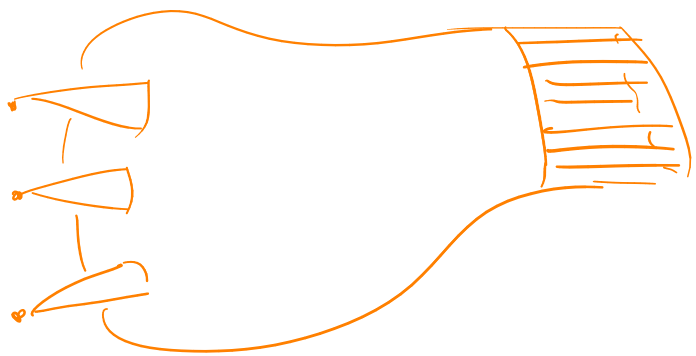
Donaldson's diagonalizability theorem.



Guillem Cazassus, University of Oxford

Refs: * Donaldson, An application of gauge theory to the topology of 4-manifolds.

* Freed-Uhlenbeck, Instantons and four-manifolds.

Def: A topological n-manifold X is a separated topological space such that every $x \in X$ admits a neighborhood homeomorphic to $\mathbb{R}^n$.

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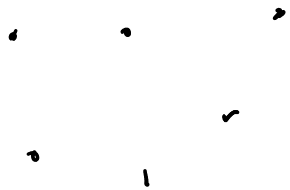
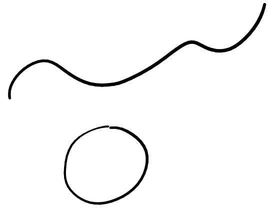
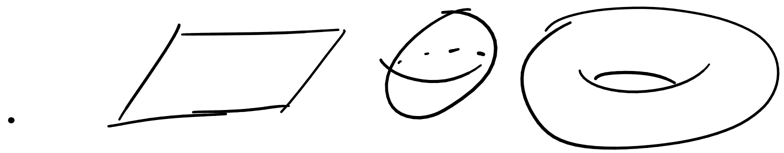
Ex: $n=0$

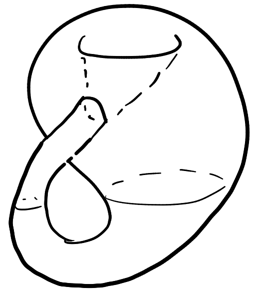


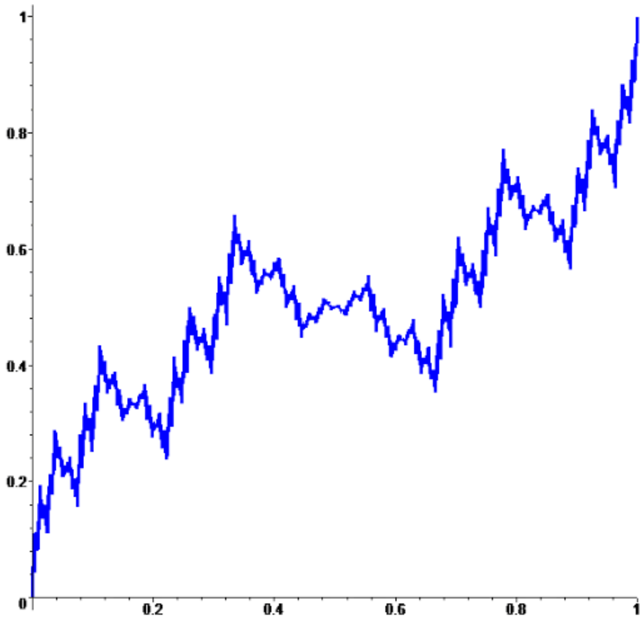
$n=1$

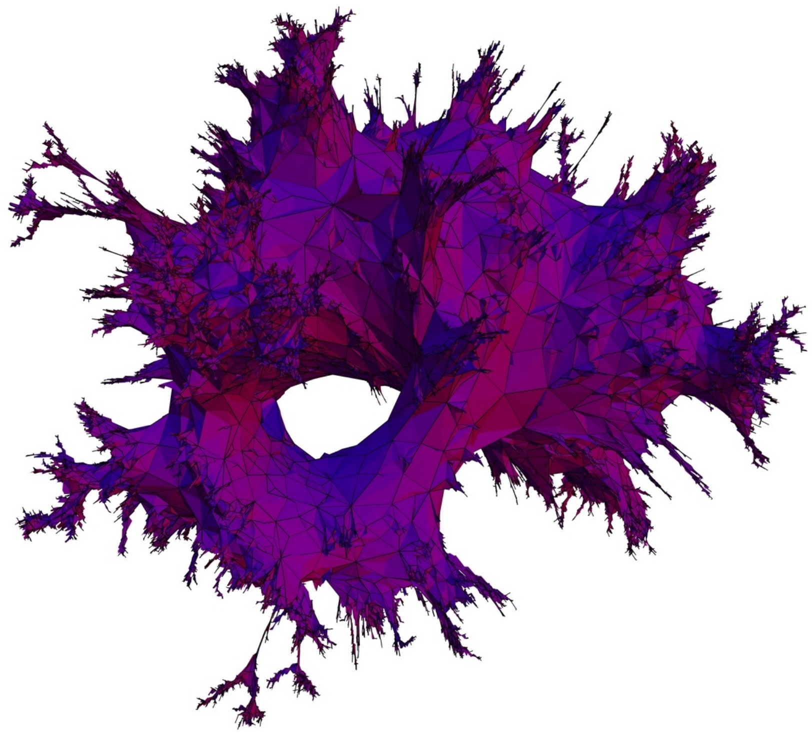


$n=2$





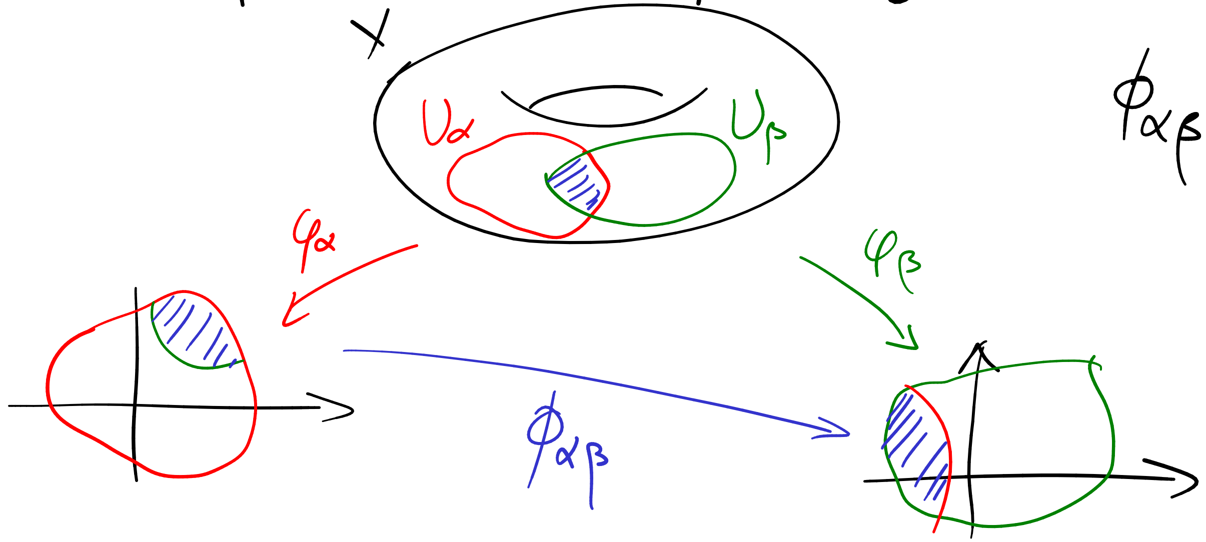




(pic. from http://bettinel.perso.math.cnrs.fr/)

Def: • A smooth atlas on a topol. manifold X is an open covering $\{U_\alpha\}_\alpha$ of X, with charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ that differ on overlaps by smooth diffeomorphisms

$$\phi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$



• A smooth structure on X is an equivalence class of smooth atlases.

Q1: (Existence) Given X topol. n manifold, does there exist a smooth structure on X?

