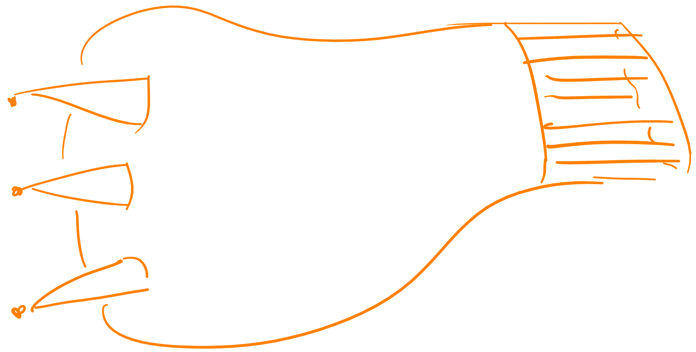
Donaldson's diagonalizability theorem.



Guillem Cazassus

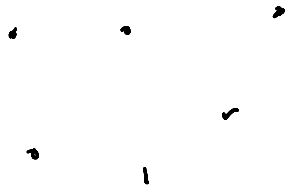
Refs: * Donaldson, An application of gauge theory to the topology of 4-manifolds.

* Freed-Uhlenbeck, Instantons and four-manifolds.

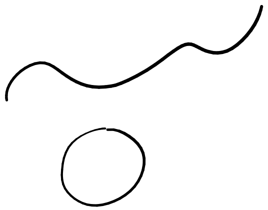
__Def__: A topological n-manifold X is a separated topological space such that every $x \in X$ admits a neighborhood homeomorphic to $\mathbb{R}^n$.

Def: A topological n-manifold X is a separated topological space such that every $x \in X$ admits a neighborhood homeomorphic to $\mathbb{R}^n$.

