightarrow] & LGr^N \arrow[d] \\ & & LGr \end{tikzcd}$$

⇒ CF graded over $\mathbb{Z}/2N\mathbb{Z}$

→ Seidel, Graded Lagrangian submanifolds

* If L_0, L_1 oriented → $\mathbb{Z}/2\mathbb{Z}$ grading:

compare orientat° of (T_xM, ω_x^n) with or. of $T_xL_0 \oplus T_xL_1$

...

prop: L_0, L_1 graded,

$$\deg_{L_0,L_1}(x) + \deg_{L_1,L_0}(x) = n \left(= \tfrac{1}{2}\dim M\right)$$

proof:

$$\begin{tikzcd}
\tilde{\Lambda}_0 \arrow[r] \arrow[rr, bend left, "\text{constant}"] & \tilde{\Lambda}_1 \arrow[r] & \tilde{\Lambda}_0 \\
\Lambda_0 \arrow[r] \arrow[u, dash, dashed] \arrow[rr, bend right, "\uparrow"'] & \Lambda_1 \arrow[r] \arrow[u, dash, dashed] & \Lambda_0 \arrow[u, dash, dashed]
\end{tikzcd}$$

$$\left.\begin{array}{l}[0,1] \longrightarrow LGr_n \\ t \longmapsto e^{-i\pi t}\mathbb{R}^n\end{array}\right\} \mu = -n$$

$$\begin{tikzcd}
\widetilde{LGr}\ \{ & \tilde{\Lambda}_0 \arrow[rr, bend left, "\gamma\quad \text{qu./htpy}"] & {} \arrow[d, squiggly] & \tilde{\Lambda}_1 \\
 & & {[\gamma]} & \\
LGr_n\ \{ & \Lambda_0 \arrow[uu, dotted] \arrow[rr, bend left] \arrow[rr, bend right, "\text{canon. short path}"'] \arrow[loop right, red, "\alpha"] & & \Lambda_1 \arrow[uu, dotted]
\end{tikzcd}$$

Rk: Maslov index vs Morse index

$f_0, f_1 : Q \to \mathbb{R}$ s.t. $f_1 - f_0$ is Morse $\to$ $\begin{cases} M = T^*Q \\ L_0 = \Gamma(df_0) \\ L_1 = \Gamma(df_1) \end{cases}$

Obs: $\forall\, x = (q,p) \in L_i$, $T_x L_i \in LGr^{\pitchfork}$

$$LGr^{\pitchfork} = \left\{ \Lambda \in LGr(T_xM) \,/\, \Lambda \pitchfork T_qQ \right\}$$

and $LGr^{\pitchfork}$ contractible

$\to$ Fix a component $E \subset \pi^{-1}(LGr^{\pitchfork}) \subset \widetilde{LGr}(TM)$,

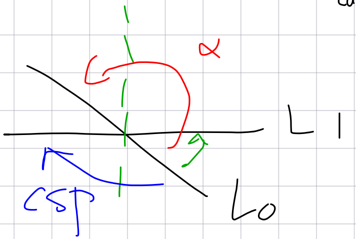
and grade L_i with $\tilde{s}_i$ s.t. $\mathrm{Im}(\tilde{s}_i) \subset E$

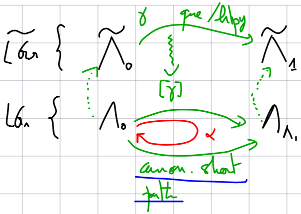
$\to$ then $\boxed{\deg_{L_0,L_1}(q,p) = \underset{\substack{\text{Morse ind.} \\ \text{for } f_0 - f_1}}{\mathrm{ind}\, q}}$ for $(q,p) \in L_0 \cap L_1$

ex: $f_1 \equiv 0$

* $f_0 = \cap$

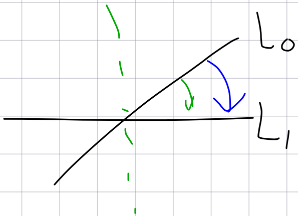
ind = 1





* $f_0 = \cup$

ind = 0



Th: [Riemann – Roch]

with strip-like ends, Floer datum, etc.

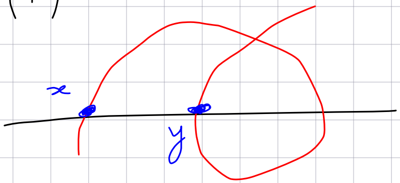
$u : D^2 \setminus \{k+1 \text{ pt's}\} \to (M; L_0, \dots L_k)$ perturbed J-hol.

$D_u \bar{\partial}$: Linearized Cauchy-Riemann operator is Fredholm,

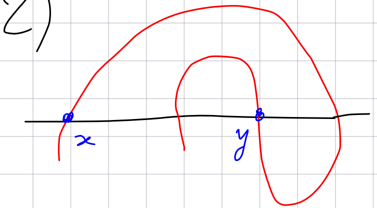
and $\operatorname{ind} D_u \bar{\partial} = \mu(u)$.

In the following examples, describe explicitly $\mathcal{M}(x,y)$: check that they are of dimension $\mu - 1$, and that Gromov compactness holds

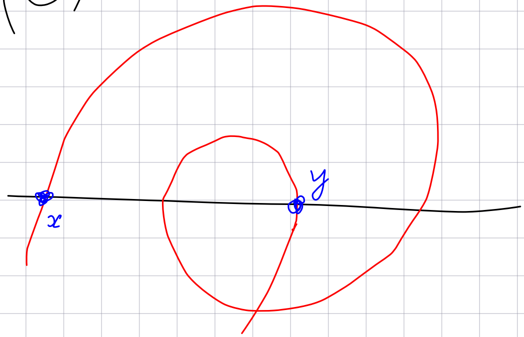
(1)



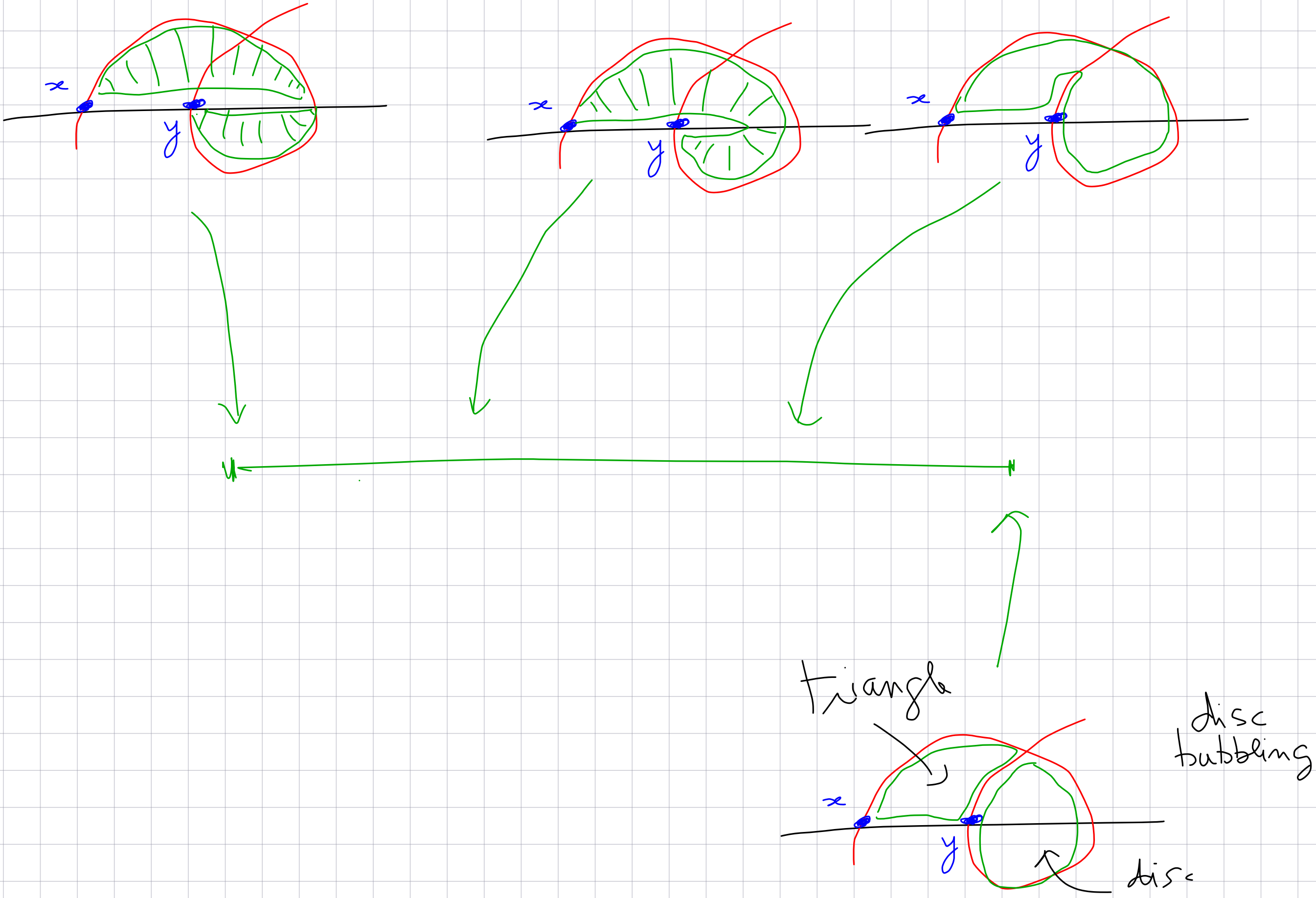
(2)



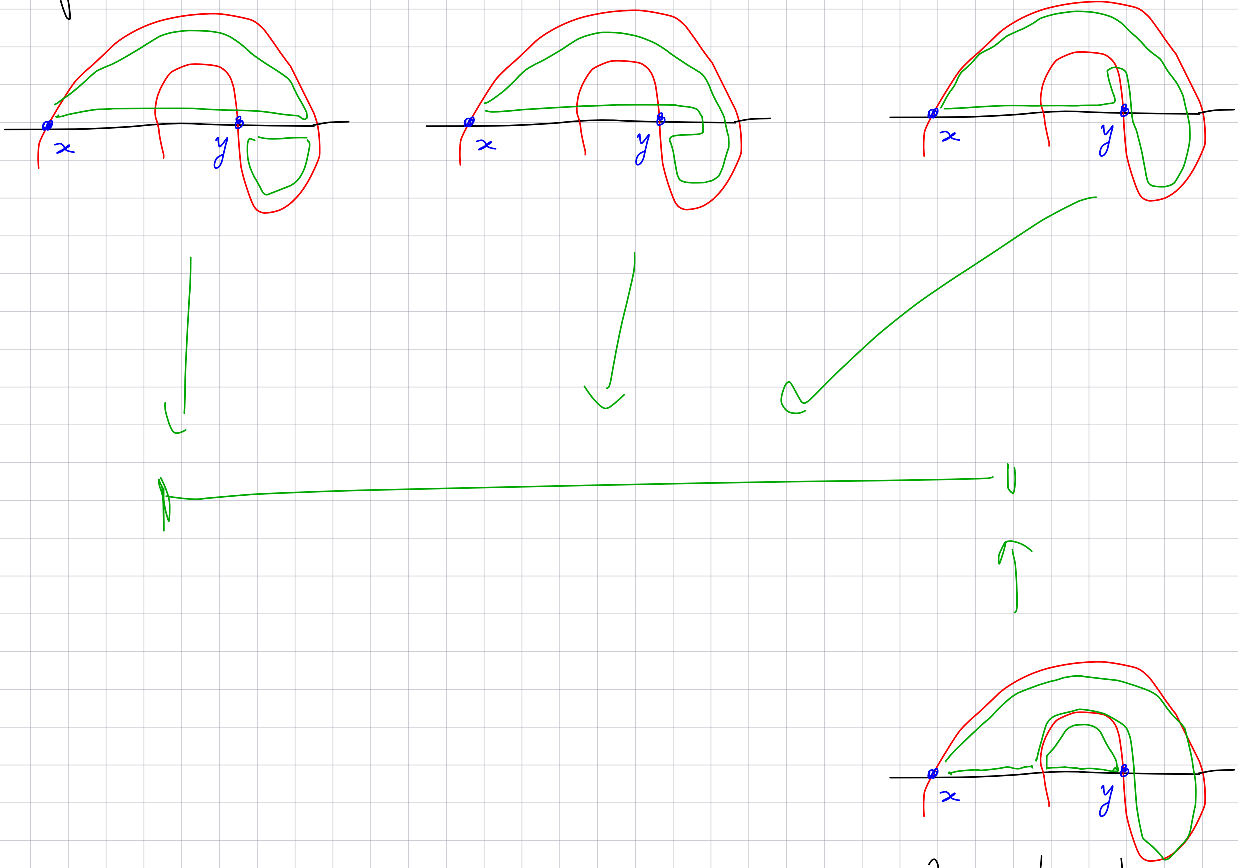
(3)



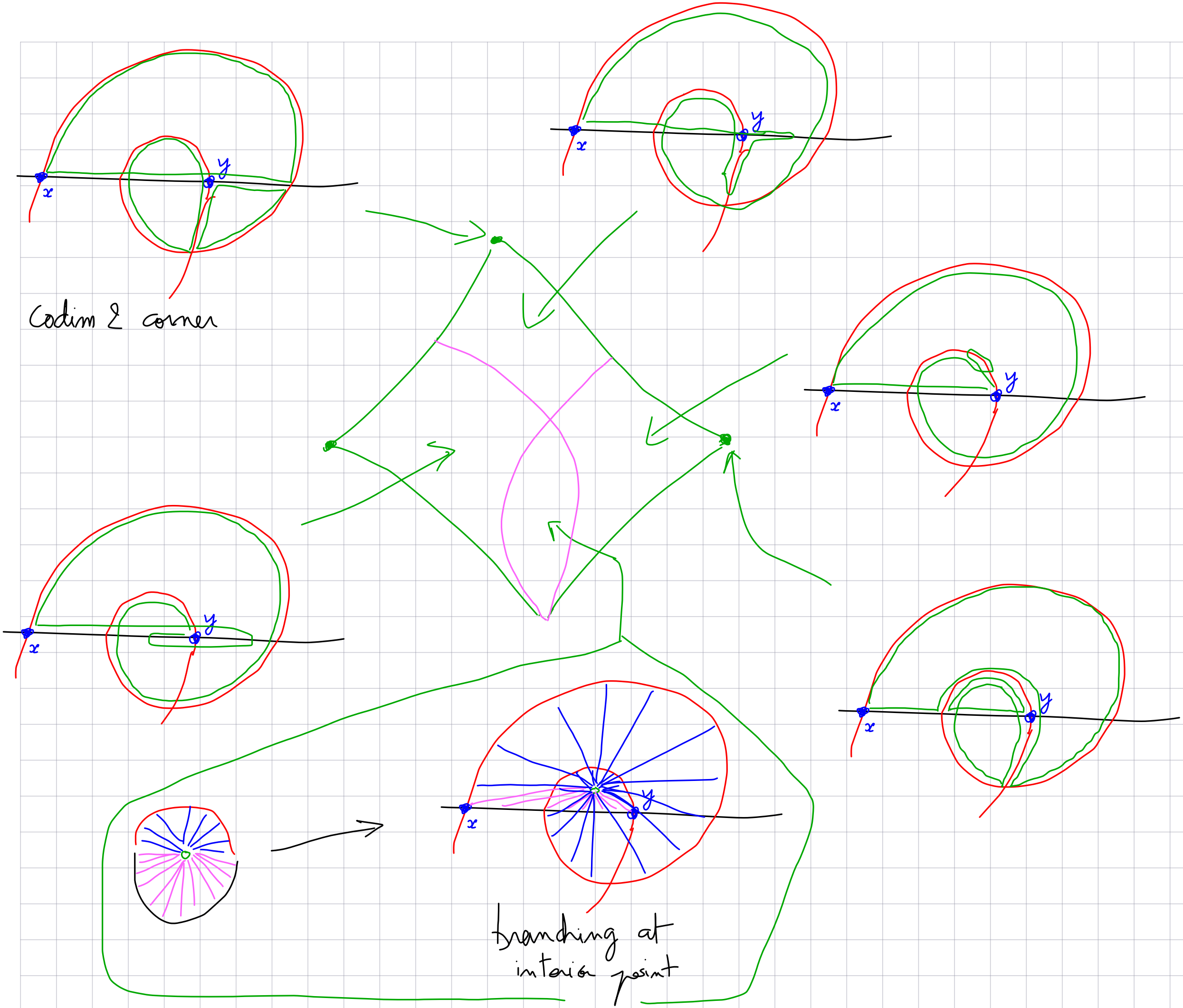
strip breaking



strip breaking



strip breaking


y
x
Codim 2 corner
y
x
y
x
y
x
y
x
y
x
branching at interior point

3.4 Families and parametrized moduli spaces

→ Seidel, Chap 9

- Families $\hat{S}$: smooth oriented surf with ∂, $\Sigma = \Sigma^+ \cup \Sigma^- \subset \partial \hat{S}$ punctures

$S = \hat{S} \setminus \Sigma$,

$$\begin{tikzcd} S \arrow[r] & \mathcal{S} \arrow[d, "\pi"] \\ & \mathcal{R} \end{tikzcd}$$

: fibre bundle, $\hat{S} \longrightarrow \hat{\mathcal{S}}$ fiberwise-compactif°

$$\begin{tikzcd} \hat{\mathcal{S}} \arrow[d, "\hat{\pi}"] \\ \mathcal{R} \end{tikzcd}$$

$+ \; I_{\mathcal{S}}$: almost cpx str on $T^{vert}\mathcal{S} = \bigcup_{r \in \mathcal{R}} T\mathcal{S}_r$

"family of pointed-boundary surfaces"

Strip-like ends for $\mathcal{S}$: $\zeta \in \Sigma$, $\rightsquigarrow$ $\epsilon_\zeta : \mathcal{R} \times Z^\pm \to \mathcal{S}$

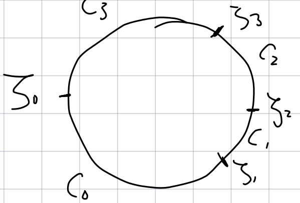
$$\begin{tikzcd} \mathcal{R} \times Z^\pm \arrow[rr] \arrow[dr] & & \mathcal{S} \arrow[dl] \\ & \mathcal{R} & \end{tikzcd}$$

that are fiberwise strip-like ends in the unparam sense

(always exist)

- Pointed discs $\hat{S} = D^2$, $\Sigma^- = \{\zeta_0\}$, $\Sigma^+ = \{\zeta_1, \dots \zeta_d\}$

ordered cyclicity as in



→ Universal family:

$$\begin{tikzcd}
\hat{S}^{d+1} \arrow[d] & = \mathrm{Conf}_{d+1}(\partial D) \times_{\mathrm{Aut}\, D} D \\
R^{d+1} & = \mathrm{Conf}_{d+1}(\partial D) / \mathrm{Aut}\, D
\end{tikzcd}$$

with $\mathrm{Conf}_{d+1}(\partial D) = \{ (\zeta_0, \dots \zeta_d) \mid \text{distinct, ordered cyclicity} \} \subset (\partial D)^{d+1}$

$\mathrm{Aut}\, D = PSL_2(\mathbb{R})$

universal: any other family $\begin{pmatrix} \hat{S} \\ \downarrow \\ R \end{pmatrix}$ is iso to a pullback by a classifying map γ:

$$\begin{tikzcd}
\hat{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{S}^{d+1} \arrow[d] \\
R \arrow[r, "\gamma"] & R^{d+1}
\end{tikzcd}$$

Def: Universal choice of strip-like ends: choice, for every $d \geq 2$, of strip-like ends $\{\epsilon_0^{d+1}, \ldots, \epsilon_d^{d+1}\}$ for $\mathcal{S}^{d+1}$

(any other family then comes equipped with strip-like ends via classifying map)

Fact: $\mathcal{R}^{d+1} \simeq T^{d+1}$ (trees)

- Deligne-Mumford-Stasheff compactification

$\overline{\mathcal{R}}^{d+1} = \coprod_T \mathcal{R}^T$, where: T: stable d-leafed tree

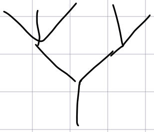
$\hookrightarrow \forall v$ vertex, $|v| \geq 3$
$\overset{\shortparallel}{\#\text{adjacent edges}}$

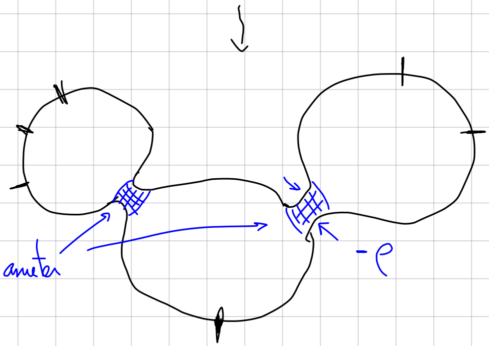
$\mathcal{R}^T := \prod_{v \text{ vertex}} \mathcal{R}^{|v|}$

glueing $\Rightarrow \overline{\mathcal{R}}^{d+1}$ equipped with topol. and smooth structure.