Q2: (Uniqueness) If so, is such a smooth str. unique? or does there exist "exotic" smooth structures, (i.e. non-diffeomorphic)

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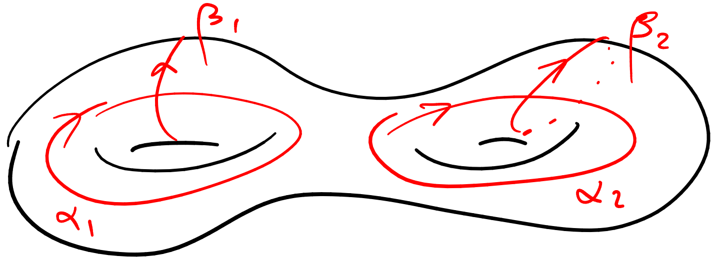
Fun facts:

- Milnor, '56: there exists an exotic structure on $\mathbb{S}^7$. (28 in total)
- Kervaire, '60: there exists a 10-manifold that doesn't admit a smooth str.
- Answer to Q2 for $X = \mathbb{R}^n$ is "Yes" ... except when $n=4$!
  $\to$ Freedman '82: $\exists$ exotic $\mathbb{R}^4$, Taubes '87: $\exists$ uncountably many such.
- "SPC4": Q2 for $X = \mathbb{S}^4$ is still open.

· Homology groups and intersection form

$k \geq 0$, $H_k(X;\mathbb{Z}) \approx \mathbb{Z}^{\# k\text{-holes in } X}$

"k-th homology group"

ex: $X =$ 

$H_1(X;\mathbb{Z}) = \mathbb{Z}[\alpha_1] \oplus \mathbb{Z}[\beta_1] \oplus \mathbb{Z}[\alpha_2] \oplus \mathbb{Z}[\beta_2]$

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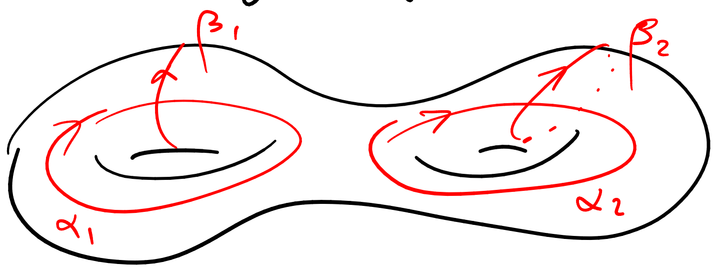
ex: $X =$ 

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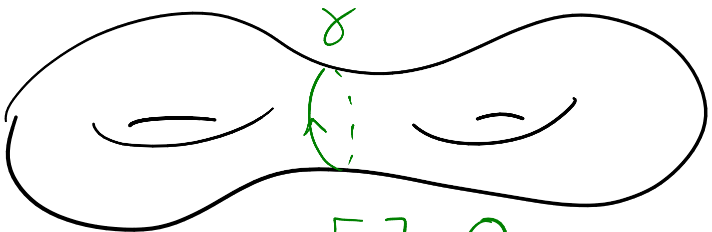
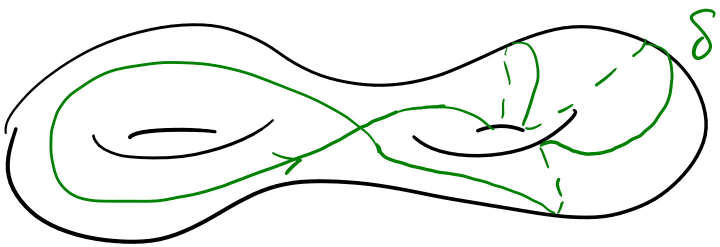


$[\gamma] = 0$



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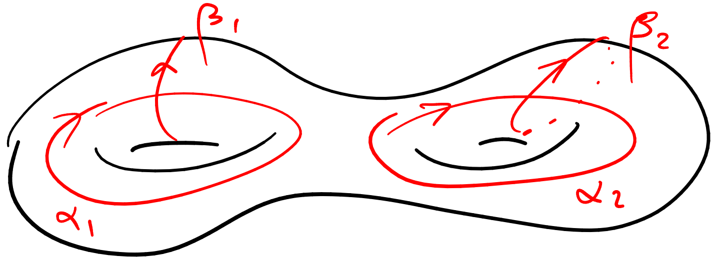
ex: $X =$ 

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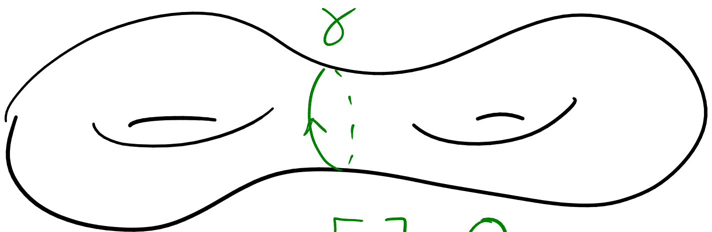

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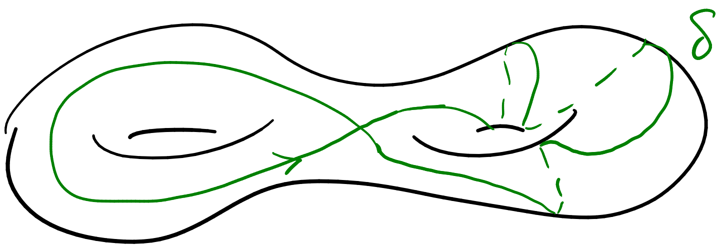

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Intersection form:

$$q_X : H_n(X;\mathbb{Z}) \times H_n(X;\mathbb{Z}) \to \mathbb{Z}$$

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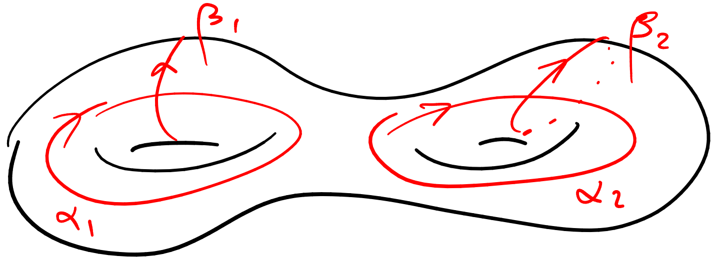
ex: $X =$ 

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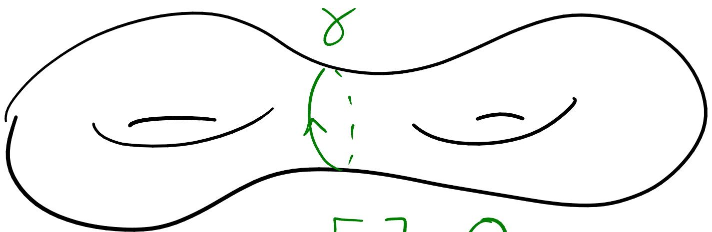

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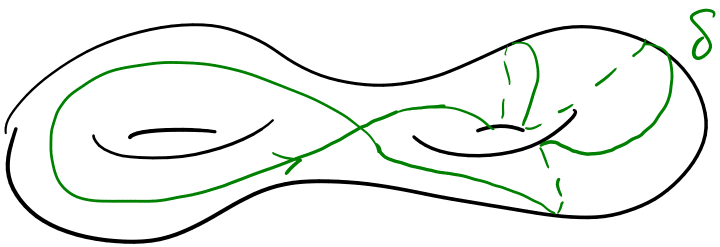

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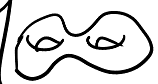
ex: q  $= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

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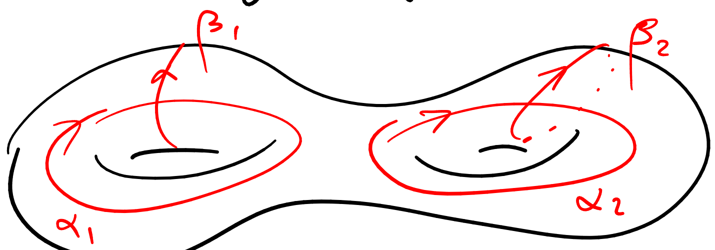
ex: $X =$ 

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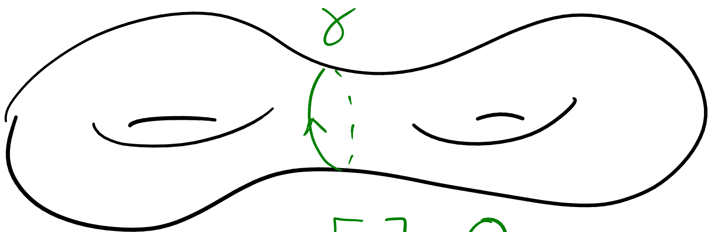