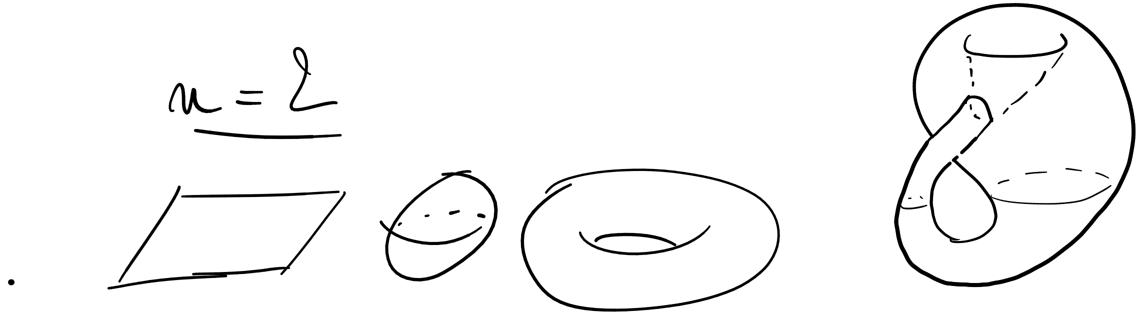
Ex: $n = 0$

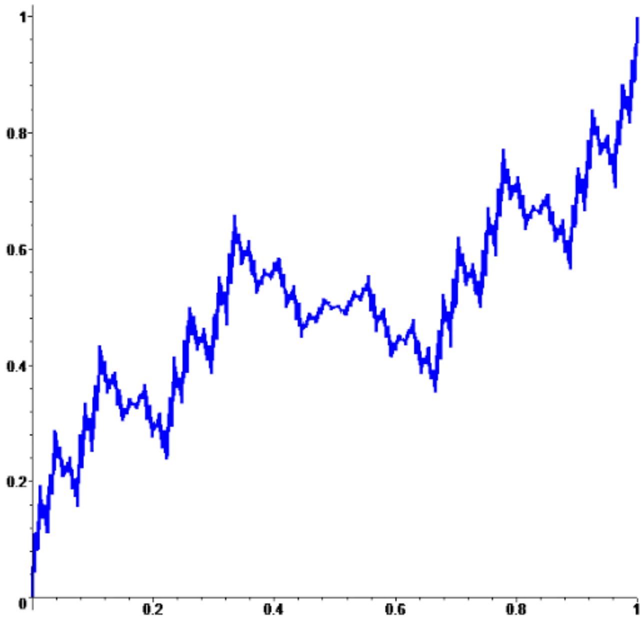


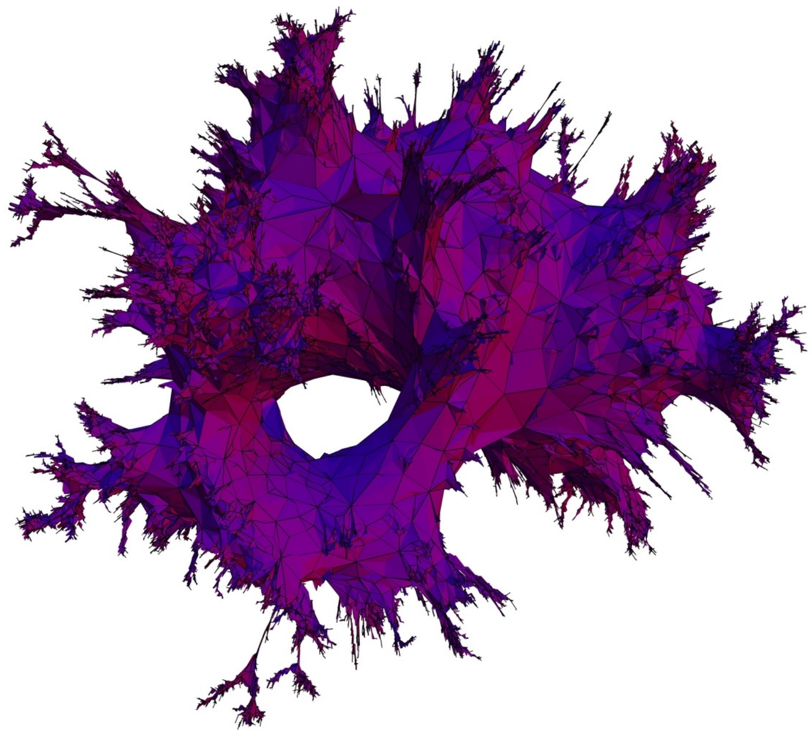
$n = 1$



$n = 2$



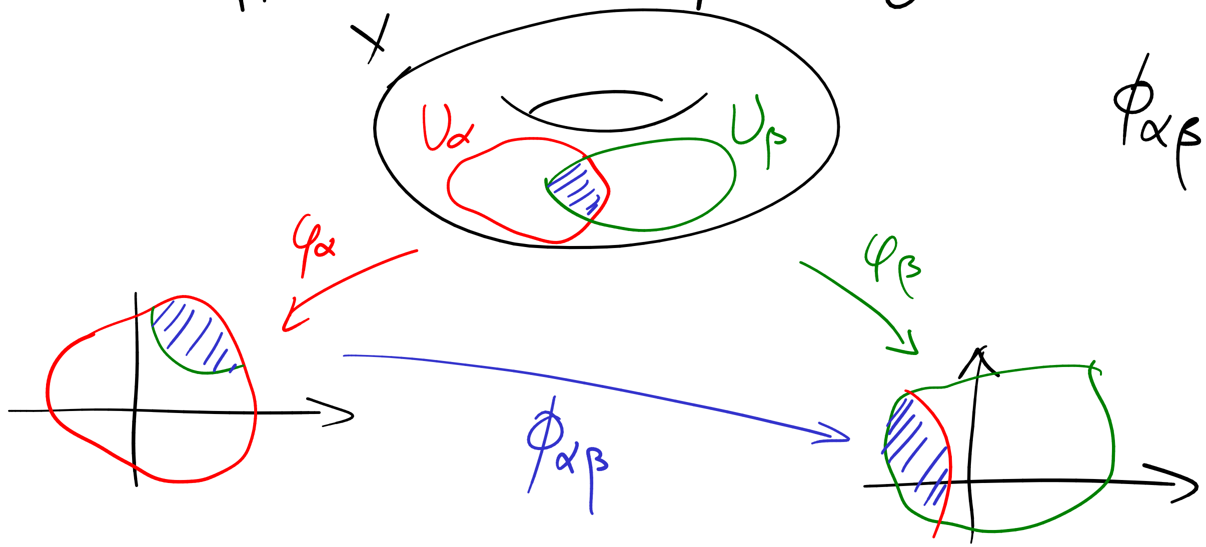




(pic. from http://bettinel.perso.math.cnrs.fr/)

<u>Def</u>: • A smooth atlas on a topol. manifold X is an open covering $\{U_\alpha\}_\alpha$ of X, with charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ that differ on overlaps by smooth diffeomorphisms

$$\phi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$



• A smooth structure on X is an equivalence class of smooth atlases.