Glueing:

$T =$





$\rho \in (-1,0)$

→ Get a family

$$\begin{tikzcd}
\mathcal{S} \arrow[d] & & \mathcal{S}^{d+1} \arrow[d] \\
\mathcal{R} = (-1,0)^{Ed^{int}(T)} \times \prod_v \mathcal{R}_v \arrow[rr] & & \mathcal{R}^{d+1}
\end{tikzcd}$$

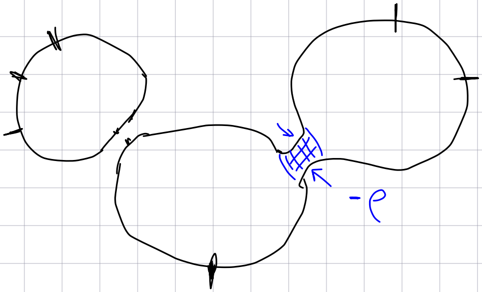
→ compactify to

$$\begin{tikzcd}
\overline{\mathcal{S}} \arrow[d] & & \mathcal{S}^{d+1} \arrow[d] \\
\overline{\mathcal{R}} = (-1,0]^{Ed^{int}(T)} \times \prod_v \mathcal{R}_v \arrow[rr] & & \overline{\mathcal{R}}_{d+1}
\end{tikzcd}$$

→ compactify to

$$\begin{tikzcd} \overline{\mathcal{J}} \arrow[d] & & \mathcal{S} \arrow[d] \\ \overline{\mathcal{R}} = (-1,0]^{\# d^{int}(T)} \times \prod_v \overline{\mathcal{R}}_v \arrow[rr] & & \overline{\mathcal{R}}_{d+1} \end{tikzcd}$$

by adding :



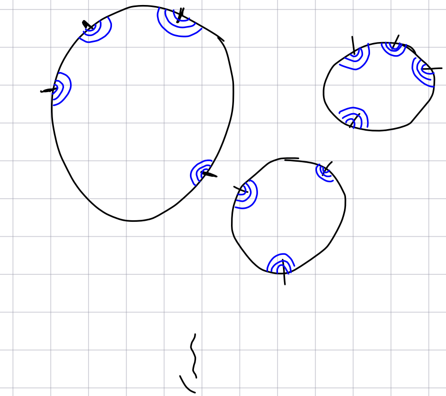
when some $\rho = 0$.

Rk: $\overline{\mathcal{J}}$ non-separated

Fact: $\overline{\mathcal{R}}^{d+1} \hookrightarrow \overline{\mathcal{M}}_{0,d+1}(\mathbb{R})$: Real locus of genus zero Deligne-Mumford.

⇒ $\overline{\mathcal{R}}^{d+1}$ compact mfd with corners

• Consistent strip-like ends



(: ends coming from the $S_v \to R_v$

(: ends coming from $S^{d+1} \to R^{d+1}$

Def. Univ choice of s.l.e is consistent if (= (via γ.

Lemma: these exist.

Rk: Does the strip model $\equiv$ for D provide explicit consistent s.l.e?

• Moduli spaces for families

M: exact sympl with convex ∂

$$\begin{tikzcd} \mathcal{S} \arrow[d] \\ \mathcal{R} \end{tikzcd}$$: family

• Lagrangian labels: locally constant $\{L_C \subset M\}$ for $C \subset \partial \mathcal{S}_r$
boundary component

• $\forall \zeta \in \Sigma \rightsquigarrow (H_\zeta, J_\zeta)$ Floer datum for $(L_{\zeta,0}, L_{\zeta,1})$

Def: Perturbation datum for $\mathcal{S}$: pair (K, J):

$K \in \Omega^1_{\mathcal{S}/\mathcal{R}}(\mathcal{S}, \mathcal{H})$: one-forms on the fibers

$J \in C^\infty(\mathcal{S}, \mathcal{J})$

satisfying as in the unparam case: • $K(\xi)|_{L_C} = 0$, $\xi \in TC \subset T(\partial S_r)$

and compat with ends: $\epsilon_\zeta^* K = H_\zeta(t)\,dt$, $J(\epsilon_\zeta(s,t)) = J_\zeta(t)$

Moduli space for a family $\begin{matrix} S \\ \downarrow \\ R \end{matrix}$, $y_\zeta \in \mathcal{C}(L_{\zeta,0}, L_{\zeta,1})$, $\zeta \in \Sigma$

$$\mathcal{M}_S(\{y_\zeta\}) = \left\{ (r,u) \;\middle|\; \begin{matrix} r \in R \\ u : S_r \longrightarrow M \end{matrix} \;\middle|\; \begin{matrix} \bullet\ u \text{ perturbed } J\text{-hol} \\ \bullet\ \text{Lag. } \partial \text{ cond.} \\ \bullet\ \text{limit to } y_\zeta \text{ at } \zeta \end{matrix} \right\}$$

→ zero of a sect' of a Banach bdle:

$U \subset R$ open, which "makes the ends constant"

$$\begin{tikzcd} \mathcal{B}_{S/U} \arrow[d] \\ U \end{tikzcd}$$

with fibers $\mathcal{B}_{S_r} = W^{1,r}_{loc}(S_r, M, L_i)$

$$\begin{tikzcd} S|_U \arrow[d] \\ U \end{tikzcd}$$

can be trivialized so that strip-like ends are indep of $r \in U$.

$$\begin{tikzcd} \mathcal{E}_{S/U} \arrow[d] \\ \mathcal{B}_{S/U} \arrow[u, bend left, "\bar\partial"'] \end{tikzcd}$$

$\bar\partial$: perturbed $\bar\partial$ operator.

$\rightarrow \bar\partial^{-1}(0) = \mathcal{M}_S \cap (U \times \mathcal{B}_S)$

$$T_{(r,u)} \mathcal{M}_S(\{y_\zeta\}) = \operatorname{Ker} D_{S,r,u}$$

$$D_{S,r,u} : T\mathcal{R}^{d+1}_r \times (T\mathcal{B}_{S_r})_u \longrightarrow (\mathcal{E}_{S_r})_u$$

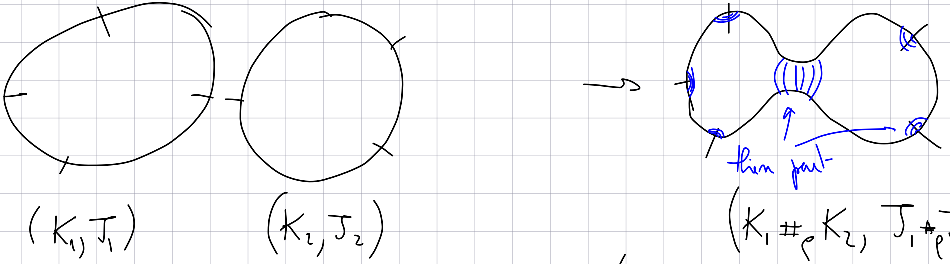
$$D_{\delta,r,u} = \left(\underset{\substack{\uparrow \\ \text{variation} \\ \text{wrt } r}}{D_1} \,\middle|\, \underset{\substack{\uparrow \\ \text{variation wrt } u \,=\, \text{usual linearized CR eq.}}}{D_2} \right)$$

$$\to \quad \operatorname{ind}\left(D_{\delta,r,u}\right) = \underset{\substack{\uparrow \\ \operatorname{ind} D_2}}{\mu(u)} - \underset{\substack{\nwarrow \\ \dim \mathbb{R}^{d+1}}}{(d-2)}$$

· Consistency of perturbation data

Def: univ choice of perturb data: $\forall d \geq 2$, $\forall L_0, \dots L_d$
$(K_{L_0,\dots L_d}, J_{L_0,\dots L_d})$ pert data for $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$

$$\mathcal{R} = (-1,0]^{Ed^{int}(T)} \times \prod_v \mathcal{R}^{|v|} \xrightarrow[\Gamma]{} \mathcal{R}^d$$



vs (K,J) inherited from the univ choice

Def: univ choice of perturb. data is consistent if

- $\exists U \subset \mathcal{R}$ neighborhood of the corner, such that pert. dat agree on the thin parts of $\mathcal{S}_r$, $r \in \mathcal{R}$
- agree everywhere on the corners $\{0\}^{Ed^{int}} \times \prod_v \mathcal{R}^{|v|}$.

Th: * One can find consist. univ choices of perturb. so that transversality holds for the zero an 1-dim moduli spaces $\mathcal{M}_{S^{d+1}}(\{y_\zeta\})$

* the zero dim ones are compact, the 1-dim can be compactified to compact 1-dim mfd with ∂, their compactification is what you think it is

→ can define μ^k over $\mathbb{Z}/2$, satisfy the A_∞-relation

→ get an A_∞-categ that depends on the choice of perturbation ...

Invariance: $L_0, L_1 \subset M$ exact lagr.,
(H^0, J^0) and (H^1, J^1): Floer datum.

→ CF^0, CF^1: corresponding chain cpx; are htpy equiv.

Proof: connect the two Floer datum by a path (H^t, J^t),

define parametrized moduli spaces, and a chain htpy as in Morse theory ... □

→ Systems of categories

$i \in I$ indexing set (= perturbations) → A^i category.

Def: * strict system if $\forall\, i_0, i_1$, have $F^{i_1 i_0} : A^{i_0} \to A^{i_1}$ st

$F^{ii} = \mathrm{id}$, $F^{i_2 i_1} \circ F^{i_1 i_0} = F^{i_2 i_0}$.

* coherent system: functors $F^{i_0 i_1}$ + functor isom: $T^{i_2 i_1 i_0}$:

$$\begin{tikzcd} A^{i_2} \arrow[r, bend left, "F^{i_2 i_1} \circ F^{i_1 i_0}"] \arrow[r, bend right, "F^{i_2 i_0}"'] \arrow[r, phantom, "\Downarrow"] & A^{i_0} \end{tikzcd}$$

st:
- * $F^{i_1 i_1} = \mathrm{id}$
- * if $i_2 = i_1$ or $i_1 = i_0$, $T^{i_2 i_1 i_0} = \mathrm{id}$
- * diag. below commutes

left- and right- compo

$$\begin{tikzcd} F^{i_3 i_2} \circ F^{i_2 i_1} \circ F^{i_1 i_0} \arrow[r, "R_{F^{i_1 i_0}}(T^{i_3 i_2 i_1})"] \arrow[d, "L_{F^{i_3 i_2}}(T^{i_2 i_1 i_0})"'] & F^{i_3 i_1} \circ F^{i_1 i_0} \arrow[d, "T^{i_3 i_1 i_0}"] \\ F^{i_3 i_2} \circ F^{i_2 i_0} \arrow[r, "T^{i_3 i_2 i_0}"'] & F^{i_3 i_0} \end{tikzcd}$$

* weak system: $A^{i_0} \simeq A^{i_1}$, unspecified iso.

Recall: $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$ functors,

natural transfo: $$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{F}"{name=U}] \arrow[r, bend right, "\mathcal{G}"'{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "T"] \end{tikzcd}$$

$\forall X \in \operatorname{ob} \mathcal{A}$, $T(X) : \mathcal{F}(X) \to \mathcal{G}(X)$ st if $X \xrightarrow{\varphi} Y$,

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "T(X)"] \arrow[d, "\mathcal{F}(\varphi)"'] & \mathcal{G}(X) \arrow[d, "\mathcal{G}(\varphi)"] \\ \mathcal{F}(Y) \arrow[r, "T(Y)"'] & \mathcal{G}(Y) \end{tikzcd}$$ commutes.

- Left-/ Right compo

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{F}"{name=U}] \arrow[r, bend right, "\mathcal{G}"'{name=D}] & \mathcal{B} \arrow[r, "\mathcal{H}"] \arrow[Rightarrow, from=U, to=D, "T"] & \mathcal{C} \end{tikzcd} \longrightarrow \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{H} \circ \mathcal{F}"{name=U}] \arrow[r, bend right, "\mathcal{H} \circ \mathcal{G}"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D, "L_{\mathcal{H}}(T)"] \end{tikzcd}$$

def by $L_H(T)(X) = H(T(X))$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "\mathcal{G}"'{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "T"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F \circ H"{name=U}] \arrow[r, bend right=50, "\mathcal{G} \circ H"'{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "R_H(T)"] \end{tikzcd}$$

def by $\quad R_H(T)(X) = T(H(X))$

<u>Ex</u> of <u>strict system</u>: $A^i \xrightarrow[H^i]{} A^{tot}$ family of full subcat,

H^i: equiv of cat.

$\forall i$, can find $\underset{\mathrm{Ob}\,A^i}{\underset{\cap}{K^i(X)}} \xrightarrow[S^i]{\sim} X$ for each $X \in \mathrm{Ob}\,A^{tot}$

s.t. $K^i(X) = X$, $S^i = id_X$ if $X \in A^i$

- K^i can be turned to a functor, $S^i \to$ natural tr $\begin{tikzcd} A^{tot} \arrow[r, bend left=40, "H^i \circ K^i"{name=U}] \arrow[r, bend right=40, "Id"'{name=D}] & A^{tot} \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$

$\to$ take $F^{i_1 i_0} = K^{i_1} \circ H^{i_0}$ and $T^{i_2 i_1 i_0} = L_{K^{i_2}}(R_{H^{i_0}}(S^{i_1}))$:

$$\begin{tikzcd} A^{i_0} \arrow[r, "H^{i_0}"] \arrow[d] & A^{tot} \arrow[dl, bend left, "K^{i_1}"] \\ A^{i_1} \arrow[ur, "H^{i_1}"] & \end{tikzcd} \qquad \begin{tikzcd} A^{i_0} \arrow[r, "H^{i_0}"'] & A^{tot} \arrow[r, bend left=50, "K^{i_1} \circ H^{i_1}"{name=U}] \arrow[r, bend right=50, ""{name=D}] & A^{tot} \arrow[r, "K^{i_2}"'] & A^{i_2} \arrow[Rightarrow, from=U, to=D, "S^{i_1}"] \end{tikzcd}$$

→ Coherent syst of A_∞ cat $\mathcal{A}^i$:

- $F^{i_1 i_0} : \mathcal{A}^{i_0} \longrightarrow \mathcal{A}^{i_1}$ A_∞-functors
- $[T^{i_2 i_1 i_0}] \in \mathrm{Hom}_{H^0(\mathrm{fun}(\mathcal{A}^{i_0}, \mathcal{A}^{i_1}))}(F^{i_2 i_1} \circ F^{i_1 i_0}, F^{i_2 i_0})$

satisfying similar axioms ... need more A_∞ algebra.

3.5 More A_∞ algebra

→ Seidel, chap 2

- Unitality

Def: An A_∞-category $\mathcal{A}$ is strictly unital if $\forall X \in \mathrm{Ob}\,\mathcal{A}$, $\exists\, e_X \in \mathrm{End}(X)$ st

- $\mu^1(e_X) = 0$
- $\mu^2(e_X, a) = (-1)^{|a|} a$, $\mu^2(a, e_X) = a$
- for $d > 2$, $\mu^d(\dots, e_X, \dots) = 0$

$\mathcal{A}$ is c-unital (c = cohomologically) if $H(\mathcal{A})$ is unital

Rks:
- strictly unital $\Rightarrow$ c-unital
- there is a 3rd notion of homotopy unital cat. but it's more complicated
- these 3 notions are nevertheless essentially equivalent.

· Functors $F = (F^1, F^2 \dots)$, $G = (G^1, G^2, \dots) \in \text{nu-fun}(\mathcal{A}, \mathcal{B})$

Q := non-unital A_∞-functors

A degree g pre-natural transformat° $T : F \longrightarrow G$ is a seq of maps $T = (T^0, T^1, \dots)$, with

$$T^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{B}}(FX_0, GX_d)[g-d]$$

when $d = 0$, $T^0 : X \longmapsto T^0 X \in \hom_{\mathcal{B}}(F(X), G(X))$

$\hom_Q^g(F, G)$:= set of such nat. pretransf. of deg g.