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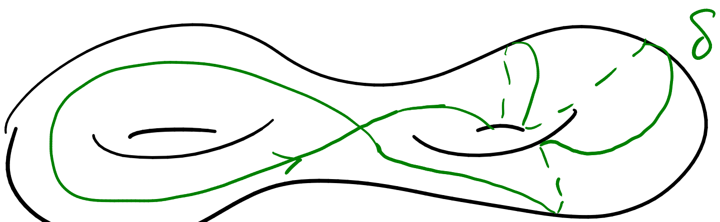

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ex: $q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

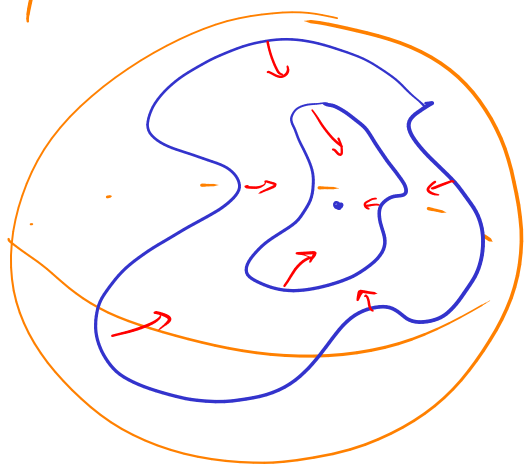
prop: If X is a closed, oriented 4-manifold, then q_X is a symmetric, unimodular quadratic form.

$\det(\mathrm{Mat}\, q_X) = \pm 1$.

Th 1: [Donaldson] X^4: smooth, closed, oriented, simply connected.

If q_X is positive definite, then $q_X \sim \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}$.

every loop in X can be contracted to a point.



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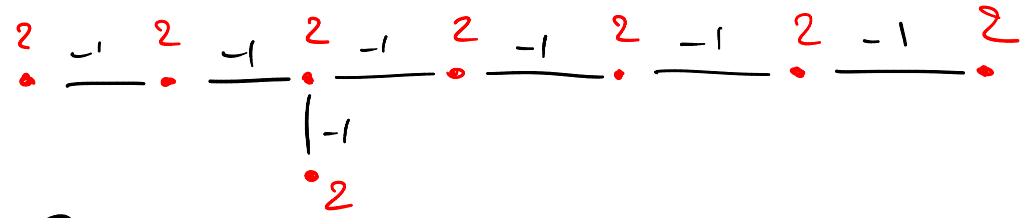
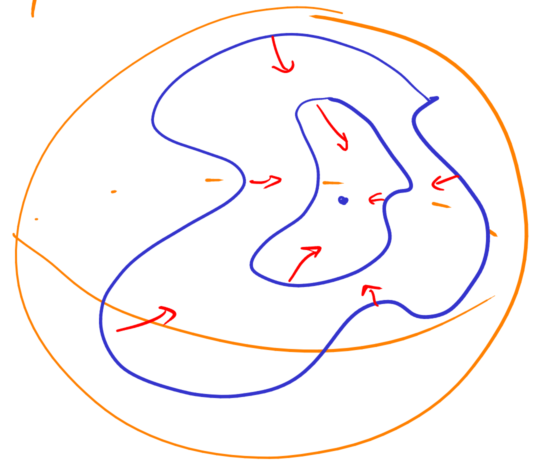


$$q_{E_8} = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$



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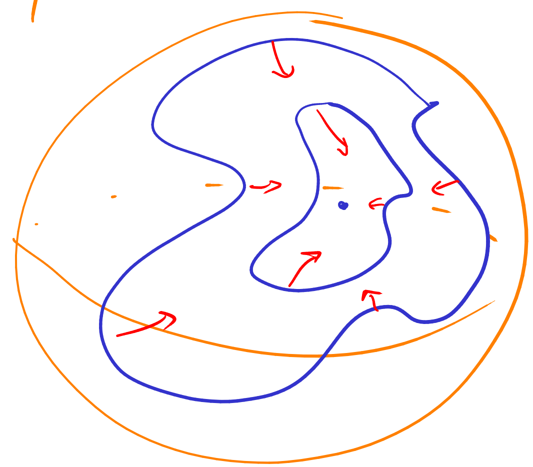

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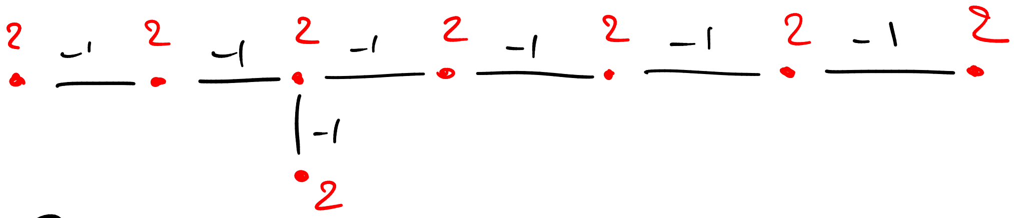

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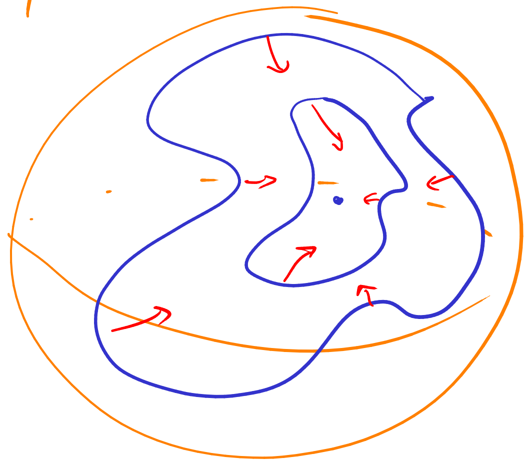

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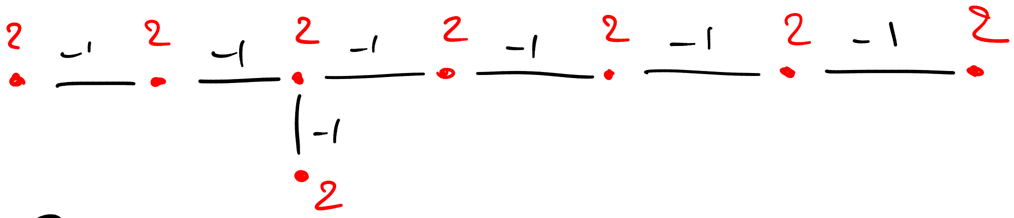

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Freedman: $\exists X^4$ topol. st $q_X = q_{E_8}$

Donaldson: X^4 not smooth.

Th 2: (Donaldson) Assume X^4 smooth, closed, oriented, simply connected, g: Riemannian metric on X

$$\mathcal{M}_{SD}(X,g) := \left\{ \begin{matrix} \text{self-dual } SU(2)\text{-instantons} \\ \text{with topological charge } k=1 \end{matrix} \right\} \Big/ \begin{matrix} \text{gauge} \\ \text{transformations} \end{matrix}$$

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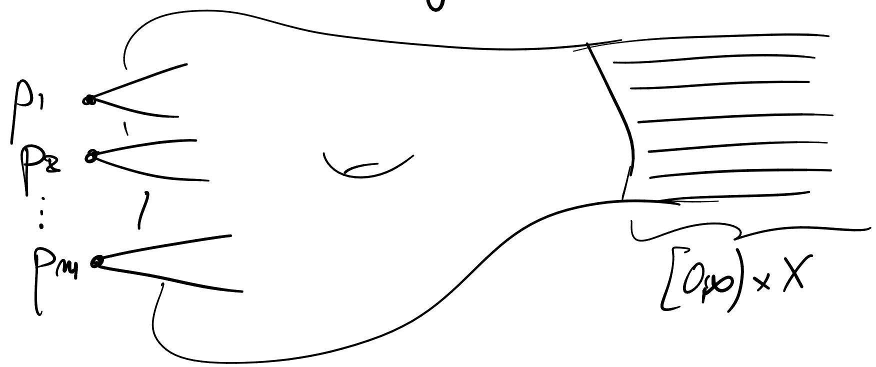
Then, for generic g: $\mathcal{M}_{SD}(X,g) \approx$