<u>Q1</u>: (Existence) Given X topol. mf manifold, does there exist a smooth structure on X?

<u>Q2</u>: (Uniqueness) <u>If</u> so, is such a smooth str. unique? or does there exist "exotic" smooth structures, (i.e. non-diffeomorphic)

<u>Q1</u>: (Existence) Given X topol. n manifold, does there exist a smooth structure on X?

<u>Q2</u>: (Uniqueness) <u>If</u> so, is such a smooth str. unique? or does there exist "exotic" smooth structures, (i.e. non-diffeomorphic)

- When $n \leq 3$, "Yes" to both questions.
- At $n=4$ and higher: strange / hard.

Q1: (Existence) Given X topol. n manifold, does there exist a smooth structure on X?

Q2: (Uniqueness) If so, is such a smooth str. unique? or does there exist "exotic" smooth structures, (i.e. non-diffeomorphic)

- When $n \leq 3$, "Yes" to both questions.
- At $n=4$ and higher: strange / hard.

Fun facts:

- Milnor, '56: there exists an exotic structure on $\mathbb{S}^7$. (28 in total)
- Kervaire, '60: there exists a 10-manifold that doesn't admit a smooth str.
- Answer to Q2 for $X = \mathbb{R}^n$ is "Yes" ... except when $n=4$!
  $\to$ Freedman '82: $\exists$ exotic $\mathbb{R}^4$, Taubes '87: $\exists$ uncountably many such.
- "SPC4": Q2 for $X = \mathbb{S}^4$ is still open.

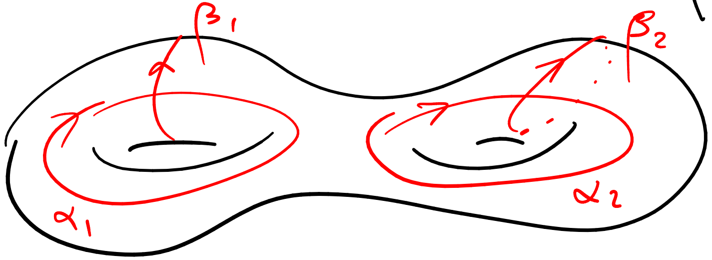
· Intersection form of closed, oriented 4-manifolds:

$$q_X : H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}.$$

$$q_X(\alpha,\beta) = (\alpha \cup \beta)[X] \qquad \left(= \int_X \alpha \wedge \beta \text{ in deRham cohomology} \right)$$

Lower dim. example: Σ: closed oriented surface

$$H^1(\Sigma;\mathbb{Z}) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\beta_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\beta_2$$



$q_\Sigma : H^1(\Sigma;\mathbb{Z}) \otimes H^1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$

has matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

If $\dim X = 4$, q_X is a symmetric unimodular quadratic form.

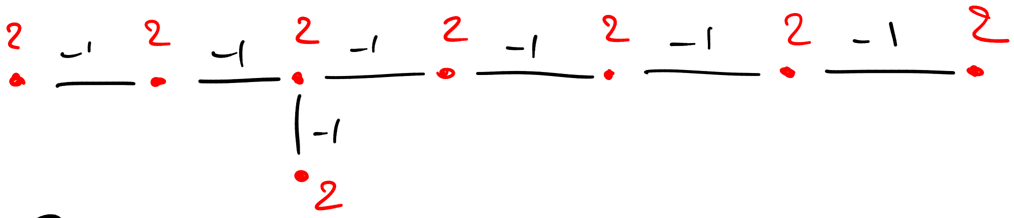
$\mathrm{Det}(\mathrm{Mat}\, q_X) = \pm 1.$

(follows from Poincaré duality)

Th 1: [Donaldson] X^4: smooth, closed, oriented, simply connected.
If q_X is positive definite, then $q_X \sim \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}$.

Th 1: [Donaldson] X^4: smooth, closed, oriented, simply connected.
If q_X is positive definite, then $q_X \sim \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}$.