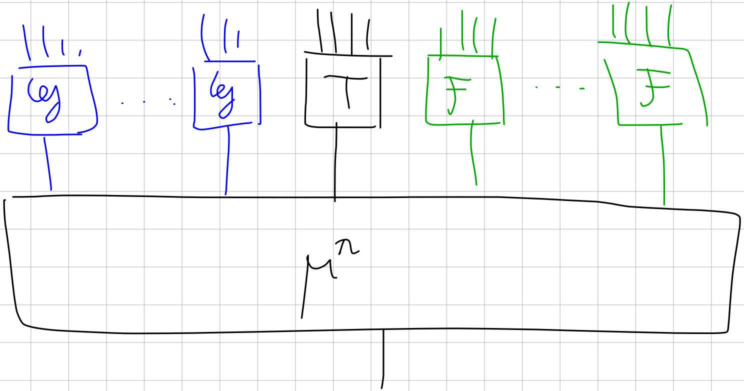
$\Rightarrow$ $Q = \text{nu-fun}(\mathcal{A}, \mathcal{B})$ is an A_∞-cat, with the μ^d

defined by ...

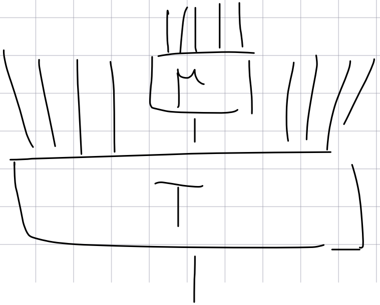
$$\left(\mu^1_Q(T)\right)^d(a_d,\dots,a_1)=\sum_{\substack{r,i\\ s_1+\dots s_r=d}}\pm\mu^r\Big(\mathcal{G}^{s_r}(a_d,\dots a_{d-s_r+1}),\dots,\mathcal{G}(\cdot\cdot),T^{s_i}(\dots),\mathcal{F}^{s_{i-1}}(\dots),\dots,\mathcal{F}^{s_1}(\dots)\Big)$$

$$-\sum_{m,n}\pm T^{d-m+1}\left(a_d,\dots,a_{n+m+1},\mu^m(\dots),a_n,\dots a_1\right)$$

$= \sum$

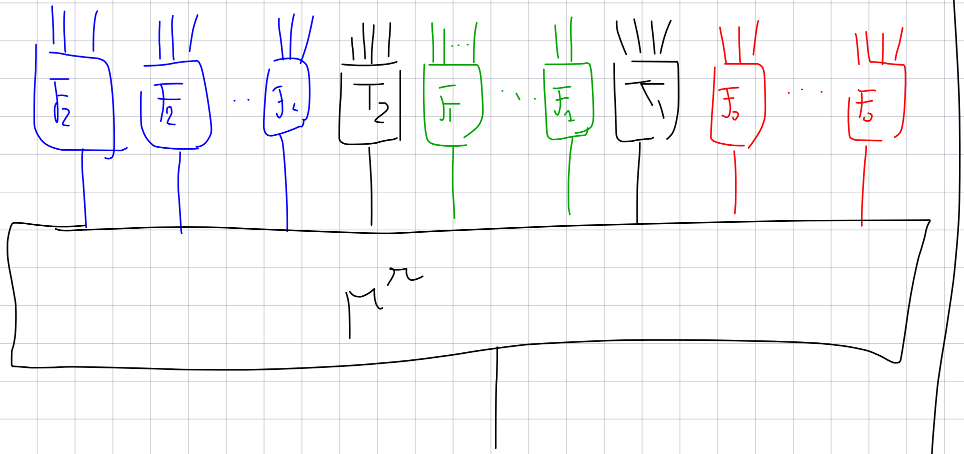


$- \sum$



$$\mathcal{F}_0 \xrightarrow{T_1} \mathcal{F}_1 \xrightarrow{T_2} \mathcal{F}_2$$

$$\left(\mu_Q^2(T_2,T_1)\right)^d(a_d,\dots a_1) = \sum_{\substack{r,i,j \\ s_1+\dots s_r=d}} \pm\, \mu^r\left(\mathcal{F}_2^{s_r}(a_d,\dots),\dots,T_2^{s_j}(\dots),\mathcal{F}_1^{s_{j-1}}(\dots),\dots,T_1^{s_i}(\dots),\mathcal{F}_0(\dots)\dots\right)$$



μ_Q^d, $d>2$ defined likewise

Def: A natural transformation is a cocycle T : $\mu^1(T) = 0$

Rk: coboundaries provide chain homotopies between them.

Rk:
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F_0"{name=U}] \arrow[r, bend right=50, "F_1"'{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "T"] \end{tikzcd} \implies \begin{tikzcd} H\mathcal{A} \arrow[r, bend left=50, "HF_0"{name=U}] \arrow[r, bend right=50, "HF_1"'{name=D}] & H\mathcal{B} \arrow[Rightarrow, from=U, to=D, "HT"] \end{tikzcd}$$

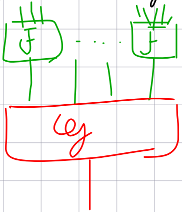
$H(T) : X \longmapsto [T^0 X]$ is a natural transform.

- Composition functors $\mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$

$$\rightarrow \begin{cases} \mathcal{L}_{\mathcal{G}} : \text{mu-fun}(\mathcal{C}, \mathcal{A}) \longrightarrow \text{mu-fun}(\mathcal{C}, \mathcal{B}) \\ \mathcal{R}_{\mathcal{G}} : \text{mu-fun}(\mathcal{A}, \mathcal{C}) \longrightarrow \text{mu-fun}(\mathcal{B}, \mathcal{C}) \end{cases}$$

* on objects : composition of A_∞-functors : $\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{A} \xrightarrow{\mathcal{G}} \mathcal{B}$

$$(\mathcal{G} \circ \mathcal{F})^d = \sum$$



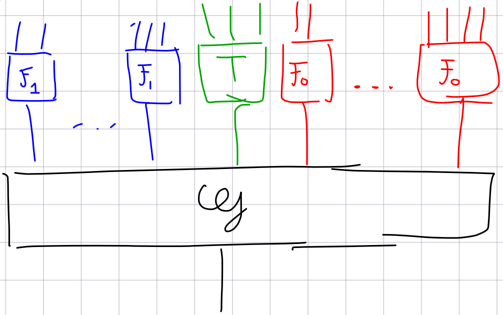
No sign here

* On morphisms : $\mathcal{L}_{\mathcal{G}} = (\mathcal{L}_{\mathcal{G}}^1, \mathcal{L}_{\mathcal{G}}^2, \dots)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathcal{F}_0"] \arrow[r, bend right, "\mathcal{F}_1"'] \arrow[r, phantom, "\Downarrow T"] & \mathcal{A} \arrow[r, "\mathcal{G}"] & \mathcal{B} \end{tikzcd}$$

$$(\mathcal{L}_{\mathcal{G}}^1 T)^d (a_d, \dots, a_1)$$

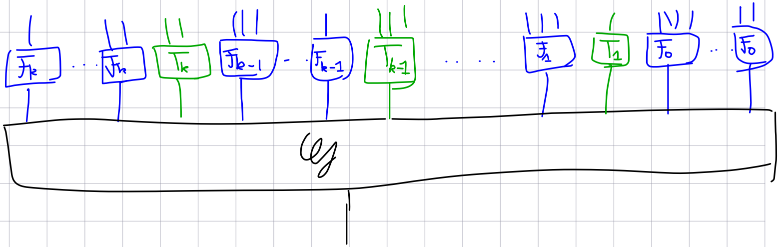
$$= \sum_{\substack{r, i \\ s_1 + \dots + s_r = d}} \pm \mathcal{G}^r \left(\mathcal{F}_1^{s_r}(a_d, \dots), \dots, T^{s_i}(\dots), \mathcal{F}_0^{s_{i-1}}(\dots), \dots \right)$$



· $\mathcal{L}_{\mathcal{G}}^{k}$ for $k \geqslant 2$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "\mathcal{F}_0"] \arrow[r, bend left=20, "\mathcal{F}_1"'] \arrow[r, bend right=60, "\mathcal{F}_k"'] \arrow[r, phantom, shift left=12, "\Downarrow T_1"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=12, "\Downarrow T_k"] & \mathcal{A} \arrow[r, "\mathcal{G}"] & \mathcal{B}
\end{tikzcd}$$

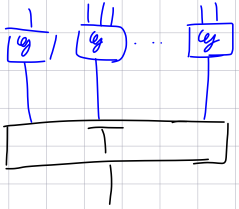
$\mathcal{L}_{\mathcal{G}}^{k}(T_k, \dots T_1) = \sum \pm$



· $R_{\mathcal{G}} = (R_{\mathcal{G}}^{1}, R_{\mathcal{G}}^{2}, \dots)$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathcal{G}"] & \mathcal{B} \arrow[r, bend left=60, "\mathcal{F}_0"] \arrow[r, bend right=60, "\mathcal{F}_1"'] \arrow[r, phantom, "\Downarrow T"] & \mathcal{C}
\end{tikzcd}$$

$(R_{\mathcal{G}}^{1} T)^{d} = \sum'$



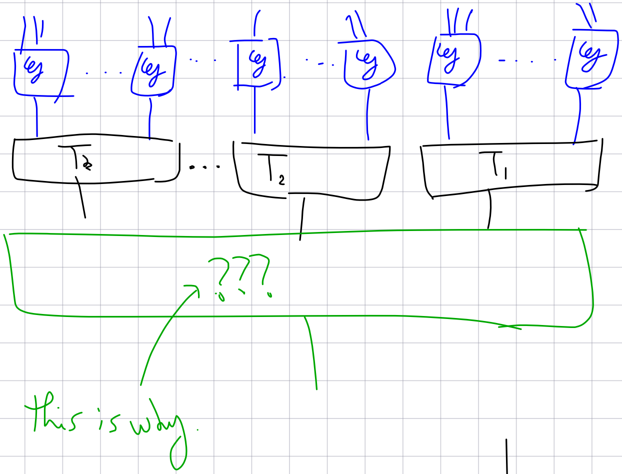
no sign!

(don't ask me why...)

- $\mathcal{R}_{\mathcal{G}}^{k} \equiv 0$ for $k \geq 2$: → Why?

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathcal{G}"] & \mathcal{B} \arrow[r, bend left=60, "F_0"{name=F0}] \arrow[r, bend left=15, "F_1"{name=F1}] \arrow[r, bend right=15, ""{name=Fk1}] \arrow[r, bend right=60, "F_k"'{name=Fk}] & \mathcal{C}
\arrow[Rightarrow, from=F0, to=F1, "T_1"]
\arrow[Rightarrow, from=Fk1, to=Fk, "T_k"]
\end{tikzcd}$$

$\mathcal{R}_{\mathcal{G}}^{k}(T_k, \dots T_1) = \sum$



Def: $\mathcal{A}, \mathcal{B}$: c-unital A_∞-cat, an A_∞-functor

- $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is c-unital if $H(\mathcal{F}) : H(\mathcal{A}) \to H(\mathcal{B})$ is unital.
- $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if $H(\mathcal{F}) : H(\mathcal{A}) \to H(\mathcal{B})$ is an equivalence.

Th: $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ quasi-equivalence, then $\exists\, \mathcal{G} : \mathcal{B} \to \mathcal{A}$ quasi-eq. such that $\mathcal{G} \circ \mathcal{F} \simeq Id_{\mathcal{A}}$ in $H^0(fun(\mathcal{A},\mathcal{A}))$

$\mathcal{F} \circ \mathcal{G} \simeq Id_{\mathcal{B}}$ in $H^0(fun(\mathcal{B},\mathcal{B}))$

Def: [c-unital module] $\mathcal{A}$: c-unital A_∞-cat

$Q = nu\text{-}mod(\mathcal{A}) = nu\text{-}fun(\mathcal{A}^{opp}, Ch)$ dg cat. of chain cpx.

Ch strictly unital $\Rightarrow$ Q strictly unital: $\mathcal{M} \in Ob\, Q \rightsquigarrow e_{\mathcal{M}} = (t^1, t^2, \dots)$ with $t^1(b) = (-1)^{|b|} b$, $t^d \equiv 0$, $d \geq 2$.

$mod(\mathcal{A}) := \{\text{c-unital } \mathcal{F} \in Q\}$

$\mathcal{M}$ is c-unital if $H(\mathcal{M})$ is unital, i.e. if $e_X \in hom^0_{\mathcal{A}}(X,X)$, then $\mu^2_{\mathcal{M}}(\cdot, e_X)$ induces id. on $H(\mathcal{M}(X))$

• Yoneda embedding $\mathcal{A}$: non-unital A_∞-cat

$$Y \in \mathrm{Ob}\,\mathcal{A} \to \mathcal{Y} \in \mathrm{Ob}(\underbrace{\text{nu-mod}(\mathcal{A})}_{Q:=}) : \begin{cases} \mathcal{Y}(X) := \hom(X,Y) \\ \mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}} \end{cases}$$

$\Longrightarrow$ non-unital A_∞-functor $\mathcal{J} = \mathcal{J}_{\mathcal{A}} : \mathcal{A} \to Q$: "Yoneda embedding"

$\mathcal{J}_{\mathcal{A}} : \mathcal{A} \to \mathrm{mod}\,\mathcal{A}$ if $\mathcal{A}$ is c-unital

$$\underset{\mathrm{Ob}\,\mathcal{A}}{\underset{\cap}{Y_0}} \xrightarrow{c} \underset{\mathrm{Ob}\,\mathcal{A}}{\underset{\cap}{Y_1}} \to \mathcal{J}^1(c) : \mathcal{Y}_0 \to \mathcal{Y}_1$$

$$\mathcal{Y}_0(X_{d-1}) \otimes \hom(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom(X_0, X_1) \to \mathcal{Y}_1(X_0)$$

$$b \otimes a_{d-1} \otimes \cdots \otimes a_1 \longmapsto \mu^{d+1}(c, b, a_{d-1}, \dots, a_1)$$

$$Y_0 \xrightarrow{c_1} Y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} Y_k \rightsquigarrow \mathcal{J}^k(c_k, \dots, c_1) : \mathcal{Y}_0 \to \mathcal{Y}_k$$

$$\mathcal{Y}_0(X_{d-1}) \otimes \hom(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom(X_0, X_1) \to \mathcal{Y}_1(X_0)$$

$$b \otimes a_{d-1} \otimes \cdots \otimes a_1 \longmapsto \mu^{d+k}(c_k, \dots, c_1, b, a_{d-1}, \dots, a_1)$$

$$X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots X_{d-1} \xrightarrow{b} Y_0 \xrightarrow{c_1} Y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} Y_k$$

3.6 Invariance

→ Systems of A_∞ categories

$i \in I$ indexing set (= perturbations) → A^i A_∞ category.

Def: * strict system if $\forall\, i_0, i_1$, have $F^{i_1 i_0} : A^{i_0} \to A^{i_1}$ st

A_∞-functors

$F^{ii} = id$, $F^{i_2 i_1} \circ F^{i_1 i_0} = F^{i_2 i_0}$.

* coherent system: $\begin{cases} \bullet\ F^{i_1 i_0} : \mathcal{A}^{i_0} \longrightarrow \mathcal{A}^{i_1} \quad A_\infty\text{-functors} \\ \bullet\ [T^{i_2 i_1 i_0}] \in \mathrm{Hom}_{H^0(fun(\mathcal{A}^{i_0}, \mathcal{A}^{i_1}))}(F^{i_2 i_1} \circ F^{i_1 i_0}, F^{i_2 i_0}) \end{cases}$

$$\begin{tikzcd} A^{i_2} \arrow[r, bend left, "F^{i_2 i_1} \circ F^{i_1 i_0}"{name=U}] \arrow[r, bend right, "F^{i_2 i_0}"'{name=D}] & \mathcal{A}^{i_0} \arrow[from=U, to=D, Rightarrow, shorten=2mm] \end{tikzcd}$$

st. * $F^{ii} = id$, * if $i_2 = i_1$ or $i_1 = i_0$, $T^{i_2 i_1 i_0} = id$

* diag. below commutes in $H^0(fun(\mathcal{A}^{i_0}, \mathcal{A}^{i_3}))$

left- and right- compo

$$\begin{tikzcd}[column sep=huge, row sep=large] F^{i_3 i_2} \circ F^{i_2 i_1} \circ F^{i_1 i_0} \arrow[r, "R_{F^{i_1 i_0}}(T^{i_3 i_2 i_1})"] \arrow[d, "\mathcal{L}_{F^{i_3 i_2}}(T^{i_2 i_1 i_0})"'] & F^{i_3 i_1} \circ F^{i_1 i_0} \arrow[d, "T^{i_3 i_1 i_0}"] \\ F^{i_3 i_2} \circ F^{i_2 i_0} \arrow[r, "T^{i_3 i_2 i_0}"'] & F^{i_3 i_0} \end{tikzcd}$$

* weak system: $A^{i_0} \simeq A^{i_1}$, unspecified iso.

As in the (non-A_∞) categ world, get a coherent syst from a family of quasi-equiv. full subcat $\mathcal{A}^i \hookrightarrow \mathcal{A}^{tot}$ ...

→ Back to (the prototypes of) Fukaya categories

$i \in I$ = { choices of regular
- consistent strip-like ends on families of discs
- Floer datum $(H_{L_0,L_1}, J_{L_0,L_1})$ for pairs (L_0, L_1)
- consistent perturbation data $(K_{L_0,\dots,L_d}, J_{L_0,\dots,L_d})$ on families, compat with }

↳ Get $\mathcal{F}(M)^{pr,i}$ ← no orientations yet... family of A_∞-cat / I.

$H\mathcal{F}(M)^{pr,i}$: strict system.

Can get a coherent syst on $\{\mathcal{F}(M)^{pr,i}\}_i$ out of

$\mathcal{F}(M)^{pr,i} \hookrightarrow \underline{\mathcal{F}(M)^{pr,tot}}$

← <u>. objects</u> (L,i), L: exact qad lag., $i \in I$

- for $(L_0, i_0) \to (L_1, i_1)$ choose regular Floer data corresp. to $(H_{L_0,L_1}, J_{L_0,L_1})$ if $i_0 = i_1$
- choose pert. data likewise for μ^d, $d \geq 2$ ...

• Group actions on A_∞-categories

Goal: $\mathrm{Aut}(M,\partial M) \circlearrowright \mathcal{F}(M)$, $\mathrm{Ham}(M,\partial M) \circlearrowright \mathcal{F}(M)$ cohomol. trivial

$$\mathrm{Aut}(M,\partial M) = \left\{ \phi : M \circlearrowleft \;/\; \phi^*\underset{\text{Liouville 1-form}}{\theta} = \underset{\text{Liouville 1-form}}{\theta} + d\underset{\substack{\text{function } M \to \mathbb{R} \\ \equiv 0 \text{ near } \partial M}}{K} \right\}$$

Def: G: group, A (A_∞-) category

- **strict G-action on A**: family of (A_∞-) functors $F^g : A \to A$, $g \in G$, st
  - $F^e = \mathrm{Id}$
  - $F^{g_2 g_1} = F^{g_2} \circ F^{g_1}$

- **Coherent G-action**
  $$\begin{cases} F^g : A \to A \\ \boxed{T^{g_2 g_1}} : F^{g_2} \circ F^{g_1} \longrightarrow F^{g_2 g_1} \in \mathrm{Hom}_{H^0(\mathrm{fun}(A,A))} \end{cases}$$

st:
- $F^e = \mathrm{Id}$, $T^{g_2 g_1} = \mathrm{Id}$ (in H^0...) if either $g_2 = e$, $g_1 = e$
- diagram commutes (in H^0)

$$\begin{tikzcd} F^{g_3} \circ F^{g_2} \circ F^{g_1} \arrow[r] \arrow[d] & F^{g_3 g_2} \circ F^{g_1} \arrow[d] \\ F^{g_3} \circ F^{g_2 g_1} \arrow[r] & F^{g_3 g_2 g_1} \end{tikzcd}$$

- Weak G-action: $F^{g_2 g_1} \simeq F^{g_2} \circ F^{g_1}$ unspecified...