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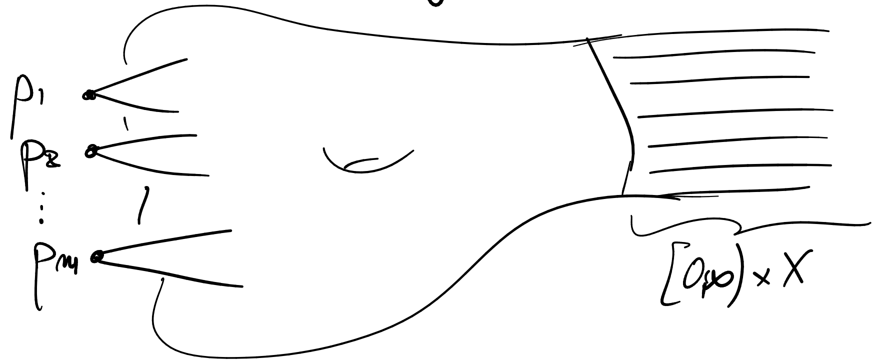
Then, for generic g: $\mathcal{M}_{SD}(X,g) \simeq$



* $\mathcal{M}_{SD}(P,g)$ is smooth, of dimension 5, away from m points $p_1, \dots, p_m$, where $m = \frac{1}{2}\#\{\alpha \in H_2(X,\mathbb{Z}) \mid q_X(\alpha,\alpha) = 1\}$,
* $\mathcal{M}_{SD}(P,g)$ is oriented.
* each p_i has a neighborhood $N_i \simeq \text{Cone}\, \mathbb{CP}^2$ (or $\overline{\mathbb{CP}}^2$)
* $\mathcal{M}_{SD}(P,g)$ has an "end" $\simeq [0,\infty)\times X$ (i.e. $\mathcal{M}_{SD} \setminus (0,\infty)\times X$ is compact)

Th 1 follows from Th 2 and some easy facts from Algebraic Topology and quadratic forms.

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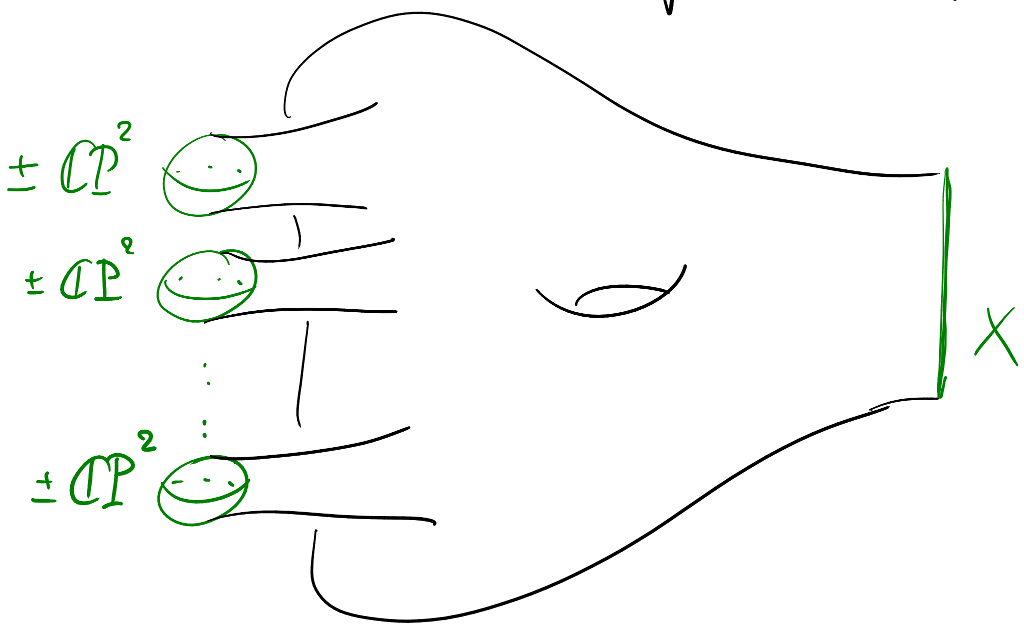
Rough idea: Cut the cones and the end of $\mathcal{M}_{SD}(X,g)$

$\rightsquigarrow$ get a "cobordism" from copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ to X



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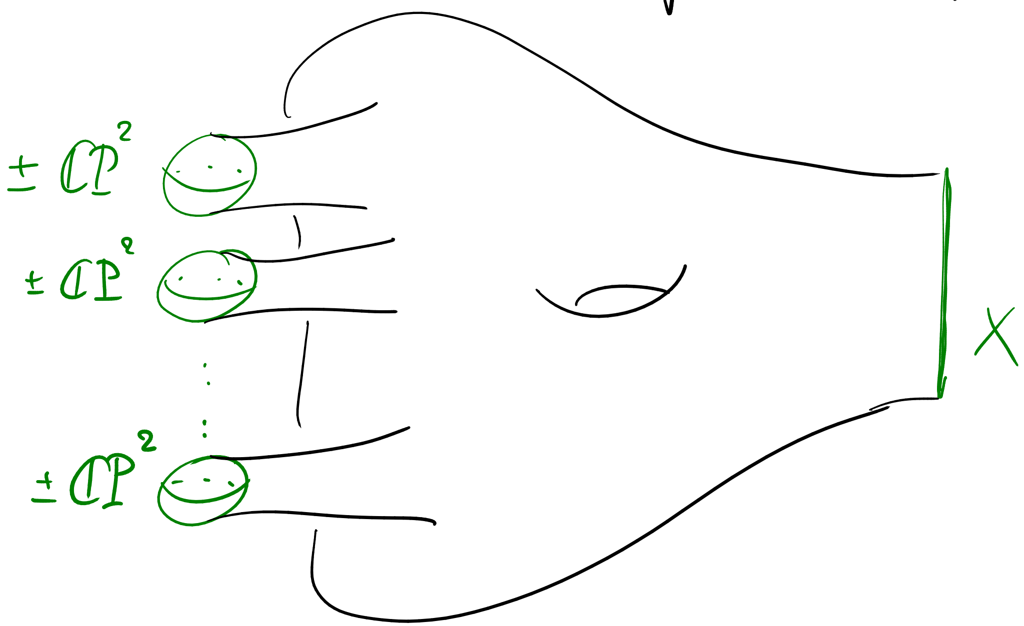

$\rightarrow$ Use the fact that $\sigma(q_X) = \#$ posit. eigenvalues $- \#$ neg. eigenvalues is a cobordism invariant. $\square$

G : Lie group

$\mathfrak{g}$: its Lie algebra

$$\left(\begin{array}{l} G = SU(2) = \{ A \in M_2(\mathbb{C}) \ / \ A \cdot \bar{A}^T = I_2 \ , \ \det A = 1 \} \\ \mathfrak{g} = \underline{su}(2) = \{ A \in M_2(\mathbb{C}) \ / \ A + \bar{A}^T = 0 \ , \ \operatorname{tr} A = 0 \} \end{array} \right)$$

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→ <u>Principal bundles</u>

$$\begin{array}{l} P \circlearrowleft G \\ \downarrow \pi \\ B \end{array}$$

s.t. locally in B:

$U \subset B$, $P_{|U} \simeq U \times G \circlearrowleft G$

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$\downarrow \pi$

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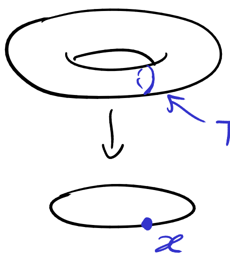
ex: $P = S^1 \times U(1)$: "trivial bundle"

$\downarrow$

S^1



$\pi^{-1}(x)$: "fiber" over $x \in B$

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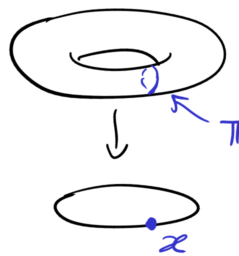
ex: • $P = \mathbb{S}^1 \times U(1)$: "trivial bundle"

$\downarrow$

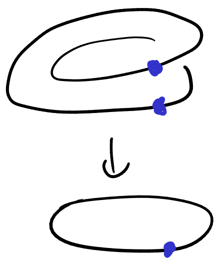
$\mathbb{S}^1$



$\pi^{-1}(x)$: "fiber" over $x \in B$

• $G = \mathbb{Z}/2\mathbb{Z}$:

$$\begin{array}{lcc} P = \mathbb{S}^1 & & \theta \\ \downarrow & & \downarrow \\ B = \mathbb{S}^1 & & 2\theta \end{array}$$



→ a nontrivial bundle.

G : Lie group

$\rightsquigarrow \mathfrak{g}$: its Lie algebra

$$\left(\begin{array}{l} G = SU(2) = \{ A \in M_2(\mathbb{C}) \;/\; A \cdot \bar{A}^T = I_2 ,\ \det A = 1 \} \\ \mathfrak{g} = \underline{su}(2) = \{ A \in M_2(\mathbb{C}) \;/\; A + \bar{A}^T = 0 ,\ \operatorname{tr} A = 0 \} \end{array} \right)$$

→ Principal bundles

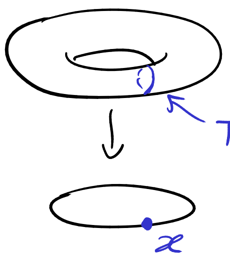
$P \circlearrowleft G$ s.t. locally in B:

$P \xrightarrow{\pi} B$ $\qquad U \subset B$, $P|_U \simeq U \times G \circlearrowleft G$

ex: • $P = S^1 \times U(1)$: "trivial bundle"

$\downarrow$

S^1



$\pi^{-1}(x)$: "fiber" over $x \in B$

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$\downarrow \qquad\qquad \downarrow$

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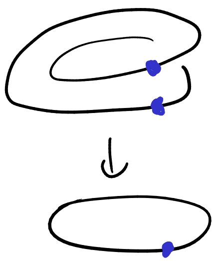

→ a nontrivial bundle.

Gauge transformations

$$\mathcal{G} = \left\{ \varphi : \begin{tikzcd} P \arrow[rr] \arrow[dr, "\pi"'] & \circlearrowleft & P \arrow[dl, "\pi"] \\ & B & \end{tikzcd} \;\middle|\; \varphi(pg) = \varphi(p) \cdot g \right\}$$

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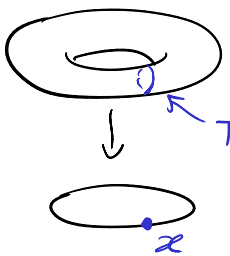

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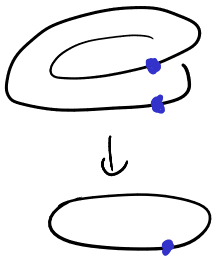

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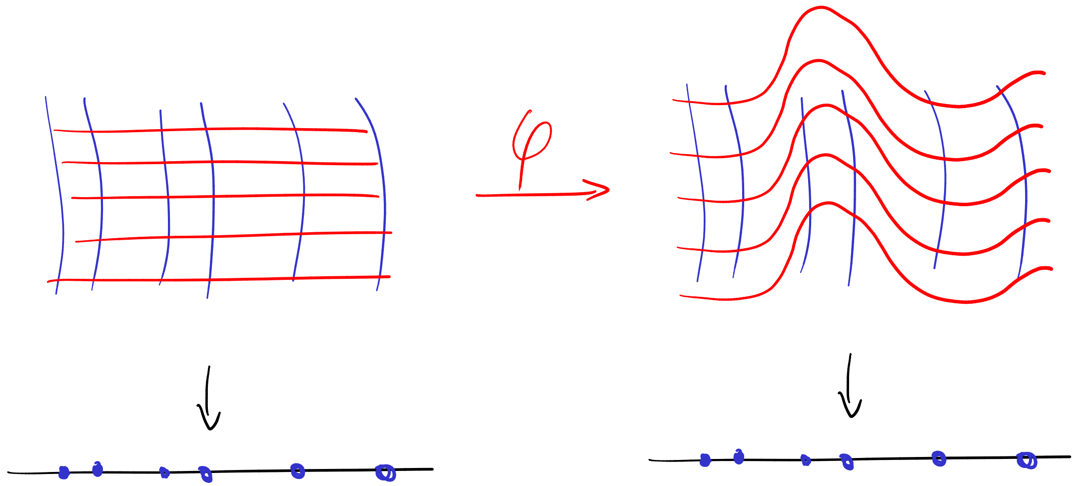
locally, $\Leftrightarrow \exists f : U \to G$:

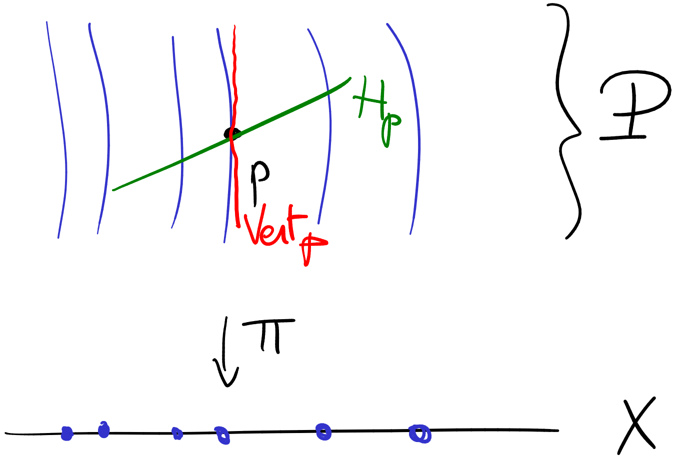
$$\varphi(b,g) = (b, f(b)\cdot g)$$



A : connexion on P : G-invariant horizontal distribution H_p :

$$T_p P = H_p \oplus \underbrace{\mathrm{Vert}_p}_{= \ker d\pi_p}$$

tangent space at p.



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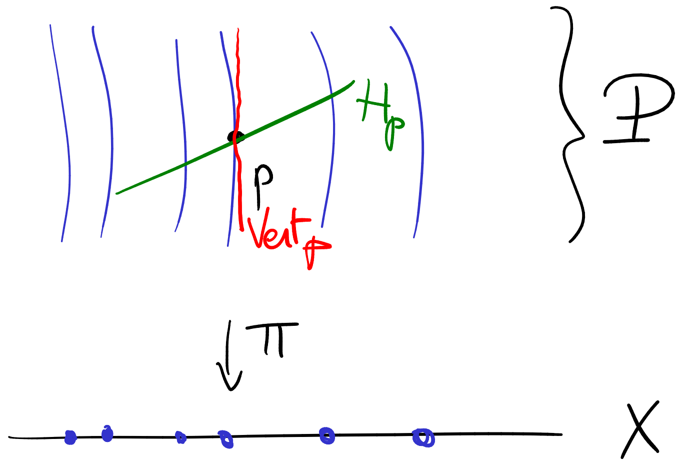

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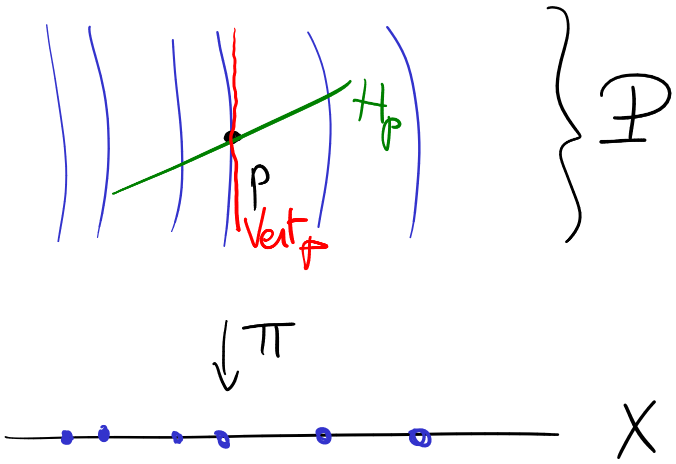

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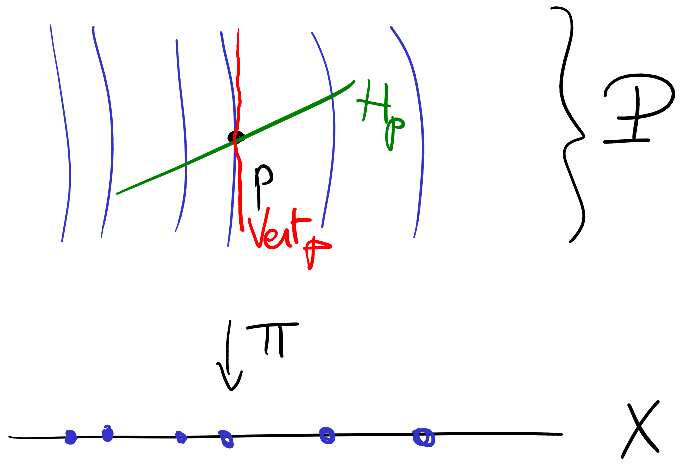