Ex: $A \hookrightarrow A^{free}$ $\Big\rangle \Rightarrow G \circlearrowright A$ coherent action

full subcat

$\uparrow$ (quasi)-equiv.

$\circlearrowleft$ strict. G

. $G = Aut(M, \partial M)$, $A = \mathcal{F}(M)^{\mu}$

$\mathcal{F}(M)^{\mu, free} :=$

- objects: (L, γ), $L \subset M$ lag, $\gamma \in G$
- choose regular and equivariant Floer/perturbat° data to define the hom and μ^d ...

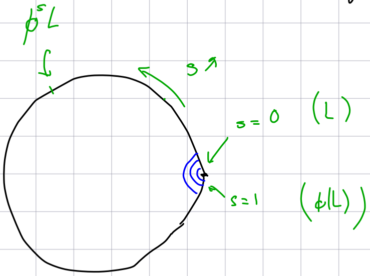
$\circlearrowleft G$

$\phi \cdot (L, \gamma) = (\phi(L), \phi\gamma)$

$\Longrightarrow$ coherent G-action on $\mathcal{F}(M)^{\mu} \hookrightarrow \mathcal{F}(M)^{\mu, free}$.

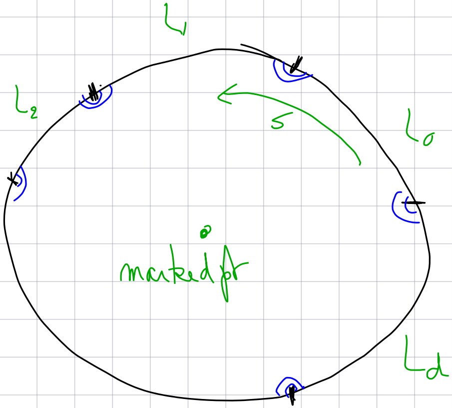
• action of Ham$(M, \partial M)$ $(\phi^s)_{0 \le s \le 1}$ Hamilt. isotopy $\phi^0 = id \to \phi^1 = \phi$

$(L, \gamma) \in Ob\, \mathcal{F}(M)^{pr, free}$ ⇝ $T^0 \in CF((L,\gamma), \phi.(L,\gamma))$ by "moving Lagrangian ∂ conditions:



More generally, can def:

$$T^d : CF((L_{d-1}, \gamma_{d-1}), (L_d, \gamma_d)) \otimes \cdots \otimes CF((L_0, \gamma_0), (L_1, \gamma_1)) \to CF((L_0, \gamma_0), \phi(L_d, \gamma_d))$$



$$\begin{tikzcd} \mathcal{F}(M)^{pr} \arrow[r, bend left, "id"{name=U}] \arrow[r, bend right, "\phi"'{name=D}] & \mathcal{F}(M)^{pr} \arrow[from=U, to=D, Rightarrow, "T"] \end{tikzcd}$$

prop: $\phi \simeq id$ in $H(fun(\mathcal{F}(M)^{pr}, \mathcal{F}(M)^{pr}))$

$\Rightarrow$ weak action of $\pi_0(Aut(M, \partial M))$.

3.7 Orientations

- Determinant line bdles

Def: • V: finite-dim vector space, $n = \dim V$

$\lambda^{top} V = \Lambda^n V$: top exterior power.

$0 \neq x \in \lambda^{top}(V) \Rightarrow$ orientation of V.

• $D : H \to K$ Fredholm op.

$$\boxed{\det D := \lambda^{top}(\operatorname{coker} D^\vee) \otimes \lambda^{top}(\ker D)}$$

prop: $\det(D_1) \otimes \det(D_2) \simeq \det(D_1 \oplus D_2)$

via canon. isom:

$$\left(\Lambda_i w_{1,i}^\vee \otimes \Lambda_i v_{1,i}\right) \otimes \left(\Lambda_i w_{2,i}^\vee \otimes \Lambda_i v_{2,i}\right) \longmapsto (-1)^{\dim(\operatorname{coker} D_2)\cdot \operatorname{index} D_1} \left(\Lambda_i w_{2,i}^\vee \wedge \Lambda_i w_{1,i}^\vee\right) \otimes \left(\Lambda_i v_{1,i} \wedge \Lambda_i v_{2,i}\right)$$

Koszul rule, all vectors are odd

prop H, K finite-dim vector spaces, $D : H \to K$

$\exists$ unique canon identification $t_D : \det D \to \det 0$ s.t.:

- compat with isom composition
- $D = 0$, $t_D = \mathrm{id}$
- $D = \mathrm{id} : H \to H$, $\dim H = 1$, then $t_D : \mathbb{R} \to H^\vee \otimes H$

  $1 \mapsto v^\vee \otimes v$ — dual basis.
- compat with $\oplus$:

$$\begin{tikzcd}
\det(D_1 \oplus D_2) \arrow[r, "t_{D_1 \oplus D_2}"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & \det(0_1 \oplus 0_2) \arrow[d, "\simeq"] \\
\det D_1 \otimes \det D_2 \arrow[r, "t_{D_1} \otimes t_{D_2}"'] & \det 0_1 \otimes \det 0_2
\end{tikzcd}$$

explicitly:

$$\begin{array}{l}
H = \mathrm{Span}\{ e_1, \dots, e_k, \overbrace{e_{k+1}, \dots, e_m}^{\mathrm{Ker}} \} \\
\qquad\quad\;\; \downarrow \qquad\quad \downarrow \\
K = \mathrm{Span}\{ f_1, \dots, f_k, \underbrace{f_{k+1}, \dots, f_n}_{\mathrm{Coker}} \}
\end{array}$$

$$t_D : (f_n^\vee \wedge \dots \wedge f_{k+1}^\vee) \otimes (e_{k+1} \wedge \dots \wedge e_m) \mapsto (f_n^\vee \wedge \dots f_1^\vee) \otimes (e_1 \wedge \dots \wedge e_m)$$

· H, K : real Banach spaces, $\mathcal{F}(H,K) := \{$ Fredholm op $H \to K \}$

→ get a line bdle

$$\begin{tikzcd} \underline{\det} \arrow[d] \\ \mathcal{F}(H,K) \end{tikzcd} \quad = \bigcup_{D} \det D$$

(use the t_D to trivialize it locally.)

· <u>Line bundles on path spaces and spectral flow</u>

<u>setting</u>: $\Sigma \subset N$ + $\begin{array}{c} \sigma \\ \downarrow \\ \Sigma \end{array}$ s^t $(\sigma^{\otimes 2})^\vee \simeq \nu_\Sigma$

hypersurf. closed mfd line bdle.

<u>prototype</u>:

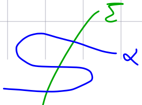
$N = \mathrm{Sym}^2(\mathbb{R}^n)^\vee = \Sigma^0 \cup \Sigma^1 \cup \dots \Sigma^k$

corank

$\Sigma = \overline{\Sigma^1} = \Sigma^1 \cup \Sigma^2 \cup \dots$ (Warning: singular)

σ = null-space on Σ^1

Path space $\mathcal{P} = \{ \alpha : [0,1] \to N \ / \ \alpha(0), \alpha(1) \notin \Sigma \}$

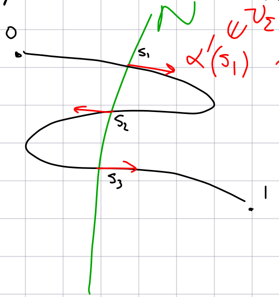


$\nu_\Sigma = (\sigma^{\otimes 2})^\vee \Rightarrow \Sigma$ co-oriented

$\Rightarrow I : \mathcal{P} \to \mathbb{Z}$ intersect^t^ number

$\alpha \mapsto \alpha \cdot \Sigma$

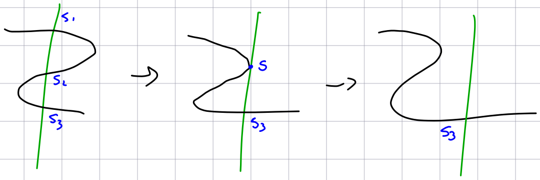
- $\begin{matrix}\delta \\ \downarrow \\ \mathcal{P}\end{matrix}$ defined by $\delta_\alpha = \sigma_{\alpha(s_1)}^{\mathrm{sign}(q_\alpha(s_1))} \otimes \cdots \otimes \sigma_{\alpha(s_\ell)}^{\mathrm{sign}(q_\alpha(s_\ell))}$



$\alpha'(s_1) \in \nu_\Sigma \simeq (\sigma^{\otimes 2})^\vee$

$\alpha'(s_1) \simeq q_{\alpha(s_1)} \in (\sigma^{\otimes 2})^\vee$ quadratic form.

Cancellation:



$$\underbrace{\sigma_{\alpha(s)} \otimes \sigma_{\alpha(s)}^{-1}} \otimes \sigma_{\alpha(s_3)} \to \sigma_{\alpha(s_3)}$$

Prop: with

$$\begin{tikzcd} S : H^*(\Sigma) \arrow[r] \arrow[d, phantom, "\wr"] & H^*(\mathcal{P}) \arrow[d, phantom, "\wr"] \\ H^{*+1}(N, N \setminus \Sigma) \arrow[r, "\substack{\mathrm{eval.} \\ \mathcal{P} \times [0,1] \to N}"'] & H^{*+1}(\mathcal{P} \times [0,1], \mathcal{P} \times \{0,1\}) \end{tikzcd}$$

- $I = S(1)$
- $w_1(\delta) = S(w_1(\sigma))$

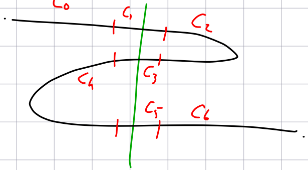
· **Spectral flow** $\psi : R \longrightarrow [0,1]$

$$\alpha \in \mathcal{P} \to d_\alpha = \frac{d}{ds} + \alpha(\psi(s)) : W^{1,2}(\mathbb{R},\mathbb{R}^n) \longrightarrow L^2(\mathbb{R},\mathbb{R}^n)$$

Lemma: . $I(\alpha) = \mathrm{ind}(d_\alpha)$

. $\delta \simeq \det(d_\alpha)$

Proof: decompose the path: and use glueing formulas:

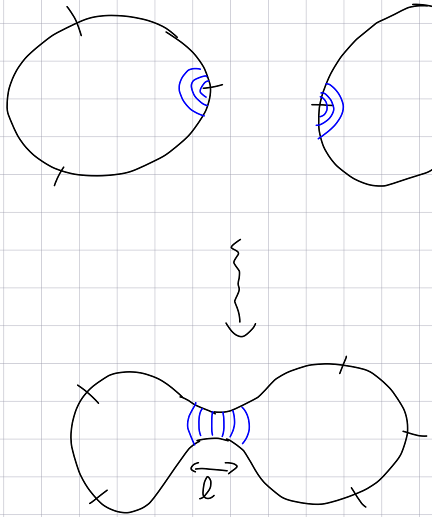


$$\mathrm{ind}(d_\alpha) = \mathrm{ind}(d_{\alpha|C_1}) + \cdots + \mathrm{ind}(d_{\alpha|C_\ell})$$

$$\det(d_\alpha) = \det(d_{\alpha|C_1}) \otimes \cdots \otimes \det(d_{\alpha|C_\ell}) \qquad \square$$

• Determinants of Cauchy-Riemann operators

$S_1, \bar{\partial}_{\nabla_1}$ $\qquad$ $S_2, \bar{\partial}_{\nabla_2}$



$$S = S_1 \#_\ell S_2 \,, \quad \bar{\partial}_{\nabla} = \bar{\partial}_{\nabla_1} \#_\ell \bar{\partial}_{\nabla_2}$$

Glueing formulas:

- $\operatorname{ind}(\bar{\partial}_{\nabla}) = \operatorname{ind}(\bar{\partial}_{\nabla_1}) + \operatorname{ind}(\bar{\partial}_{\nabla_2})$
- $\det(\bar{\partial}_{\nabla}) \simeq \det(\bar{\partial}_{\nabla_1}) \otimes \det(\bar{\partial}_{\nabla_2})$

- $\bar\partial$-operators on line bdles

S: Riem. surf with ∂

(no punctures)

$$\begin{tikzcd}
\text{Hermitian line bdle} & \text{real sub-bundle} \\
E \arrow[d] & F \arrow[l, hook'] \arrow[d] \\
S & \partial S \arrow[l, hook']
\end{tikzcd}$$

$F \simeq \rho_2 \cup \dots \cup \rho_\ell$ in a trivializ-

$\rho_i : S^1 \to \mathbb{RP}^1$

$\mu(\rho_i) = \deg(\rho_i)$

$\partial S = \partial_1 S \cup \dots \cup \partial_\ell S$

Lemma: If $\mu(\rho_1) + \dots + \mu(\rho_\ell) < 0$, then $\bar\partial_\nabla$ is injective.

Proof: X: non-zero sol$^{\text{o}}$ of $\bar\partial_\nabla X = 0$,

$\Rightarrow$ has isolated zeros of order $\nu(z) \geq 1$

and $2 \sum_{\substack{z \in \operatorname{int} S \\ \text{zero}}} \nu(z) + \sum_{\substack{z \in \partial S \\ \text{zero}}} \nu(s) = \mu(\rho_1) + \dots + \mu(\rho_\ell)$ $\square$

- $S = \hat S \setminus Z$ with strip-like ends

Lemma: If $\mu(\rho_1) + \dots + \mu(\rho_\ell) - |Z^-| < 0$, then $\bar\partial_\nabla$ is injective

Proof. Fourier expansion on the strip-like ends...

• Index theory and the Lagrangian Grassmannian

$$\mathrm{Gr}\, V = L\mathrm{Gr}(V) \simeq U(n)/O(n)$$

tautol. bundle :
$$\begin{tikzcd} (V \times \mathrm{Gr} \supset T) \arrow[r, leftrightarrow] \arrow[d] & (V \supset L) \\ \mathrm{Gr}\, V & \end{tikzcd}$$

⇝ Maslov Class $\mu \in H^1(\mathrm{Gr}\, V; \mathbb{Z})$ Rk, $w_1 = \mu \mod 2$.

⇝ 2nd Stiefel Whitney class $w_2 \in H^2(\mathrm{Gr}\, V; \mathbb{Z}/2)$

- μ induces $\pi_1(\mathrm{Gr}\, V) \xrightarrow{\sim} \mathbb{Z}$
- w_2 induces $\pi_2(\mathrm{Gr}\, V) \xrightarrow{\sim} \mathbb{Z}/2$ when $n = \dim V \geq 3$.

- $\underline{n=1}$ $\mathrm{Gr}\, V = U(1)/O(1) = S^1$, $\pi_2 = 0$

• $\underline{n=2}$: $\mathrm{Gr}\, V = U(2)/O(2) \simeq S^1 \underset{\mathbb{Z}/2}{\times} S^2$, $\pi_2 = \mathbb{Z}$

$\Omega\, \mathrm{Gr}\, V$ = based loop space $\{\rho : (S^1, *) \to (\mathrm{Gr}\, V, *)\}$

$\mathcal{L}\, \mathrm{Gr}\, V$ = free loop space $\{\rho : S^1 \to \mathrm{Gr}\, V\}$

$\Omega_k \mathrm{Gr}\, V = \{\rho \mid \mu(\rho) = k\}$ (connected comp)

$\mathcal{L}_k \mathrm{Gr}\, V = \{\rho \mid \mu(\rho) = k\}$

$$\pi_1(\Omega_k Gr V) = \pi_2(Gr V) \underset{n \geq 3}{\simeq} \mathbb{Z}/2$$

$$\pi_1(\mathcal{L}_k Gr V) = \pi_2(Gr V) \times \pi_1(Gr V) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$$

$$\Rightarrow H^1(\mathcal{L}_k Gr V ; \mathbb{Z}/2) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \mathbb{Z}_2 \cdot T(w_2) \oplus \mathbb{Z}_2 \cdot U(\mu)$$

Generators: $ev : S^1 \times \mathcal{L} Gr V \longrightarrow Gr V$

$(z, \rho) \longmapsto \rho(z)$

$$\begin{tikzcd} ev^* : H^*(Gr V) \arrow[r] \arrow[d, "T"'] \arrow[rrd, "U"] & H^*(S^1 \times \mathcal{L} Gr V) \arrow[d, phantom, "\wr"] & \\ H^{*-1}(\mathcal{L} Gr(V)) & \oplus & H^*(\mathcal{L} Gr(V)) \end{tikzcd}$$

Rel. with $\bar\partial$ op.

. I_V : compat. acs on V

. S : compact Riemann surf, $\partial S \simeq S^1$

$$\begin{tikzcd} S \times V = E \arrow[d] & F_\rho \arrow[l, hook'] \arrow[d] \\ S & \partial S \arrow[l, hook'] \end{tikzcd}$$

$F_\rho \longleftarrow$ Lag sub-bundle $(F_\rho)_{e^{2\pi s}} = \rho(s)$

$\rightsquigarrow D_{S,\rho}$: Linearized CR op.