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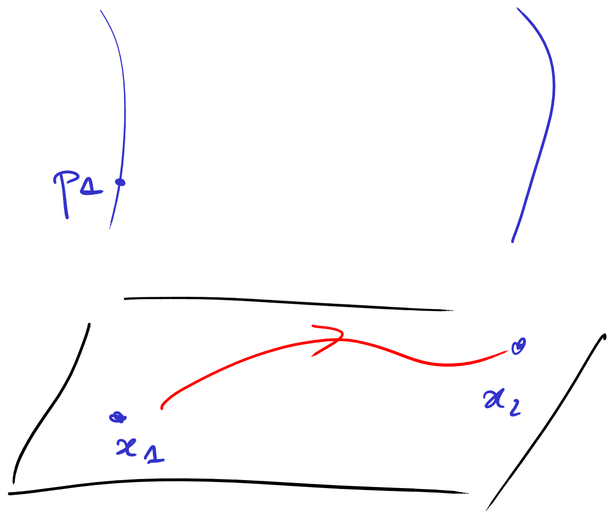
Rk : $X = \mathbb{R}^4$, $G = U(1)$, $\mathfrak{g} = \mathbb{R}$.

ϕ : electric potential

$\vec{A} = (A_1, A_2, A_3)$: magnetic potential

$\rightsquigarrow$ $A = \phi \cdot dt + A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3$

Paths in X can be lifted:



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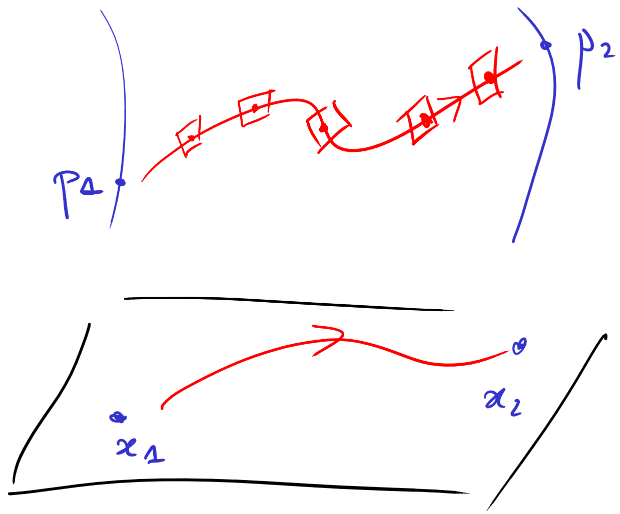

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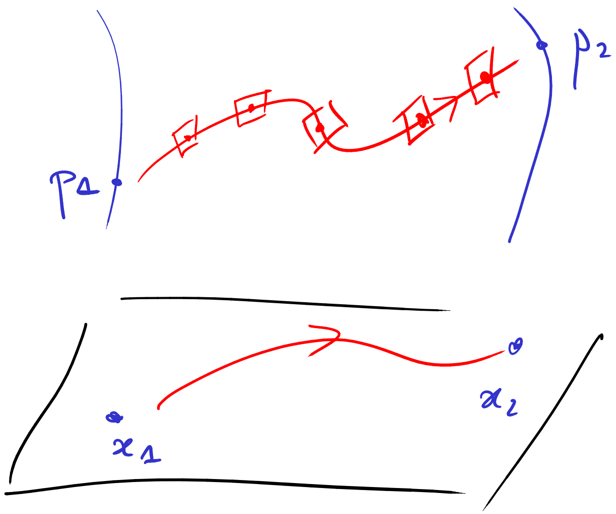
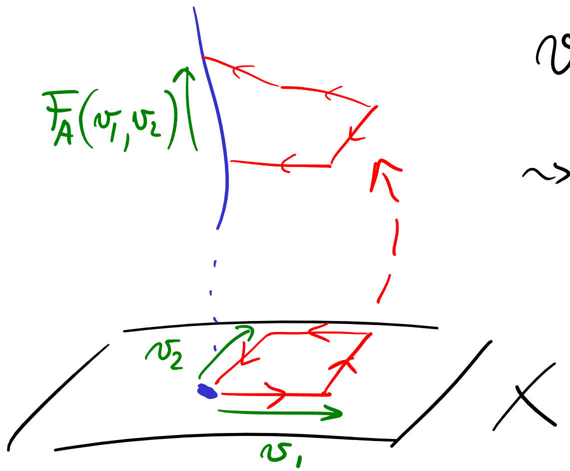


<u>Curvature</u> of <u>A</u>: F_A $\left(\in \underset{\text{locally}}{\Omega^2(U)} \otimes \mathfrak{g}\right)$

↳ Measures "local holonomy"



$v_1, v_2 \in T_x X$

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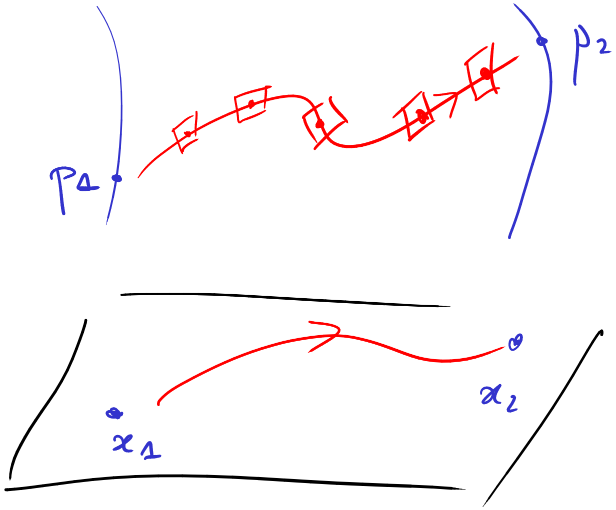

In coordinates $(x_1, \dots, x_n)$:

$$A = A_1\, dx_1 + \cdots + A_n\, dx_n$$

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<u>Curvature</u> of <u>A</u>: F_A ($\in \Omega^2(U) \otimes \mathfrak{g}$ locally)

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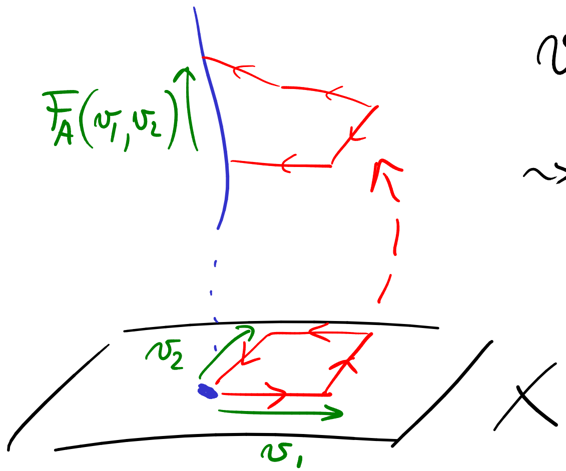

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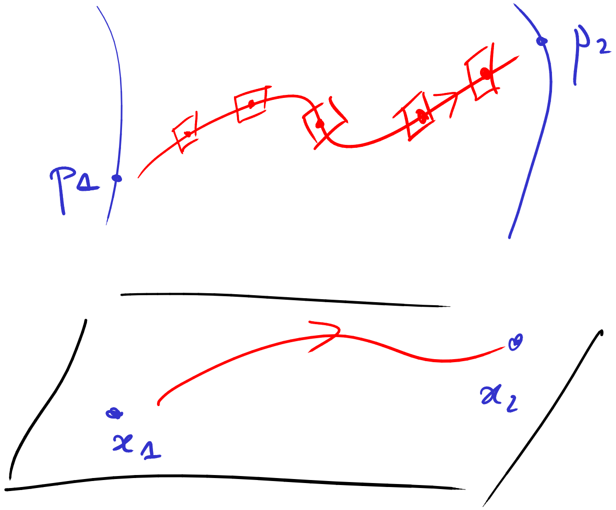

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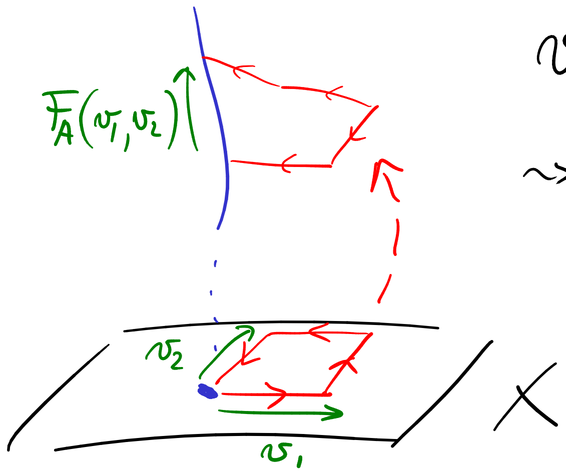

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Remark: $A \simeq$ Electromagnetic potential

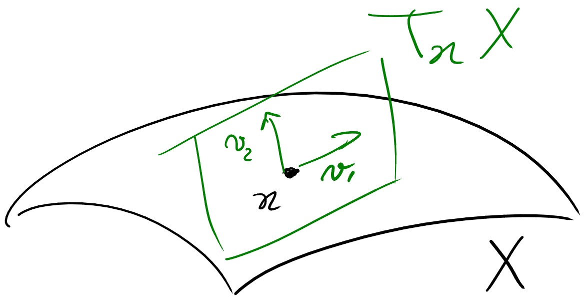
$\Rightarrow F_A \approx$ Electromagnetic field:

$$\vec{E} = (F_{01}, F_{02}, F_{03})$$

$$\vec{B} = (F_{23}, -F_{13}, F_{02})$$

Self-duality

- X oriented ($\Leftrightarrow T_x X$ oriented)
- X endowed with a Riemannian metric g : scalar product $/ T_x X$



$\rightarrow g_x(v_1, v_2) \in \mathbb{R}$

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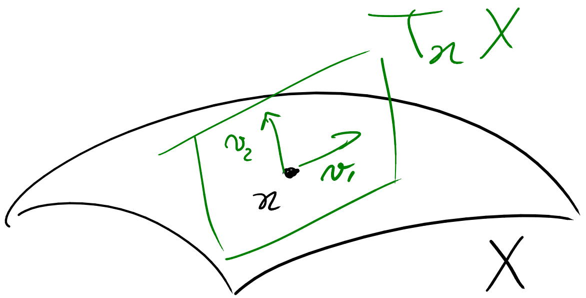

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$$*: \Lambda^k T_x X^* \longrightarrow \Lambda^{n-k} T_x X^*$$

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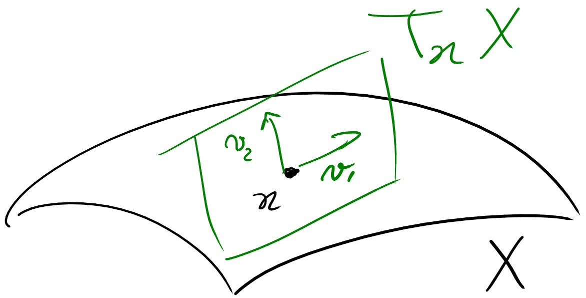

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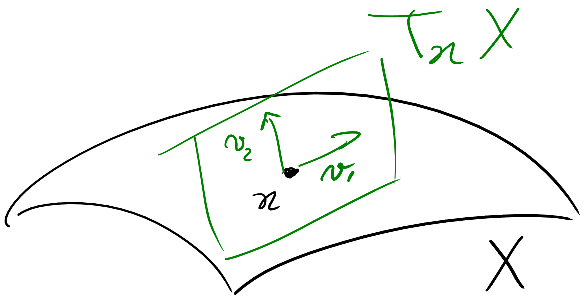

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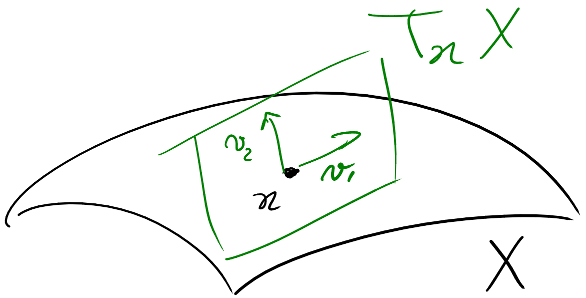

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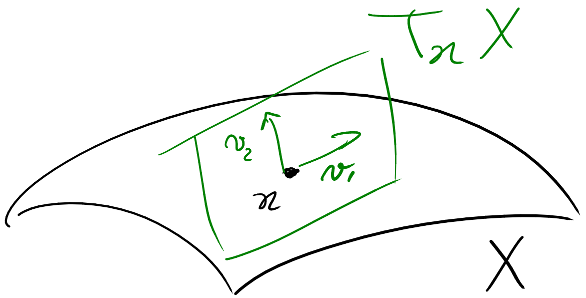

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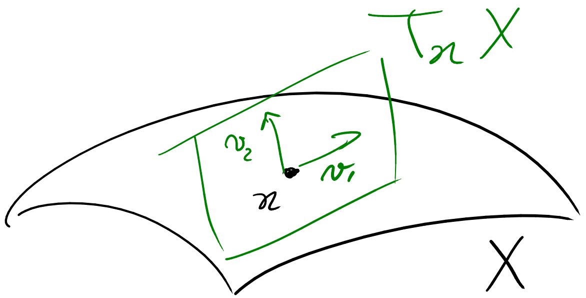

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<u>Rk</u>: Particular sol^ts of Yang-Mills equations: $D * F_A = 0$.

Geometrically:

- $H \subset T_x X$ oriented 2-plane
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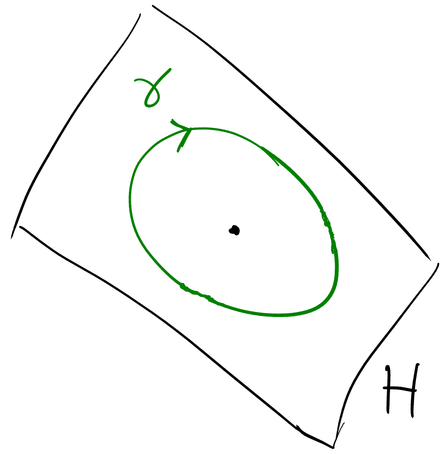

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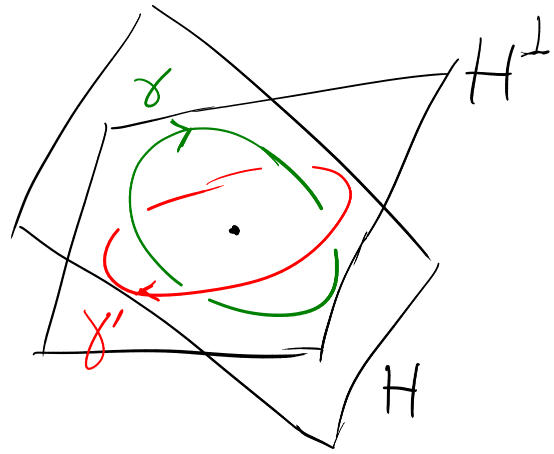

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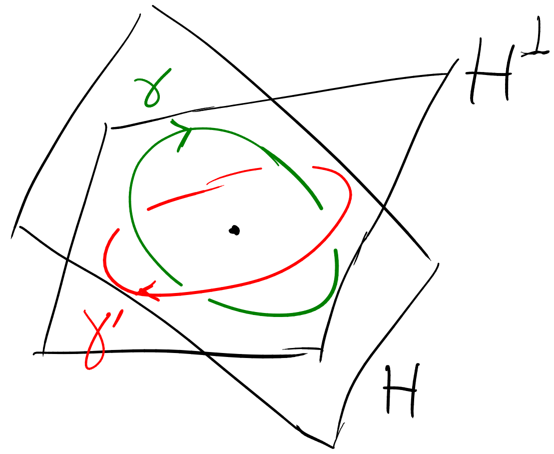

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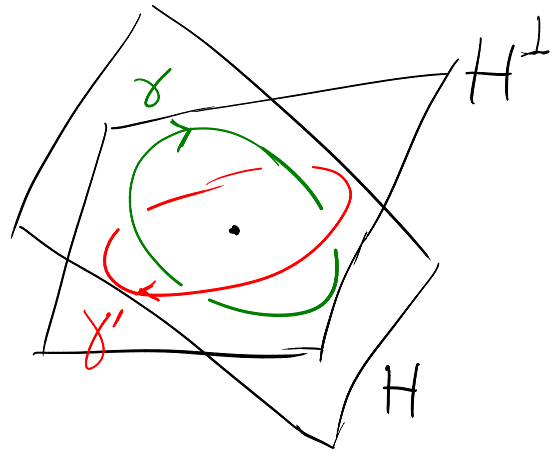