$\rightsquigarrow$

$$\begin{tikzcd} \underline{\det}_S \arrow[d] & \det(D_{S,\rho}) \arrow[l, hook'] \arrow[d, dash] \\ \mathcal{L} Gr & \rho \arrow[l, phantom, "\ni"] \end{tikzcd}$$

Lemma:

. $\operatorname{ind}(D_{S,\rho}) = n\chi(S) + \mu(\rho)$ (Riemann-Roch)

Rk: no punctures on ∂S

. $w_1(\underline{\det}_S) = T(w_2) + (T(\mu) - 1) U(\mu)$

• The Arnol'd stratification

$$\mathrm{Gr}^2 V := \overset{(\Lambda_0\,,\,\Lambda_1)}{\mathrm{Gr}\,V \times \mathrm{Gr}\,V} = \underset{\{\Lambda_0 \pitchfork L_1\}}{\underset{\|}{\Sigma^0}} \cup \underbrace{\Sigma^1 \cup \Sigma^2 \cup \dots \cup \Sigma^n}_{\Sigma :=}$$

$\dim(\Lambda_0 \cap \Lambda_1)$

on each Σ^d, $\sigma^d \hookrightarrow \Lambda_0 \cap \Lambda_1$: rank d v.b

$$\begin{tikzcd} \sigma^d \arrow[d] \\ \Sigma^d \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \sigma = \sigma^1 \arrow[d] \\ \Sigma \end{tikzcd} \text{ line bdle}$$

⇒ by last time's construction, get

- I_Z : "Maslov index for paths"
- δ_Z : Real line bdle

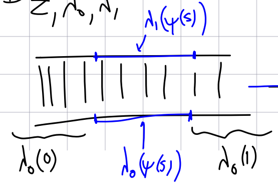
$$\begin{tikzcd} \delta_Z \arrow[d] & \\ \mathcal{P}^2(\mathrm{Gr}\,V) \arrow[r, "I_Z"] & \mathbb{Z} \end{tikzcd}$$

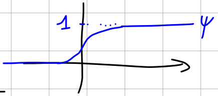
paths with end points $\notin \Sigma_1$

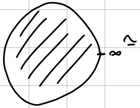
strip $\mathbb{R} \times [0,1]$

Lemma: $I_Z(\lambda_0, \lambda_1) = \mathrm{ind}(D_{Z,\lambda_0,\lambda_1})$, with $D_{Z,\lambda_0,\lambda_1}$ the lin. CR of:

$(\delta_Z)_{\lambda_0\lambda_1} = \det(D_{Z,\lambda_0,\lambda_1})$





- The 1-punctured disc $H =$  $\simeq$ upper half-plane

$$\mathcal{P}^- \mathrm{Gr}\, V := \left\{ \lambda : [0,1] \to \mathrm{Gr}\, V \;\middle|\; \begin{array}{l} \cdot\ \lambda(0) \pitchfork \lambda(1) \\ \cdot\ (\lambda, \lambda(1)) \text{ has neg. definite crossing form at } s=1 \end{array} \right\}$$

$\hookrightarrow (\lambda_0, \lambda_1) : \underset{\overset{\rotatebox{90}{$\in$}}{s}}{[0,1]} \longrightarrow \mathrm{Gr}^2(V)$

choose $\phi_{k,r,s} : \lambda_k(s) \longrightarrow \lambda_k(r)$, $k = 0,1$

s.t. $\phi_{k,s,s} = \mathrm{id}$

$$q_{\lambda_0,\lambda_1}(s) : \lambda_0(s) \cap \lambda_1(s) \longrightarrow \mathbb{R}$$
$$v \longmapsto -\left(\frac{d}{dr}\right)\Big|_{r=s} \omega\left(\phi_{0,r,s}(v), \phi_{1,r,s}(v)\right)$$

δ_H pulled back from

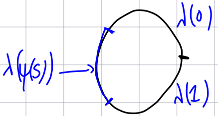
$$\begin{tikzcd} \delta_H \arrow[d] & \\ \mathcal{P}^- \mathrm{Gr}\, V \arrow[r, "I_H"] & \mathbb{Z} \end{tikzcd}$$

$$\mathcal{P}^- \mathrm{Gr}\, V \longrightarrow \mathcal{P}^2 \mathrm{Gr}\, V$$
$$\lambda \longmapsto (\lambda, \lambda(1))\big|_{[0,s']}$$

st $\lambda(s) \pitchfork \lambda(1)$ for $s \geq s'$

Lemma
- $I_H(\lambda) = \mathrm{ind}(D_{H,\lambda})$
- $(\delta_H)\big|_\lambda = \det(D_{H,\lambda})$

with $D_{H,\lambda}$ lin. CR op assoc to:



- Abstract frame structures

$(\mu, w_2) \Longleftrightarrow$ htpy class

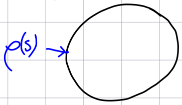
$$\begin{tikzcd} & & \mathbb{Z} \arrow[r, hook] & E \arrow[d] & \mathbb{RP}^\infty \arrow[r, hook] & E \arrow[d] \\ \mathrm{Gr}\,V \arrow[rrr] & & & K(\mathbb{Z},1) & \times & K(\mathbb{Z}_{/2},2) \end{tikzcd}$$

$\rightsquigarrow$ pullback:

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{RP}^\infty \arrow[r, hook] & \mathrm{Gr}\,V^\# \arrow[d, dashed] & \supset \Lambda^\# \arrow[d, dashed] \\ & \mathrm{Gr}\,V & \supset \Lambda \end{tikzcd}$$

$\Lambda^\#$: "abstract linear frames"

Lemma: $\rho^\# \in \mathcal{L}\,\mathrm{Gr}\,V^\# \rightsquigarrow D_{D,\rho}$: lin CR op assoc with :



Then $\cdot\ \mathrm{ind}\, D_{D,\rho} = n$

$\cdot\ \det(D_{D,\rho}) \simeq \lambda^{top}(\rho(0))$ canon.

$(\Lambda_0^\#, \Lambda_1^\#)$ st $\Lambda_0 \pitchfork \Lambda_1 \rightsquigarrow$ $\cdot\ i(\Lambda_0^\#, \Lambda_1^\#) = I_H(\lambda)$ "absolute index"

$\cdot\ o(\Lambda_0^\#, \Lambda_1^\#) = (S_H)_\lambda$ "orientation space"

for λ induced by a path $\lambda^\#$ in $\mathrm{Gr}\,V^\#$ connecting $\Lambda_0^\#$ and $\Lambda_1^\#$

$$\begin{tikzcd}
S \times V = E \arrow[d] & E_v \arrow[l, hook'] \arrow[d] \\
\hat{S} \setminus \Sigma = S & \partial S \arrow[l, hook']
\end{tikzcd}$$

surf with strip-like ends

Lag. sub-bundle from $v : \partial S \to Gr_n V$ loc. cstant on the ends ζ

$+ \ \Lambda_{\zeta,0} \pitchfork \Lambda_{\zeta,1}$

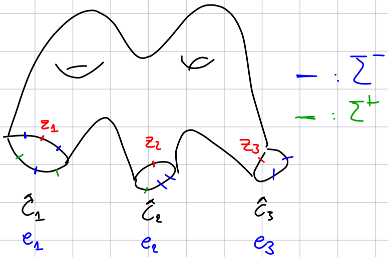
$+$ lifts to $Gr_n V^\#$:

$$\begin{tikzcd}
\partial S \arrow[r, "v"] \arrow[dr, "v^\#"'] & Gr_n V \\
 & Gr_n V^\# \arrow[u]
\end{tikzcd}$$

($\Rightarrow$ Get lifts $\Lambda^\#_{\zeta,0}, \Lambda^\#_{\zeta,1}$ at each ends)

<u>Prop</u>: $\hat{C}_1, \ldots, \hat{C}_k$ comp. of $\hat{S}$, $e_j = |\Sigma^- \cap \hat{C}_j|$ number of incoming points

, $z_j \in \hat{C}_j \setminus \Sigma$ base pt



$$\cdot \ \operatorname{ind}(D_{S,v}) = n\left(\chi(S) - |\Sigma'^-|\right) + \sum'_{\zeta \in \Sigma'^\pm} \mp i\left(\Lambda^\#_{\zeta,0}, \Lambda^\#_{\zeta,1}\right)$$

$$\cdot \ \det(D_{S,v}) \simeq \bigotimes_j \left(\Lambda^{top}(v_{z_j})^{\otimes(1-e_j)} \otimes \bigotimes_{\zeta \in \Sigma^\pm \cap \hat{C}_j} o\left(\Lambda^\#_{\zeta,0}, \Lambda^\#_{\zeta,1}\right)^{\mp 1}\right.$$

• Pin structures

(V, q) : v. space + quadratic form $= (\mathbb{R}^n, \|\cdot\|^2)$

. $TV = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \dots$ Tensor algebra

. $Cl(V,q) = TV / {v^2 - q(v)}$: Clifford algebra $\simeq \Lambda^* V = Cl(V, q=0)$ as vector spaces.

Def: $Pin_n \subset Cl(\mathbb{R}^n)^\times$: multipl. subgroup gen by $S^{n-1} \subset \mathbb{R}^n$

$\mathbb{R}^n \subset T\mathbb{R}^n$ im $\mathbb{R}^n \subset Cl(\mathbb{R}^n)$

Adjoint rep: $Cl(\mathbb{R}^n) \circlearrowright \mathbb{R}^n$: $w \cdot v = -v w v$ $\Leftrightarrow$ reflexion along $v^\perp \subset \mathbb{R}^n$ if $v \in S^{n-1}$

→ induces central extention: $1 \to \mathbb{Z}_2 \to Pin_n \xrightarrow{p} O_n \to 1$

Rk: $Spin_n = p^{-1}(SO_n)$

Def: a Pin-str on a vector bundle $\begin{matrix} F^n \\ \downarrow \\ B \end{matrix}$ is:

- a principal Pin_n-bundle $\begin{matrix} P^\# \\ \downarrow \\ B \end{matrix}$
- an isom. $F \simeq P^\# \times_{Pin_n} \mathbb{R}^n$

Rk: $\begin{matrix} F \\ \downarrow \\ B \end{matrix}$ ⇝

$$\begin{tikzcd} O_n \arrow[r, hook] & O(F) \arrow[d] \\ & B \end{tikzcd}$$

orthonormal frame bdle

→ get a 2-fold cover.

$$\begin{tikzcd} Pin_n \arrow[r] \arrow[d] & P^\# \arrow[d] \\ O_n \arrow[r, hook] & O(F) \arrow[d] \\ & B \end{tikzcd}$$

Prop [existence & uniqueness of Pin-str]

- F admits a Pin-str $P^\#$ iff $w_2(F) = 0 \in H^2(B;\mathbb{Z}/2)$
- if $w_2 F = 0$, $\{\text{Pin-str on } F\}/\text{isom} \simeq H^1(B;\mathbb{Z}_2)$

$$H^1(B;\mathbb{Z}/2) \circlearrowright \{\text{Pin-str on } F\}/\text{isom}$$

def. by :

$$\beta \cdot P^\# = P^\# \times_{\mathbb{Z}/2} S(\beta) \leftarrow \text{sphere bdle of } \beta$$

(2-fold cover of B)

$$\overset{\text{(notation)}}{=} P^\# \otimes \beta$$

as an affine space over $H^1(B;\mathbb{Z}/2) \simeq \{\text{line bdles}/B\}/\text{isom}$

$$w_1(\beta) \longleftarrow\!\shortmid \beta$$

Rk: F (+ riem. metric) $\rightsquigarrow \mathrm{Pin}_n \to \mathrm{Pin}(F) \subset C\ell(F)$ bundle of groups.

$$\begin{tikzcd} \mathrm{Pin}(F) \arrow[d] \\ B \end{tikzcd}$$

$\times$

Pin-str $\}$ $P^\#$ principal bdle $\rightsquigarrow \mathrm{Pin}_n \to \mathrm{Aut}\,P^\#$: bundle of groups.

$$\begin{tikzcd} \mathrm{Aut}\,P^\# \arrow[d] \\ B \end{tikzcd}$$

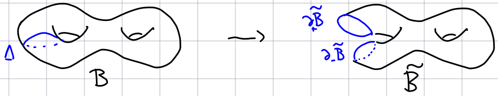
Prop: $\mathrm{Aut}\,P^\# \simeq \mathrm{Pin}(F \otimes \Lambda^{top}(F))$.

Def : <u>w - Twisted Pin - structure</u> , $w \in H^2(B;\mathbb{Z}/2)$

assume
$$\begin{tikzcd} \beta \arrow[d] \\ \Lambda \arrow[r, hook] & B \end{tikzcd}$$
line bdle representing w

smooth hypersurface

$$\left\{ \begin{array}{l} \bullet\ w|_{B\setminus\Lambda} = 0 \in H^2(B\setminus\Lambda;\mathbb{Z}/2) \\ \bullet\ w|_{\Lambda} = w_1(\beta) \in H^2(\Lambda;\mathbb{Z}/2) \end{array} \right]$$
$\rightsquigarrow$ can do it more Gally with Cech cohomol...



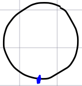
$\rightsquigarrow$ an w-twisted Pin-structure on $\begin{array}{c} F \\ \downarrow \\ \tilde{B} \end{array}$ is :

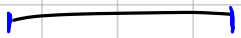
$$\left\{ \begin{array}{l} \bullet\ \text{a Pin-str } \tilde{P}^\# \text{ on } \tilde{B} \\ \bullet\ \text{an isom} \end{array} \right.$$
$$\begin{tikzcd} \tilde{P}^\#|_{\partial_+\tilde{B}} \arrow[r, phantom, "\simeq"] \arrow[d] & \tilde{P}^\#|_{\partial_-\tilde{B}} \otimes \beta \arrow[d] \\ \partial_+\tilde{B} \arrow[r, phantom, "\simeq"] & \partial_-\tilde{B} \end{tikzcd}$$

Prop . exist iff $w_2(F) + w = 0$

. classified by $H^1(B;\mathbb{Z}/2)$ as before ex. $B = S^1$, $\rho \in \mathcal{L}_k \mathrm{Gr} V$ $(\mu(\rho) = k)$

$$\begin{tikzcd} \rightsquigarrow F_\rho \arrow[r, hook] \arrow[d] & D^2 \times V \arrow[d] \\ S^1 = \partial D \arrow[r, hook] & D \end{tikzcd} \qquad \begin{tikzcd} \rightsquigarrow \tilde{F}_\rho \arrow[d] \\ {[0,1]} = \tilde{B} \end{tikzcd}$$





a twisted Pin-str for ρ is
- a Pin-str $\tilde{P}^\#$ on $\tilde{F}_\rho$
- an isom $P^\#_1 \simeq \tilde{P}_0 \otimes \Lambda^{top}(\rho(1))^{\otimes k}$

Lemma: twisted Pin-str for ρ

$$\Rightarrow \det(D_{D\rho}) \simeq \Lambda^{top}(\rho(0))$$

Proof : • $\underline{k=0}$: can assume ρ is constant

$\Rightarrow F_\rho$ trivial $\Rightarrow$ | $\ker D_{D,\rho} \simeq \rho(0)$
coker $D_{D\rho} = 0$

$\nearrow$ trivial Pin-str$^\wedge$ $\det(D_{D,\rho}) \underset{id}{\simeq} \Lambda^{top}(\rho(0))$

$\searrow$ nontriv Pin-str$^\wedge$ $\det(D_{D,\rho}) \underset{-id}{\simeq} \Lambda^{top}(\rho(0))$

• $\underline{k=-1}$ take orthog splitting $V = V_+ \oplus V_-$

$$\begin{matrix} & \cup & \cup \subset \dim_{\mathbb{C}} = 1 \\ & \Lambda_+ & \Lambda_- \\ & \text{lag.} & \end{matrix}$$

$$\rho(s) := \Lambda_+ \oplus \left(e^{-\pi s I_V} \cdot \Lambda_-\right)$$

$\rightarrow$ an x° on F_ρ, with monodromy $Id_{\Lambda_+} \oplus (-Id_{\Lambda_-}) = A$

v. space

$\underline{\text{sub-lemma}}$: $A \in O(F)$ involution

$$\left\{\begin{matrix}\text{preimages of } A \\ \text{in } Pin(F)\end{matrix}\right\} \xleftrightarrow{1:1} \text{orientations of } F^{anti} = Ker(Id + A) = \Lambda^-$$

$A^\# = v_1 \cdot \ldots \cdot v_r \longleftarrow\!| \; v_1 \wedge \ldots \wedge v_r$

· twisted Pin-str : lift of A to an isom $\tilde{P}_0^\# \otimes \lambda^{top}(\rho(1)) \simeq \tilde{P}_1^\#$

· $\Leftrightarrow$ a pre-image $A^\# \in \mathrm{Pin}(\Lambda \otimes \lambda^{top}\Lambda) \times_{\mathbb{Z}/2} S(\lambda^{top}(\Lambda))$
of $A \otimes \mathrm{id}$

$\Leftrightarrow$ an orientat° of Λ_-

now
$$\left| \begin{array}{l} \mathrm{Ker}\, D_{D_\rho} \simeq \Lambda_+ \\ \mathrm{Coker}\, D_{D,\rho} = 0 \end{array} \right.$$

$\Rightarrow$ orientation of Λ_- $\rightsquigarrow$ isom $\mathrm{Det}(D_{D,\rho}) \simeq \lambda^{top}(\Lambda)$

· <u>other k's</u> : ↗ Pin-str only dep. on k mod 2

↘ glue k times the prev constr... $\square$

• Branes from Pin-structures

$\eta_V^2 : \Lambda^{top}_{\mathbb{C}}(V)^{\otimes 2} \xrightarrow{\sim} \mathbb{C}$ quadratic complex vol form

↪ squared phase map $\alpha_V : Gr_n V \longrightarrow S^1$

$$\Lambda \longmapsto \frac{\eta_V(\overbrace{v_1 \wedge \cdots \wedge v_n}^{\text{basis of } \Lambda})^2}{|\eta_V(v_1 \wedge \cdots \wedge v_n)|^2}$$

Def: Linear brane: triple $(\Lambda, \alpha^\#, P^\#)$ st

- $\Lambda \in Gr_n V$
- $\alpha^\# \in \mathbb{R}$ st $e^{2i\pi\alpha^\#} = \alpha_V(\Lambda)$ $(\Leftrightarrow$ grading $\tilde{\Lambda} \in \widetilde{Gr_n} \Lambda)$
- $P^\#$ principal homog Pin-space, with $\Lambda \simeq P^\# \times_{Pin_n} \mathbb{R}^n$

Def:

$$\begin{tikzcd} E & F \arrow[l, hook'] \arrow[dl] \\ B \arrow[from=u] & \end{tikzcd}$$

E : symplectic v.b. with $2c_1(E)=0$

$\Rightarrow \eta_E^2 : \Lambda^{top}_{\mathbb{C}}(E) \longrightarrow \mathbb{C} \quad \Rightarrow \alpha_E : G_n(E) \longrightarrow S^1$

F: Lagrangian subbundle

a <u>brane str.</u> on F is

- $\alpha^\# : B \longrightarrow \mathbb{R}$ s.t. $\exp(2i\pi\alpha^\#(b)) = \alpha_E(F_b)$
- $\underset{B}{\overset{P^\#}{\downarrow}}$: Pin-str. on F.

↳ exist iff

$$\begin{cases} 0 = \mu_F \in H^1(B;\mathbb{Z}) \\ 0 = w_2(F) \in H^2(B;\mathbb{Z}/2) \end{cases}$$

↳ Affine space over $H^0(B;\mathbb{Z}) \oplus H^1(B;\mathbb{Z}/2)$

<u>Rk:</u> $\xi_E \subset \Lambda^{top}_{\mathbb{C}}(E)$: Real line bdl on which $\pm\eta_E \in \mathbb{R}$ (square root of η_E^2)

brane str $\Rightarrow$ F oriented relat. to ξ_E

Abstract vs concrete Brown Gr_n

$$\left\{ \text{lifts } \begin{tikzcd} & Gr\,V^{\#} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Gr\,V \end{tikzcd} \right\} \Big/ \text{htpy} \quad \xleftrightarrow{1:1} \quad \left\{ (\alpha^{\#}, P^{\#}) \right\} \Big/ \text{isom}$$

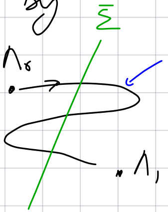
$$\begin{tikzcd} Gr\,V^{\#} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_1 \times U_2 \arrow[d] & & BPin_n \arrow[ll] \arrow[d] \\ Gr\,V \arrow[r] \arrow[dr, "\alpha_V"'] \arrow[ddr] & K(\mathbb{Z},1) \times K(\mathbb{Z}/2,2) \arrow[d, phantom, "\cong" rotate=90] & & BO_n \arrow[ll] \\ & S^1 & & \\ & \left\{ \begin{matrix} \text{Grassmannian} \\ \text{of } n\text{-planes} \end{matrix} \right\} \arrow[uurr] & & \end{tikzcd}$$

$$\left.\begin{aligned} \Lambda_0^\# &= (\Lambda_0, \alpha_0^\#, P_0^\#) \\ \Lambda_1^\# &= (\Lambda_1, \alpha_1^\#, P_1^\#) \end{aligned}\right\}$$ Linear branes with $\Lambda_0 \pitchfork \Lambda_1$

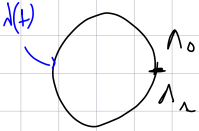
by

$i(\Lambda_0^\#, \Lambda_1^\#)$

$o(\Lambda_0^\#, \Lambda_1^\#)$



CR op



$\bar\partial_\nabla$: "orientation operator" for $(\Lambda_0^\#, \Lambda_1^\#)$

$$\boxed{\begin{aligned} i(\Lambda_0^\#, \Lambda_1^\#) &= \operatorname{ind}(\bar\partial_\nabla) \\ o(\Lambda_0^\#, \Lambda_1^\#) &\simeq \det(\bar\partial_\nabla), \text{ canonically} \end{aligned}}$$

Shift: $\Lambda^\# = (\Lambda, \alpha^\#, P^\#) \rightsquigarrow S^\sigma \Lambda^\# = (\Lambda, \alpha^\# - \sigma, P^\# \otimes \lambda^{top}(\Lambda)^{\otimes \sigma})$

$\sigma \in \mathbb{Z}$

Prop: $i(\Lambda_0^\#, S^\sigma \Lambda_1^\#) = i(\Lambda_0^\#, \Lambda_1^\#) - \sigma$

$o(\Lambda_0^\#, S^\sigma \Lambda_1^\#) \simeq o(\Lambda_0^\#, \Lambda_1^\#)$ canon.

3.8 Definition of the Fukaya category

(M, ϕ_M) exact symplectic with $2c_1(M) = 0$

$+ \; I_M$: a.c.s compatible.

$+ \; \eta_M^2 : \Lambda_{\mathbb{C}}^{top}(TM)^{\otimes 2} \longrightarrow \mathbb{C}$ quadratic cpx vol. form.

$\hookrightarrow \alpha_M : Gr(TM) \longrightarrow S^1$

· **objects** : Lagrangian branes $L^\# = (\overset{\nearrow}{L}, \underbrace{\alpha^\#, P^\#}_{\text{brane str. on } TL \downarrow L})$

exact lag.

$x \in L \rightsquigarrow \Lambda_x^\# = (T_xL, \alpha^\#(x), P_x^\#)$ lin. brane on T_xM.

· **morphisms** $hom(L_0^\#, L_1^\#) = CF^{\overset{\nearrow}{*}}(L_0^\#, L_1^\#) = \bigoplus_{y \in L_0 \cap L_1} |o(y)|_{\mathbb{K}}$

graded by $i(y)$

with $\rightarrow i(y) = i(\Lambda_{0,y}^\#, \Lambda_{1,y}^\#)$ $\quad \rightarrow \mathbb{K}$: field , δ : real line , $\{o, o'\}$: orientat°s of δ

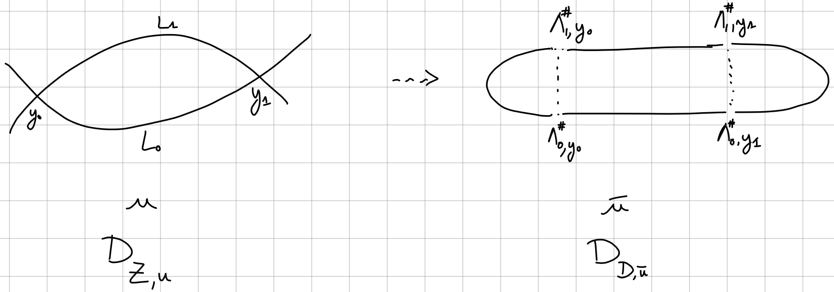
$\rightarrow o(y) = o(\Lambda_{0,y}^\#, \Lambda_{1,y}^\#)$ $\quad |\delta|_{\mathbb{K}} = \{ k\,o + k'\,o' \,/\, k + k' = 0 \} \subset \mathbb{K}o \oplus \mathbb{K}o'$

"$\mathbb{K}$-normalization of δ"

(need to fix Floer, perturb datum, etc. .)

• μ^d maps

→ $\mu^1(a) = (-1)^{\deg a} \partial a$, with $\gamma^{y_0,y_1} = \sum_{u \in \mathcal{M}^*_Z(y_0,y_1)} |c_u|_{\mathbb{K}} : |o(y_1)|_{\mathbb{K}} \to |o(y_0)|_{\mathbb{K}}$



Gluing: $\det(D_{D,\bar{u}}) \simeq \delta(y_0) \otimes \det(D_{Z,u}) \otimes \delta(y_1)$

↑ oriented by the Pin str.

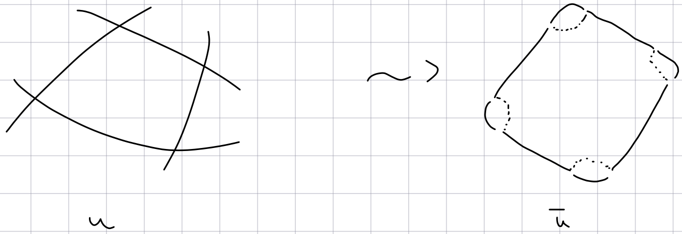
$\Rightarrow \det(D_{Z,u})$ oriented rel. to $\delta(y_0), \delta(y_1)$

(u regular: $\det(D_u) = \lambda^{top}(\mathrm{Ker}\, D_u)$)

$\Rightarrow T_{[u]}\mathcal{M}^*_Z(y_0,y_1)$ also, from $\mathbb{R} \to \mathrm{Ker}\, D_{Z,u} \to T_{[u]}\mathcal{M}^*_Z(y_0,y_1)$

$\Rightarrow$ Get isom $c_u : o(y_1) \xrightarrow{\sim} o(y_0)$

$d \geq 2$



$$\mathrm{Det}\left(D_{D,\bar{u}}\right) \simeq \delta(y_0) \otimes \det\left(D_{S,u}\right) \otimes \delta(y_1)^\vee \otimes \cdots \otimes \delta(y_d)^\vee$$

$\rightarrow$ family
$$\begin{tikzcd} \mathcal{G}^d \arrow[d] \\ \mathcal{R}^d \end{tikzcd}$$
$$\rightsquigarrow \Lambda^{top} T_{(r,u)} \mathcal{M}_{\mathcal{G}^d}(y_0; \dots) \simeq \Lambda^{top}(T_r \mathcal{R}^d) \otimes o(y_0) \otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee$$

$$\begin{tikzcd} \cdot\ \mathcal{R}^{d+1} = \mathrm{Conf}_{d+1}(\partial D) \big/ \mathrm{Aut}\, D & \text{oriented by} \arrow[dl] \\ \{z_0, z_1, z_2 \text{ fixed}\} \subset \partial D^{d-1} \arrow[u, "\text{slice}"] & \end{tikzcd}$$

Rk: $\bar{\mathcal{R}}^{d+2-m} \times \bar{\mathcal{R}}^{m+1} \hookrightarrow \partial \bar{\mathcal{R}}^{d+1}$
changes orientat° by $(-1)^{m(d-n)+m+m}$

$\Rightarrow$ get $c_{(r,u)} : o(y_1) \otimes \cdots \otimes o(y_d) \xrightarrow{\sim} o(y_0)$

$\Rightarrow$ def μ^d by $\sum'_{(r,u) \in \mathcal{M}_{\mathcal{G}^d}} |c_{(r,u)}|_{\mathbb{K}} \cdot (-1)^{i(y_1) + 2i(y_2) + \cdots + d\, i(y_d)}$

Th: this is an A_∞- structure

3.9 The PSS isomorphism

Th: $L^\#$: exact Lagrangian brane $\subset M$

($+$ perturbation & Floer datum)

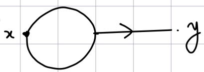
· $\hom_{\mathcal{F}(M)}(L^\#, L^\#) \simeq \mathrm{Morse}(L)$ (quasi-isom of A_∞-Algebras)

· in particular, $HF^*(L^\#, L^\#) \simeq H^*(L, \mathbb{K})$

· **Proof** main ingredient: Piunikhin–Salamon–Schwarz isom

$$CF^*(L^\#, L^\#) \underset{SSP}{\overset{PSS}{\rightleftarrows}} CM^*(L, (f, X))$$

· $\langle PSS\, x, y \rangle = \# \{ \;\; \}$



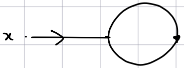
pairs (u, γ), $u : D^2 \to M$ inhomog J-hol, with L.B.C

$\gamma : \mathbb{R}_+ \to L$ flow line with $\lim_{+\infty} \gamma = y$

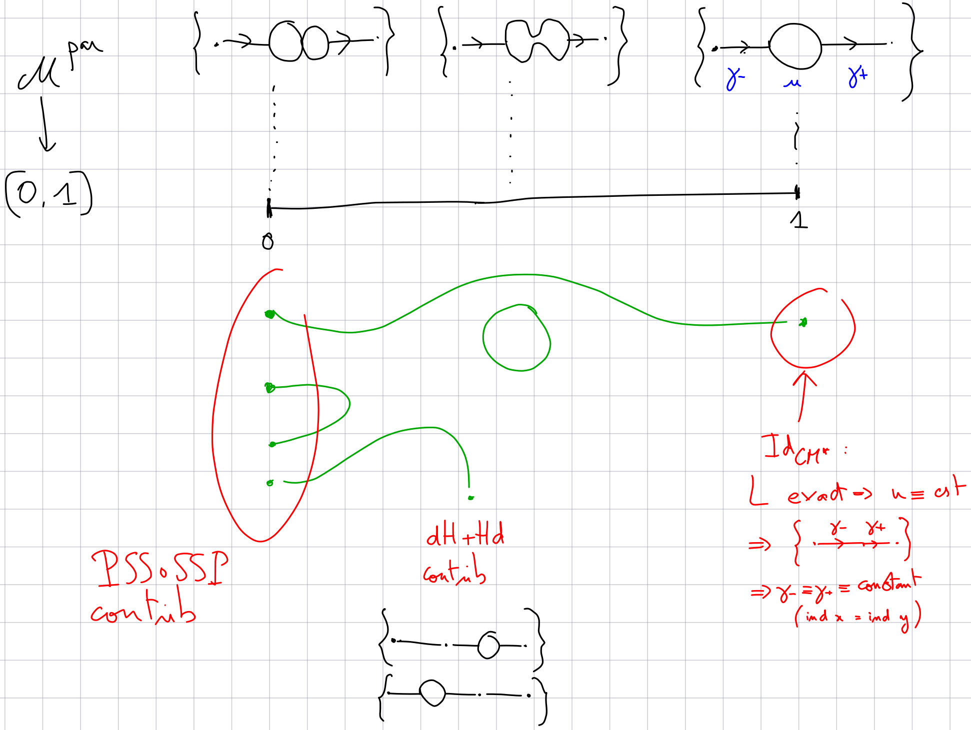
st $\lim_{-1} u = x$

$\lim_{1} u = \gamma(0)$.

· $\langle SSP\, x, y \rangle = \# \{ \;\; \}$



$$PSS \circ SSP = Id_{CM^*} + dH + Hd$$



Id_{CM^*} :

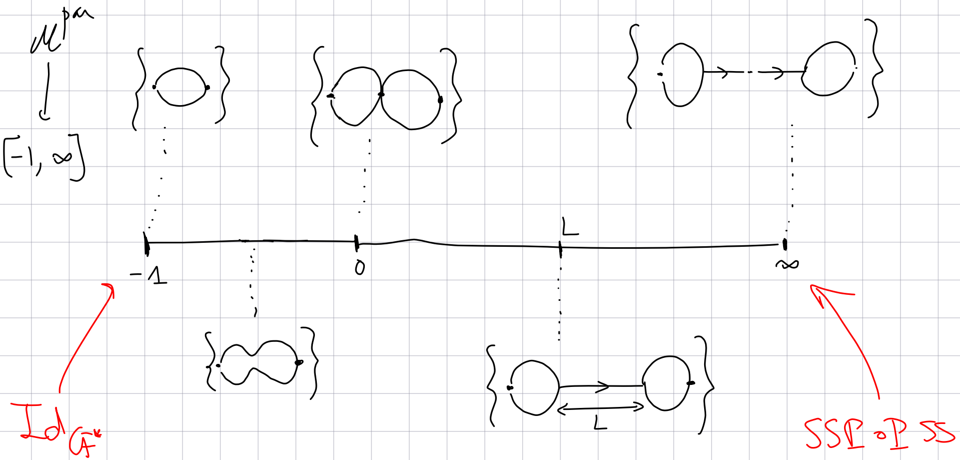
L exact ⇒ $u \equiv cst$

$\Rightarrow \{ \cdot \xrightarrow{\gamma_-} \xrightarrow{\gamma_+} \cdot \}$

$\Rightarrow \gamma_- \equiv \gamma_+ \equiv$ constant

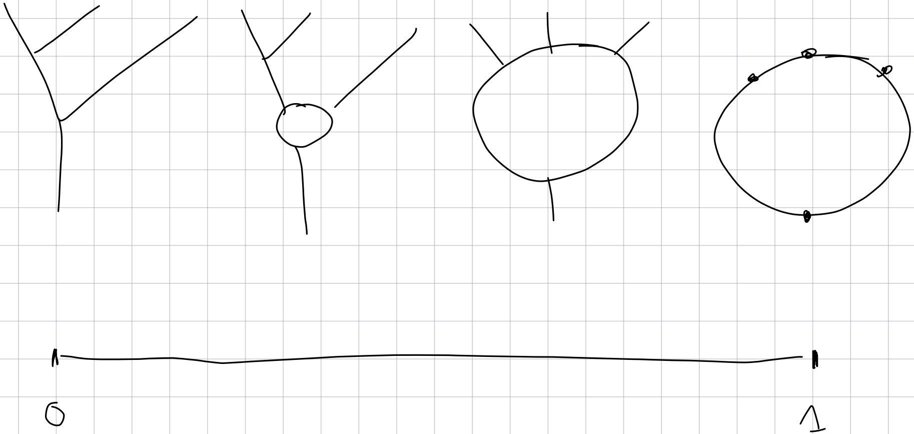
(ind x = ind y)

$$SSP \circ PSS = Id_{CF^*} + dH + Hd$$



higher μ^d maps : look at families S interpolating between trees & discs

$$\begin{tikzcd} S \arrow[d] \\ \mathcal{J}^{d+1} \times [0,1] \\ \mathcal{R}^{d+1} \arrow[u, equal] \end{tikzcd}$$



3.10 Surfaces

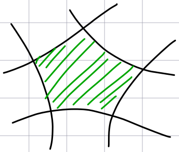
$M = (S, I_S)$ Riemann surface

$\underline{L}^\# = (L_1^\#, L_2^\#, \dots, L_m^\#)$: exact Lag branes in G^{al} position : $\begin{cases} L_i \pitchfork L_j \\ L_i \cap L_j \cap L_k = \emptyset \end{cases}$

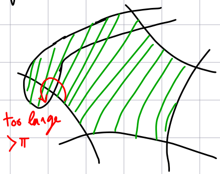
$\mathcal{F}(\underline{L}^\#)^{\rightarrow}$: directed full A_∞-subcat :
- $\underline{ob}$: L_i's
- $hom(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) & \text{if } i < j \\ \mathbb{K} \cdot e_{L_i} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$

$\underline{Th}$: [Automatic regularity] No need to perturb! For the Floer/pert. datum, one can just take $H=0$, $J = I_S$, all the moduli spaces involved are regular.

$\Rightarrow$ count honest holom. polygons, which are explicit by Unif. theorem:



<u>Counts</u>

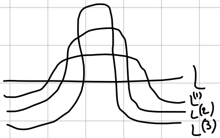


<u>Doesn't count</u>: wrong index

Rk: one can get the non-directed full subcat $\mathcal{F}(\underline{L})$ by fixing a Hamiltonian H_i for each L_i, and replace L_i by $L_i^{(k)} = \phi^k_{H_i}(L_i)$ so that

for any sequence $i_0, i_1, \dots, i_d$,

$L_{i_0}, L_{i_1}^{(1)}, \dots, L_{i_d}^{(d)}$ are in G^{al} position.



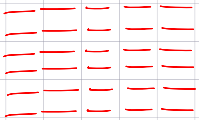
Brane structures:

now, $\Lambda^{top}_{\mathbb{C}}(TM) = TM$.

$\eta_M^2 : TM \otimes_{\mathbb{C}} TM \longrightarrow \mathbb{C}$: nowhere vanishing quadratic differential.