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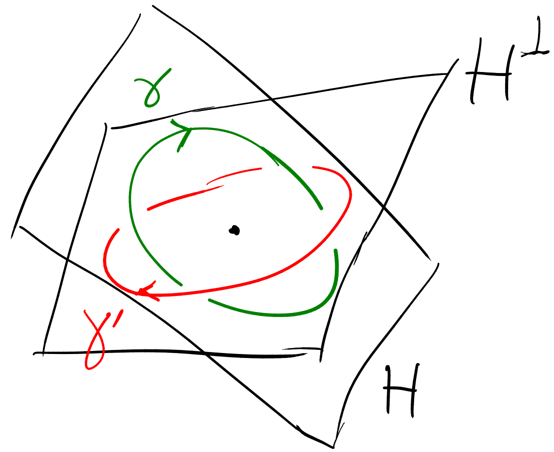

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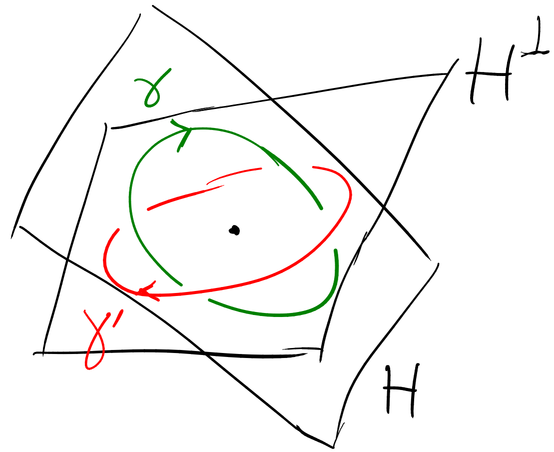

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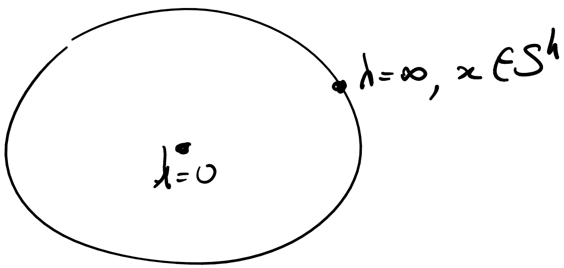
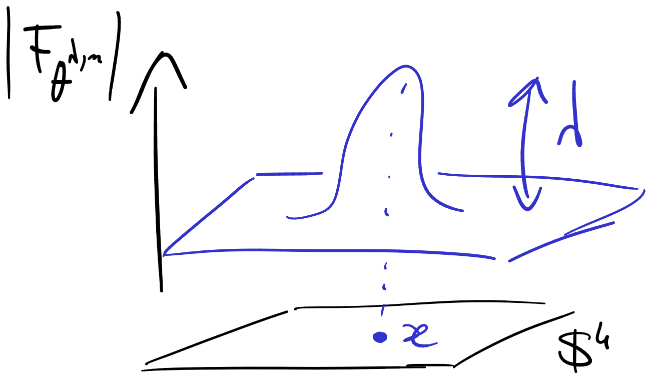
Def: $\mathcal{M}_{SD}(X, g) = \{ A \text{ conn.}/P \text{ with } k=1 \;/\; *F_A = F_A \} / \mathcal{G}$.

Ex: $X = S^4$ + round metric

Th: [Atiyah - Drinfeld - Hitchin - Manin]

$$\mathcal{M}_{SD}(S^4, g_{round}) \simeq \underset{\text{(open ball)}}{B^5} \simeq \left\{ \Theta^{\lambda, x} \;\middle|\; \underset{\text{"scale"}}{\lambda \geq 0} \,,\; \underset{\text{"center"}}{x \in S^4} \right\}$$

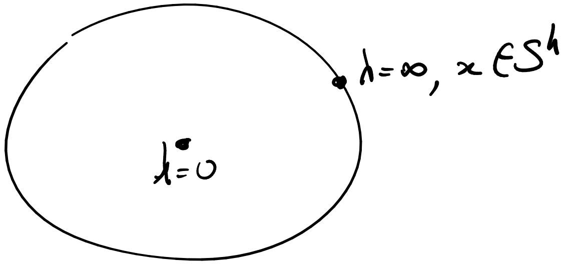


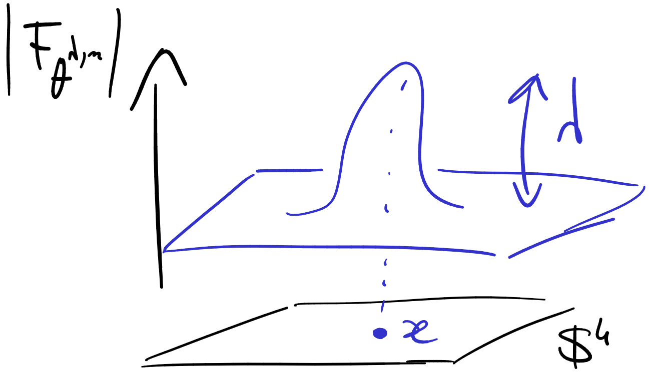


Ex: $X = \mathbb{S}^4$ + round metric

Th: [Atiyah - Drinfeld - Hitchin - Manin]

$$\mathcal{M}_{SD}(\mathbb{S}^4, g_{round}) \simeq \underset{\text{(open ball)}}{B^5} \simeq \Big\{ \Theta^{\lambda,x} \;\Big|\; \underset{\text{"scale"}}{\lambda} \geq 0 \,,\; \underset{\text{"center"}}{x} \in \mathbb{S}^4 \Big\}$$





Rk: no cones / $\mathbb{CP}^2$ here, since q_X trivial. ($H_2(\mathbb{S}^4) = \{0\}$).

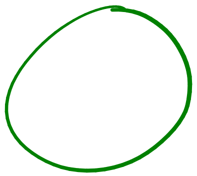
In general, correspond to "reducible connexions".

Some recent developments involving instantons:
(instanton homology)

- Knot detection:

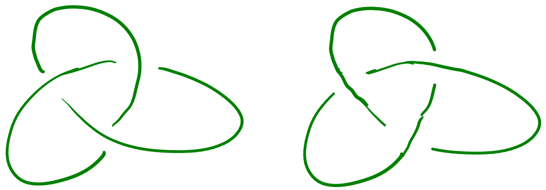
Reduced Khovanov homology detects:

× The unknot



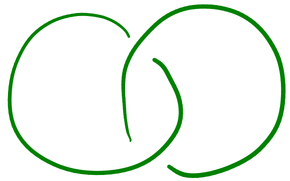
(Kronheimer-Mrowka)

× The trefoil knots



(Baldwin-Sivek)

× the Hopf link



(Baldwin-Sivek-Xie)

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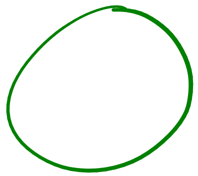

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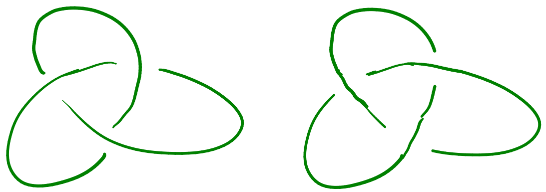

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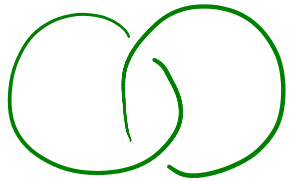

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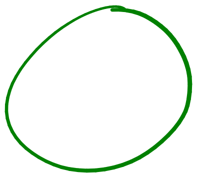

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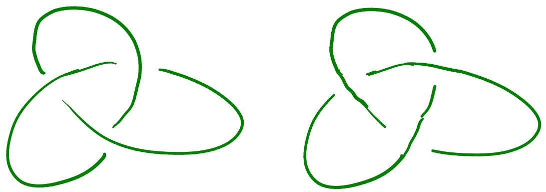

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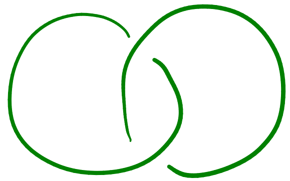

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- (Conjecturally) Proof of the four-color theorem (Kronheimer-Mrowka)