↳ $\alpha_M : \mathbb{RP}(TM) \longrightarrow S^1$

↳ $\xi_M = \alpha_M^{-1}(1)$: unoriented line field on M



$L^\# \Leftrightarrow (L, H, \beta)$ with :

- $$\begin{tikzcd} H \subset \pi^*(TM|_L) \arrow[d] & TM|_L \arrow[d] \\ {[0,1]\times L} \arrow[r, "\pi"'] & L \end{tikzcd}$$ with $\begin{cases} H_0 = \xi_M|_L \\ H_1 = TL \end{cases}$

$\Leftrightarrow$ grading : can lift $$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp(2i\pi\cdot)"] \\ {[0,1]\times L} \arrow[ru, dashed, "a"] \arrow[r, "\alpha_M(H)"'] & S^1 \end{tikzcd}$$ with $a(0,x) = 0$

$\to$ take $a^\# = a(1,\cdot)$

- $$\begin{tikzcd} \beta \arrow[d] \\ L \end{tikzcd}$$ real line bdle $\Longleftrightarrow$ Pin_1 - structure

$$\begin{tikzcd} \overset{\{\pm e, \pm v\}}{\overset{\shortparallel}{Pin_1}} \arrow[r, shift left] & \overset{\{e, v\}}{\overset{\shortparallel}{O_1}} \arrow[l, dashed, bend left, shift left] \end{tikzcd}$$ splits

$e \longleftarrow\!\!\shortmid\ e$

$v \longleftarrow\!\!\shortmid\ v$

$\Rightarrow$ any O_1 - bdle has a canon. lift $P_0^\#$

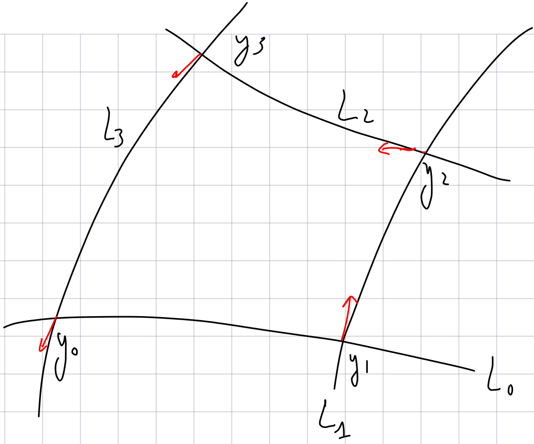
$\Rightarrow$ $\{\text{line bdles}\} \overset{1:1}{\longleftrightarrow} \{Pin_1\text{-str}\}$

$\beta \longmapsto P_0^\# \otimes \beta$

- $L_0^\#, L_1^\#, y \in L_0 \cap L_1 \to$ $$\begin{tikzcd} T_yL_0 & \xi_{M_y} \arrow[l, "H_0"'] \arrow[r, "H_1"] & TL_1 \\ \arrow[from=1-1, to=1-3, bend right, "\text{total angle } \measuredangle(y) \in \mathbb{R} \setminus \pi\mathbb{Z}"'] \end{tikzcd}$$

$$\boxed{\begin{aligned} i(y) &= \left[\frac{\measuredangle(y)}{\pi}\right] + 1 \\ o(y) &= \hom(\beta_{0|y}, \beta_{1,y}) \otimes (TL_1)_y^{\otimes i(y)} \end{aligned}}$$

next lowest integer



- $\beta_i|_{y_i} \simeq \beta_i|_{y_{i+1}}$ via // transport along the L_i edge.
- likewise, $\beta_d|_{y_0} \simeq \beta_0|_{y_0}$
- orient $T_{y_i}L_i$ and $T_{y_0}L_d$ as indicated $\rightarrow \theta_i$

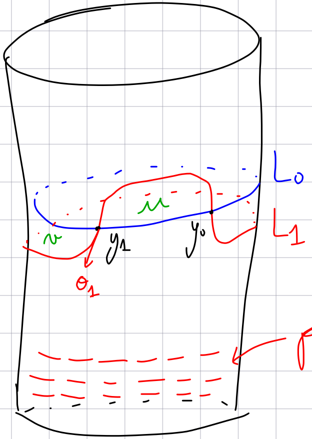
↪ $\tilde{c}_{\tilde{u}} : o(y_d) \otimes \cdots \otimes o(y_1) \longrightarrow o(y_0)$

$$\hom(\beta_d|_{y_d}, \beta_{d-1}|_{y_d}) \otimes TL_d^{\otimes i(y_d)} \otimes \cdots \otimes \hom(\beta_2|_{y_2}, \beta_1|_{y_2}) \otimes TL_2^{\otimes i(y_2)} \otimes \hom(\beta_1|_{y_1}, \beta_0|_{y_1}) \otimes TL_1^{\otimes i(y_1)}$$

$$\longrightarrow \hom(\beta_d|_{y_0}, \beta_0|_{y_0}) \otimes TL_d^{\otimes i(y_0)}$$

$$v_d \otimes \theta_d^{\otimes i(y_d)} \otimes \cdots v_1 \otimes \theta_1^{\otimes i(y_1)} \mapsto (v_1 \circ \cdots \circ v_d) \otimes \theta_0^{\otimes i(y_0)}$$

ex :



$H_0 : \xi_M \to TL_0$ trivial htpy

$H_1 : \xi_M \to TL_1$ obtained by deforming $L_1 \to L_0$

β_0, β_1 : trivial

$i(y_0) = 0$ $\quad$ $i(y_1) = 1$

$o(y_0) = \mathbb{R}$ $\quad$ $o(y_1) = (TL_1)_{y_1} = \mathbb{R}\,\theta_1$

in $CF(L_0, L_1)$, $\quad u : y_0 \to y_1 \quad c_u : 1 \mapsto \theta_1$

$v : y_0 \to y_1 \quad c_v : 1 \mapsto -\theta_1 \quad \Rightarrow \partial = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

3.11 More general approaches

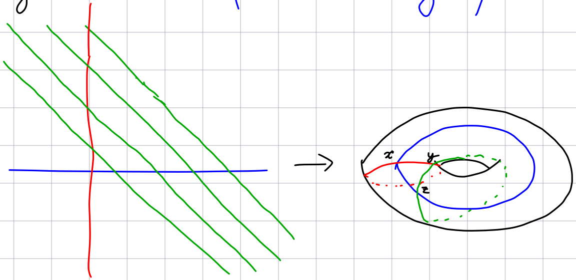
Ref: FOOO's book.

Goal: * remove exactness assumptions on M & L

* eventually, allow Lagrangian immersions (→ Akaho - Joyce)

Main issues:

① Failure of compactness: ex:



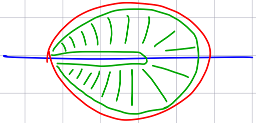
(Recall: in the exact setting,

all strips from x → y have same area (= A(y) - A(x))

& same for polygons

Fix: use a Novikov ring to keep track of the area.

② disc bubbling



$\Rightarrow \partial^2 \neq 0$

Fix: • work in the curved A_∞-setting

• use "bounding cochains"

③ Failure of transversality for 1-punctured discs

 ↻ Aut acts transitively on int(D) => cannot use domain-dep^t perturbations

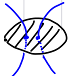
Fix: Kuranishi structures. (controversial...)

Other approaches: (not yet fully implemented)

→ Polyfoldz (HWZ)

→ Pardon's work

→ Stabilizing divisors (Cieliebak - Mohnke)



introduce marked pts to kill the action of Aut

① Novikov coefficients

$\mathbb{K} = \mathbb{R}, \mathbb{Q} \dots$ ground field

Def: • Novikov ring: $\Lambda_0 = \left\{ \sum_{i=0}^{\infty}{}' \, a_i T^{b_i} \;\middle|\; a_i \in \mathbb{K},\ b_i \in \mathbb{R}_{\geq 0},\ \lim_{i\to\infty} b_i = +\infty \right\}$

• Novikov field $\Lambda = \mathrm{Frac}(\Lambda_0) = \left\{ \sum_{i=0}^{\infty}{}' \, a_i T^{b_i} \;\middle|\; a_i \in \mathbb{K},\ b_i \in \mathbb{R}_{\geq 0},\ \lim_{i\to\infty} b_i = +\infty \right\}$

Rk: other variants exist.

Def: • $CF(L_0^\#, L_1^\#; \Lambda) = \bigoplus_{x \in L_0 \cap L_1} |o(x)|_\Lambda \simeq \bigoplus_x \Lambda . x$

$= \int_D u^* \omega$ sympl. area

• $\langle \partial x, y \rangle = \sum_u{}' \, \varepsilon(u) \, . \, T^{A(u)}$

$\uparrow$

$= \pm 1 \leftarrow$ in fact, migth have to be in $\mathbb{Q} \setminus \mathbb{Z} \dots$ (vfc's...)

• $\langle \mu^d(a_d, \dots, a_1), a_0 \rangle = \sum_u{}' \, \varepsilon(u) . T^{A(u)} \in \Lambda_0$ by Gromov compactness

$\hookrightarrow \forall K,$

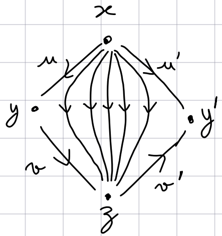
$\mathcal{M}(x,y) \cap \{ A(u) \leqslant K \}$ is compact

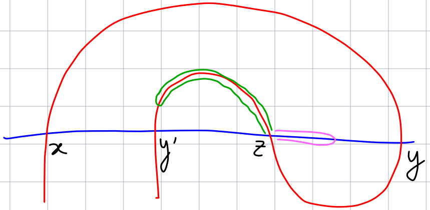
Rk: Univ. coeff th doesn't apply

$\partial^\Lambda \neq \partial^{\mathbb{Z}} \otimes id_\Lambda$

Prop: (Ignoring issue 2) $\partial^2 = 0$ and A_∞-relations continue to hold

Proof: $\partial^2 = 0$





$$\langle \partial^2 x, z \rangle = \sum_{x \xrightarrow{u} y \xrightarrow{v} z}' \pm T^{A(u)+A(v)}$$

Key: If $I \subset \overline{\mathcal{M}}(x,z)$ is an interval component such that $\partial I = \{(u,v),(u',v')\}$, then

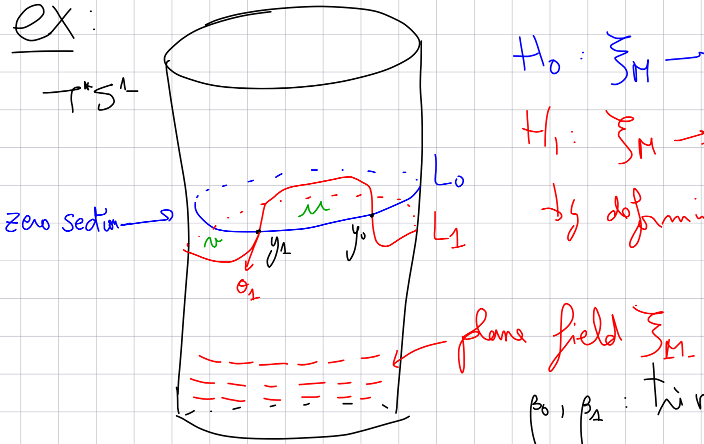
$A(u) + A(v) = A(u') + A(v') = A(w)$, with $w \in \mathrm{int}\, I$

Result (would) follow from $\partial \overline{\mathcal{M}}(x,z) = \bigcup_y \mathcal{M}(x,y) \cdot \mathcal{M}(y,z)$

Same for A_∞-rel. □

ex :

T^*S^1



$H_0 : \xi_M \to TL_0$ trivial htpy

$H_1 : \xi_M \to TL_1$ obtained by deforming $L_1 \to L_0$

β_0, β_1 : trivial

$i(y_0) = 0 \qquad i(y_1) = 1$

$o(y_0) = \mathbb{R} \qquad o(y_1) = (TL_1)_{y_1} = \mathbb{R}\,\theta_1$

in $CF(L_0^\#, L_1^\#)$, $\quad u : y_0 \to y_1 \quad c_u : 1 \longmapsto \theta_1$

$v : y_0 \to y_1 \quad c_v : 1 \longmapsto -\theta_1$

$$\Rightarrow \quad 0 \to CF^0(L_0^\#, L_1^\#) \longrightarrow CF^1(L_0^\#, L_1^\#) \longrightarrow 0$$

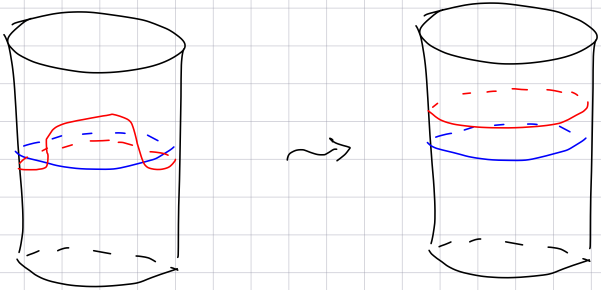
$$y_0 \longmapsto (T^{A(u)} - T^{A(v)}).\, y_1$$

$$\Rightarrow HF^*(L_0^\#, L_1^\#; \Lambda) = \begin{cases} \Lambda \oplus \Lambda & \text{if } A(u) = A(v) \ (\Leftrightarrow L_1 \text{ exact.}) \\ 0 & \text{if } A(u) \neq A(v) \end{cases}$$

Prop: $HF(L_0^\#, L_1^\#, \Lambda)$ invariant under Hamilt isotopies.

Warning: wrong if one works over the ring Λ_0

ex: $A(u) > A(v) \Longrightarrow$ L_0 and L_1 are displaceable



$$HF^0(L_0^\#, L_1^\#; \Lambda_0) = 0 \qquad HF^0(L_0^\#, L_1^\#; \Lambda_0) = 0$$

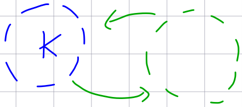
$$HF^1(L_0^\#, L_1^\#; \Lambda_0) = \frac{\Lambda_0}{(T^{A(u)} - T^{A(v)})\Lambda_0} \qquad HF^1(L_0^\#, L_1^\#; \Lambda_0) = 0$$

Λ_0-torsion is sensitive to Hamilt. isotopies.

Rk: exact Lagrangians are not displaceable from themselves ($HF(L^\#, L^\#) \simeq H^*(L)$)

$\Longrightarrow$ **Th:** [Gromov] No compact exact Lagrangians in $\mathbb{R}^{2n}$.

Proof: can always displace $K \subset \mathbb{R}^{2n}$



□

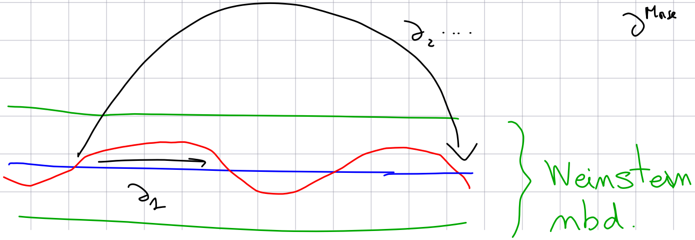
$\exists\, \alpha > 0$ s.t. $A(u) = \tau \mu(u)$
$u \in \pi_2(M,L)$

"minimal Maslov number"
$N_L = \min\{\mu(u) \,/\, u \in \pi_2(M,L), A(u) > 0\}$

Th: [Oh spectral sequence] $L \subset M$ positively monotone + $N_L > 2$

There's a spectral sequence whose E_2-page $\simeq H^*(L;\mathbb{Z}_2)$ and E_∞-page $\simeq HF^*(L^\#, L^\#; \mathbb{Z}_2)$

Idea of proof: filter by energy: $\partial^{Floer} = \overbrace{\underset{\underset{\partial^{Morse}}{\wr}}{\partial_1}}^{\text{small energy}} + \partial_2 + \cdots$



Th: [Poźniak, Schmaschke]

$L_0, L_1 \subset M$ posit. mon. + $N_{L_i} > 2$

$L_0 \cap L_1$ *clean* (i.e. $L_0 \cap L_1$ smooth, and $T_x(L_0 \cap L_1) = T_x L_0 \cap T_x L_1$)

Then $H^*(L_0 \cap L_1) \Longrightarrow HF^*(L_0, L_1)$ ($\mathbb{Z}_2$-coeffs)

Conj (Biran - Cornea, Wide/narrow dichotomy)

$L \subset M$ positively monotone + $N_L > 2$

Then, either $HF(L,L) \simeq \begin{cases} H^*(L) : L \text{ is "wide"} \\ 0 \quad : L \text{ is "narrow"} \end{cases}$

Believed to be false, but no known counter-examples.

② Curved A_∞-structures & bounding cochains

Recall: $(A, m = (m^0, m^1, \dots))$ is a curved A_∞-algebra if

$m^d : A^{\otimes d} \longrightarrow A$ of deg $2-d$ satisfies:

- $\mu^1 \circ \mu^1 \pm \mu^2(\mu^0, -) \pm \mu^2(-, \mu^0) = 0$
- $\mu^1 \circ \mu^2 \pm \mu^2(\mu^1(-), -) \pm \mu^2(-, \mu^1(-)) \pm \mu^3(\mu^0, -, -) \pm \dots = 0$
- $\mu^2(\mu^2(-,-),-) \pm \mu^2(-, \mu^2(-,-)) \pm \mu^1 \circ \mu^3 \pm \mu^3(\mu^1(-), -, -) \pm \dots \pm \mu^4(\mu^0, -, -, -) \pm \dots = 0$

etc...

"Bulk deformations"

Def: Deformat° of A_∞-str: (A, m) curved A_∞-alg, $b \in A^1$

Assume for simplicity that $m^d \equiv 0$ for large d's

In $\mathcal{E}^{al}$, ask $b = \sum a_i T^{b_i} \cdot x_i$, (generator $\downarrow$ x_i)

with $b_i > 0, \forall i$

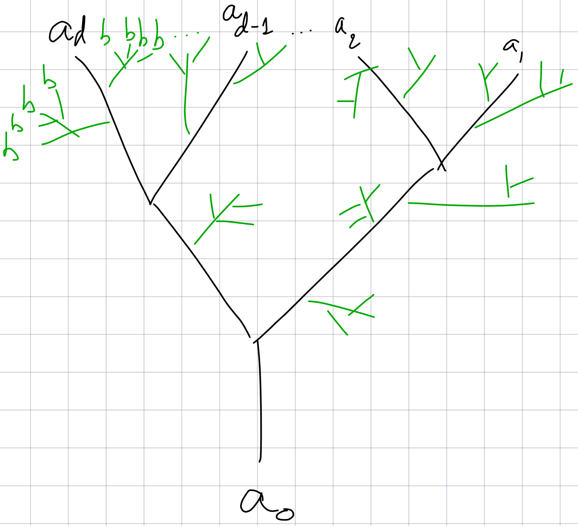
$\Rightarrow$ ensures convergence in

$\rightsquigarrow$ get a new curved A_∞-algebra (A, m_b), with

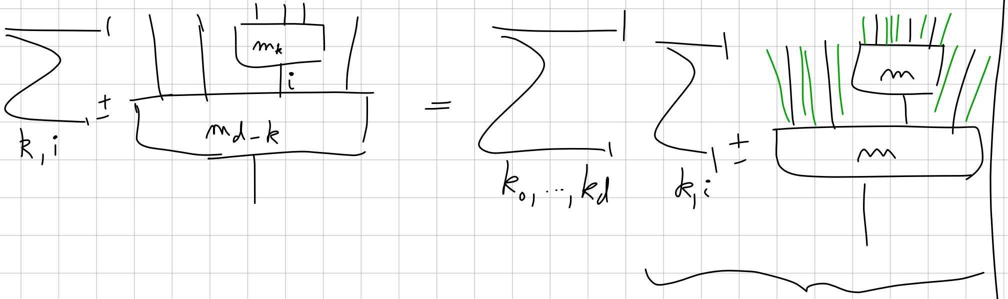
$$m_b^d(a_d, \dots, a_1) = \sum_{k_i \geqslant 0} m^{d + \sum k_i}(\underbrace{b, \dots, b}_{k_d}, a_d, \underbrace{b, \dots, b}_{k_{d-1}}, a_{d-1}, \dots \underbrace{b, \dots, b}_{k_1}, a_1, \underbrace{b, \dots, b}_{k_0})$$

$$m_b^d(a_d,\dots,a_1) = \sum_{k_i \geqslant 0} m^{d+\sum_i k_i}\Big(\underbrace{b,\dots,b}_{k_d}, a_d, \underbrace{b,\dots,b}_{k_{d-1}}, a_{d-1},\dots \underbrace{b,\dots,b}_{k_1}, a_1, \underbrace{b,\dots,b}_{k_0}\Big)$$

ex: Mistletoe $b \in CM^2(M, f, X)$



Proof of A_∞-rel for m_d:



$= 0$ from $(d + k_0 + \dots)$'s A_∞-rel.

□

Def b is a bounding cochain if

(i) (A, m_b) is flat

$\Updownarrow$

(ii) b satisfies the Maurer-Cartan equation $\underbrace{m^0 + m^1(b) + m^2(b,b) + \cdots}_{= m_b^0} = 0$

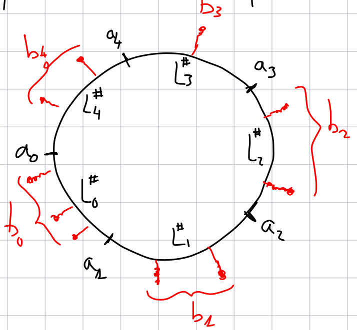
Def: $\mathcal{A}$: curved A_∞-cat $\to \mathcal{A}'$ (flat) A_∞-cat:

obj: $(L^\#, b)$, with . $L^\# \in \mathrm{Ob}\,\mathcal{A}$

. $b \in \hom^1(L^\#, L^\#)$ bounding cochain

* $\hom_{\mathcal{A}'}\left((L_0^\#, b_0), (L_1^\#, b_1^\#)\right) = \hom_{\mathcal{A}}\left(L_0^\#, L_1^\#\right)$

* $\mu_{\mathcal{A}'}^d(a_d, \dots, a_1) = \mu^{d+k_0+\dots+k_d}\big(\underbrace{b_d, \dots, b_d}_{k_d}, a_d, \underbrace{b_{d-1}, \dots, b_{d-1}}_{k_{d-1}} \dots \underbrace{b_1, \dots, b_1}_{k_1}, a_1, \underbrace{b_0 \cdots b_0}_{k_0}\big)$



Rk: quilt theory suggests that bounding cochains can retain important piece of informations that are not contained in L

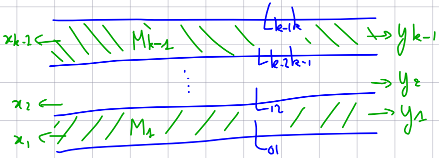
Brief intro to quilted Floer homol.

$\underline{L} = \{ pt \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \xrightarrow{L_{23}} \cdots M_{k-1} \xrightarrow{L_{(k-1)k}} pt \}$ sequence of Lagrangian corresp. $(L_{ij} \subset M_i^- \times M_j)$ (+ brane structures...)

ex: $\{ pt \xrightarrow{L_0} M \xrightarrow{L_1} pt \} \Leftrightarrow L_0, L_1 \subset M$.

↳ $CF^*(\underline{L})$: * generated by $\mathcal{I}(\underline{L}) = \{ \underline{x} = (x_0, \dots, x_k) \,/\, (x_i, x_{i+1}) \in L_{i,i+1} \}$

* ∂ counts quilted strips:



Th: [Wehrheim-Woodward] Assume:

* standard assumptions on M_i's & L_{ij}'s so that $\partial^2 = 0$ (exactness or monotonicity)
* composition $M_{i-1} \xrightarrow{L_{(i-1)i}} M_i \xrightarrow{L_{i(i+1)}} M_{i+1}$ is "embedded" ← non-generic assumption

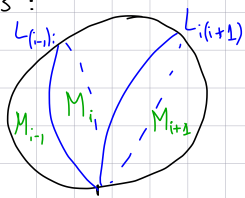
Then $HF(\underline{L}) \simeq HF(\dots, L_{(i-1)i} \circ L_{i(i+1)}, \dots)$

Conj: [Bottman-Wehrheim] if compo is only "cleanly immersed"

↳ means self-intersec^tns are clean, generic assumption

then $HF(\underline{L}) \simeq HF(\dots, (L_{(i-1)i} \circ L_{i(i+1)}, b), \dots)$

with $b \in CF(L_{(i-1)i} \circ L_{i(i+1)}, L_{(i-1)i} \circ L_{i(i+1)})$ def by counting "figure 8 bubbles":

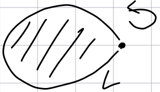


3.12 Kuranishi structures and vfc's

References:

- * Kontsevich: Enumeration of rational curves via torus action
- * Fukaya, Ono: Arnold conjecture and Gromov-Witten invariants
- * Li, Tian: Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties
- * FOOO's book
- * FOOO: Technical details on Kuranishi str. and vfc's.
- * McDuff-Wehrheim: Smooth Kuranishi atlases with trivial isotropy.
- * Joyce: Kuranishi homology & Kuranishi cohomology
- * A fight to fix Geometry's foundations. quantamagazine.org

Goal: $L^\# \subset M$ not exact $\rightsquigarrow$ $\mu^0_{L^\#} \in CF^*(L^\#, L^\#)$



$$\circlearrowright \mathrm{Aut}(D) = G = \left\{ \begin{matrix} H \to H \\ z \mapsto az+b \end{matrix} \;\middle|\; \begin{matrix} a>0 \\ b \in \mathbb{R} \end{matrix} \right\}$$

Standard strategy:

$$\begin{tikzcd} G \circlearrowright \mathcal{E} \arrow[d] & \text{Banach bdle} \\ G \circlearrowright \mathcal{B} \arrow[u, bend right, "s"'] & \leftarrow \text{Banach mfd} \end{tikzcd}$$

s ($= \bar\partial$: Fredholm section)

$\rightsquigarrow$ ① perturb

Warning: $G \circlearrowright \mathcal{B}$ is not smooth...

$$\begin{tikzcd} G \circlearrowright \mathcal{E} \arrow[d] \\ G \circlearrowright \mathcal{B} \arrow[u, bend right, "s+p"'] \end{tikzcd}$$

$\rightsquigarrow$ ② $\mathcal{M}_p = (s+p)^{-1}(0) \big/ G$

Initial moduli problem

Pb: might not have p such that

- G-equivariant
- $(s+p)^{-1}(0)$ transversely cut out

$\Rightarrow$ Fails here

Kuranishi method:

$$\begin{tikzcd} G \arrow[r, phantom, "\circlearrowright"] & \mathcal{E} \arrow[d] & \text{Banach bdle} \\ G \arrow[r, phantom, "\circlearrowright"] & \mathcal{B} \arrow[u, bend right, "s"'] & \text{Banach mfd} \arrow[l] \end{tikzcd}$$

$s \left(= \bar{\partial} : \text{Fredholm section} \right)$

Initial moduli problem

Step 1: finite dim approx

near $u \in s^{-1}(0)$: $(V_u, E_u, \Gamma_u, s_u, \psi_u)$: "Kuranishi nbd"

$$\begin{tikzcd} \Gamma_u \arrow[r, phantom, "\circlearrowright"] & E_u \arrow[d] & \text{finite dim vecta bdle} \\ \Gamma_u \arrow[r, phantom, "\circlearrowright"] & V_u \arrow[u, bend right, "s_u"'] & \text{finite dim. mfd.} \end{tikzcd}$$

s_u smooth sect°

Γ_u finite group

such that $s_u^{-1}(0)/\Gamma_u \xrightarrow[\psi_u]{\text{loc. homeo}} s^{-1}(0)/G = \mathcal{M}$ ← Unperturbed moduli space is endowed with a structure of "Kuranishi space".

ψ_u Kuranishi map

Step 2: use abstract perturbations to produce a

"virtual fundamental cycle" $VC(\mathcal{M}, ev) \in C_*(L; \mathbb{Q})$ from $ev : \mathcal{M} \to L$

Step 1: finite dim approx

$$\begin{tikzcd} & \mathcal{E} \arrow[d, shift right] & \text{Banach bdle} \\ G \circlearrowright & \mathcal{B} \arrow[u, bend right, "s \; (= \bar\partial \,:\, \text{Fredholm section})"'] & \leftarrow \text{Banach mfd} \end{tikzcd}$$

① take $\tilde{E} \subset \mathcal{E}$ finite dim. sub-bundle such that:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, shift left] & \mathcal{E}/\tilde{E} \arrow[dl, shift left] \\ \mathcal{B} \arrow[u, bend left, "s"] \arrow[ur, bend right, "\tilde{s}"'] & \end{tikzcd}$$

★ $\tilde{s} = s/\tilde{E}$ is surjective near $u \in s^{-1}(0)$

★ $\tilde{E}$ is G-invariant

② restrict to $\tilde{V} = \tilde{s}^{-1}(0) = s^{-1}(\tilde{E})$:

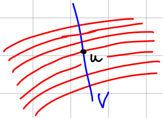
$$\begin{tikzcd} G \circlearrowright \tilde{E} \arrow[d, shift right] \\ G \circlearrowright \tilde{V} \arrow[u, bend right, "s_{|\tilde{V}}"'] \end{tikzcd}$$

here, $s_{|\tilde{V}}^{-1}(0) \xrightarrow[\text{loc. homeo}]{\sim} s^{-1}(0)$

③ Slice the G-action

★ Assume first Stab(u) trivial

take $V_u \subset \tilde{V}$ a local slice through u, and



$$\begin{tikzcd} E_u = \tilde{E}_{|V} \arrow[d, shift right] \\ V_u \arrow[u, shift right, "s_u = s_{|V_u}"'] \end{tikzcd}$$

$\to s_u^{-1}(0) \xrightarrow[\text{loc. homeo}]{\sim} s^{-1}(0)/G$

* Assume $\Gamma_u = \mathrm{Stab}(u)$ discrete

→ do the same, get a residual Γ_u - action :

$$\begin{tikzcd} \Gamma_u \circlearrowright E_u \arrow[d] \arrow[r, phantom, "\subset"] & \tilde{E} \arrow[d] \\ \Gamma_u \circlearrowright V_u \arrow[u, bend left, "s_u"] \arrow[r, phantom, "\subset"] & \tilde{V} \end{tikzcd}$$

$$\Rightarrow \quad s_u^{-1}(0)/\Gamma_u \xrightarrow[\text{loc. hom.}]{\psi_u} s^{-1}(0)/G$$

* If $\dim(\mathrm{Stab}_u) \geq 1$, u must be constant : doesn't count ($\mu(u) = 0 \cdots$)

<u>Upshot:</u> $\mathcal{M}$ has been endowed with a Kuranishi Atlas

<u>Def</u> [Kuranishi neighborhood] of $u \in X$ is $(V_u, E_u, \Gamma_u, s_u, \psi_u)$

$$\begin{tikzcd} \Gamma_u \circlearrowright E_u \arrow[d] & & \\ \Gamma_u \circlearrowright V_u \arrow[u, bend right, "s_u"'] & s_u^{-1}(0) \arrow[l, hook'] \arrow[d] & \\ & s_u^{-1}(0)/\Gamma_u \arrow[r, "\psi_u"] & X \end{tikzcd}$$

- V_u : smooth finite-dim mfd
- $E_u \to V_u$: finite-dim vector bdle "obstruction bundle"
- Γ_u : finite group "isotropy group" acting compat.l V_u & E_u
- s_u : smooth section
- ψ_u : $s_u^{-1}(0)/\Gamma_u \longrightarrow \underset{\overset{\cup}{u}}{\mathcal{U}_u} \subset X$ homeo "Kuranishi map"

Def: $p \in X$, $q \in \operatorname{Im} \psi_p$ a coordinate change from $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ to $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ is a triple $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$:

$$\begin{tikzcd}
\Gamma_q \circlearrowright E_q \arrow[r, "\hat{\phi}_{pq}"] \arrow[d] & E_p \circlearrowleft \Gamma_p \arrow[d] \\
\Gamma_q \circlearrowright V_q \arrow[r, "\phi_{pq}"] & V_p \circlearrowleft \Gamma_p \\
\Gamma_q \arrow[r, "h_{pq}"'] & \Gamma_p
\end{tikzcd}$$

- $h_{pq} : \Gamma_q \hookrightarrow \Gamma_p$ injective group mph.
- $\phi_{pq} : V_q \hookrightarrow V_p$: equivariant smooth embedding
- $\hat{\phi}_{pq} : E_q \to E_p$: equiv. embedding of vector bundles covering ϕ_{pq}

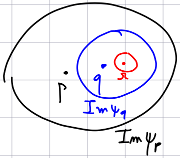
such that
$$\begin{tikzcd}
E_q \arrow[r, "\hat{\phi}_{pq}"] & E_p \\
V_q \arrow[u, "s_q"] \arrow[r, "\phi_{pq}"] & V_p \arrow[u, "s_p"']
\end{tikzcd}$$
and
$$\begin{tikzcd}
s_q^{-1}(0)/\Gamma_q \arrow[rr] \arrow[dr] & & s_p^{-1}(0)/\Gamma_p \arrow[dl] \\
& X &
\end{tikzcd}$$
commute

Def: A Kuranishi str. on X is
- a germ of Kuranishi nbd $(V_p, \ldots)$, $\forall p \in X$
- a germ of coord. changes $(\hat{\phi}_{pq}, \ldots)$, $\forall q \in \operatorname{Im} \psi_p$

such that
- $\operatorname{vdim} X := \dim V_p - \operatorname{rk} E_p$ indep. on $p \in X$.

"Virtual dimension"

- if p, q, r are like:



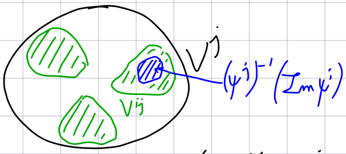
then $\alpha_{pq} \circ \alpha_{qr} = \alpha_{pr}$, for $\alpha = \hat{\phi}, \phi, h$

Rk: if $E_p = 0 \; \forall p$, $\Longleftrightarrow$ orbifold

Rk: using germs apparently produces issues, new constructions don't use germs anymore

Def: A good coordinate system consist in:

- $(I, <)$ ordered indexing set
- $\{(V^i, E^i, \Gamma^i, s^i, \psi^i)\ ;\ i \in I\}$ family of K. nbd covering $X = \bigcup_i \mathrm{Im}\,\psi^i$
- for $j < i$ st $\mathrm{Im}\,\psi^i \cap \mathrm{Im}\,\psi^j \neq \emptyset$, $(V^{ij}, \hat{\phi}^{ij}, \phi^{ij}, h^{ij})$



→ V^{ij} : Γ_j-invariant open nbd of $(\psi^j)^{-1}(\mathrm{Im}\,\psi^i)$ in V^j

→ $(\hat{\phi}^{ij}, \phi^{ij}, h^{ij})$ coord change

$$(V^{ij}, E^j|_{V^{ij}}, \Gamma^j, s^j|_{V^{ij}}, \psi^j|_{V^{ij}}) \to (V^i, E^i, \Gamma^i, s^i, \psi^i)$$

+ obvious transitivity assumpt's for $k < j < i$

$$\begin{tikzcd}
 & (V^j, \ldots) \arrow[dr] & \\
(V^k \ldots) \arrow[ur] \arrow[rr] & & (V^i, \ldots)
\end{tikzcd}$$

Prop [FOOO] can always find such a good coord. syst, for which $\mathrm{Im}(\psi^i) \subset \mathcal{U}_\alpha$, for any prescribed open cover $\{\mathcal{U}_\alpha\}_\alpha$ of X

Virtual chains

$$\begin{tikzcd} X \arrow[r, "g"'] & Y \end{tikzcd} \rightsquigarrow VC(X,g) \in C_*(Y;\mathbb{Q})$$

Kuranishi space

strongly smooth

orbifold

Def: $g: X \to Y$ is strongly smooth, if, in a good coord. syst / X, $g|_{Im\psi^i}$ is induced by a smooth $g^i: V^i \to Y$

$$\begin{tikzcd} V^i \arrow[rr, "g^i"] \arrow[d, phantom, "\cup"] & & Y \\ (s^i)^{-1}(0) \arrow[r, "\psi^i"'] & Im\psi^i \arrow[ur, "g"'] & \end{tikzcd}$$

- **Construction of $VC(X,g)$**

Assume first can find perturbation $\tilde{s}_i = s_i + p$ $\begin{tikzcd} E^i \arrow[d] \\ V^i \arrow[u, bend left] \end{tikzcd}$ that are both Γ_i-equiv and transverse to the zero sec.

↳ take $\tilde{s}_i^{-1}(0)$, triangulate it, $\boxed{\text{glue everything together}}$ to get a cycle in $C_*(Y;\mathbb{Z})$

this is where things get complicated.

Problem: cannot always find such $\tilde{s}_i$'s

Solution: Approximate s_i by "multisections", and average their zero loci.

Def: an <u>n-multisection</u> is a Γ_i-equiv. continuous section of

$$\begin{tikzcd}
\mathfrak{S}_n E^i = (E^i \times \cdots \times E^i)/\mathfrak{S}_n \arrow[d] \\
V^i \arrow[u, bend left, "s"]
\end{tikzcd}$$

symmetric group

s is <u>liftable</u> if lifts to $E^i \times \cdots \times E^i$

$$\begin{tikzcd}
 & (E^i)^n \arrow[dl] \\
\mathfrak{S}_n E^i \arrow[d] & \\
V_i \arrow[u, bend left, "s"] \arrow[uur, dashed, "\tilde{s}=(s_1,\dots,s_n)"'] &
\end{tikzcd}$$

with s_j smooth, transverse to 0, but need not be Γ_i-equiv.

<u>FOOO</u>: can find liftable n-multi sect° that C^0-approximate $\underbrace{(s^i, \dots, s^i)}_{n}$ for large enough n

$\Rightarrow$ take $\frac{1}{n} \sum_{j=1}^{n} s_j^{-1}(0)$ triangulate it

... and glue everything in $C_*(Y;\mathbb{Q})$.