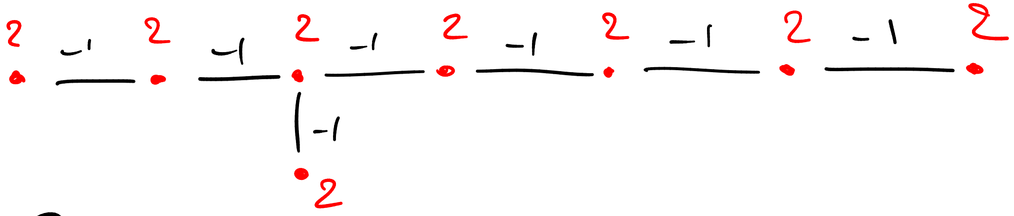
Rk: E_8:



$$q_{E_8} = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

pos. definite, but not $\sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Th 1: [Donaldson] X^4: smooth, closed, oriented, simply connected.
If q_X is positive definite, then $q_X \sim \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ over $\mathbb{Z}$.

Rk: E_8 : 

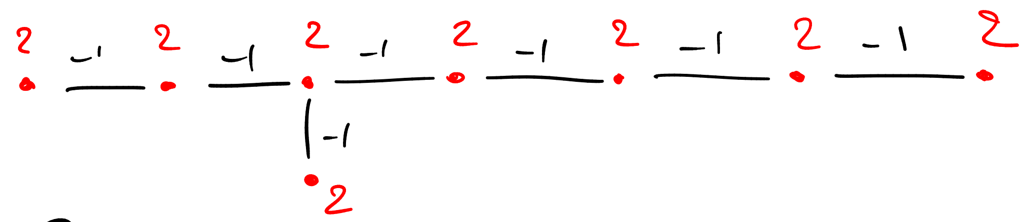
$$q_{E_8} = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

pos. definite, but not $\sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Freedman: $\exists X^4$ topol. st $q_X = q_{E_8}$

Th 1: [Donaldson] X^4: smooth, closed, oriented, simply connected. If q_X is positive definite, then $q_X \sim \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ over $\mathbb{Z}$.

Rk: E_8:



$$q_{E_8} = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

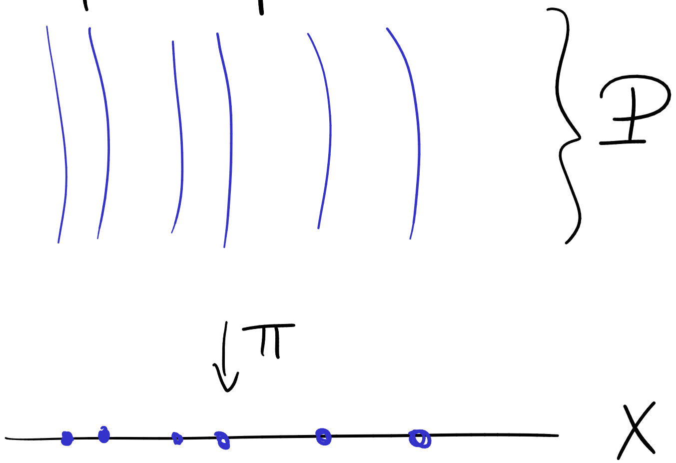
pos. definite, but not $\sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Freedman: $\exists X^4$ topol. st $q_X = q_{E_8}$

Donaldson: X^4 not smooth.

→ **Gauge theory** X^4_+ Riem. metric g , $G = SU(2)$
$\mathfrak{g} = \underline{\mathfrak{su}}(2)$

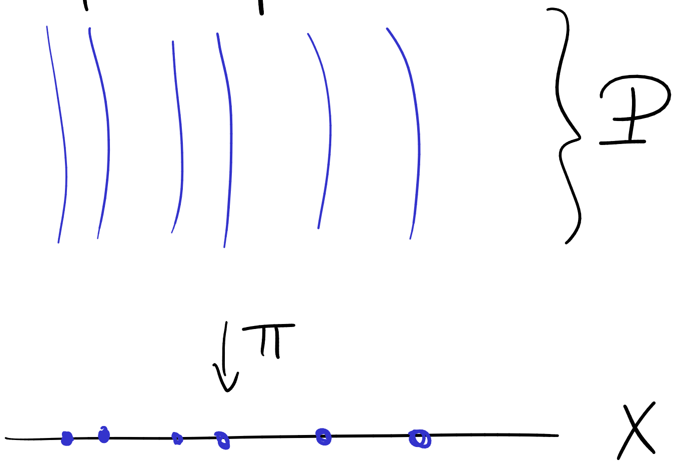
$$\begin{array}{c} P \circlearrowleft G \\ {\scriptstyle \pi} \downarrow \\ X \end{array}$$: principal bundle



→ $$\begin{array}{c} V = P \times_G \mathbb{C}^2 \\ \downarrow \\ X \end{array}$$: rank 2 $\mathbb{C}$-vector bundle

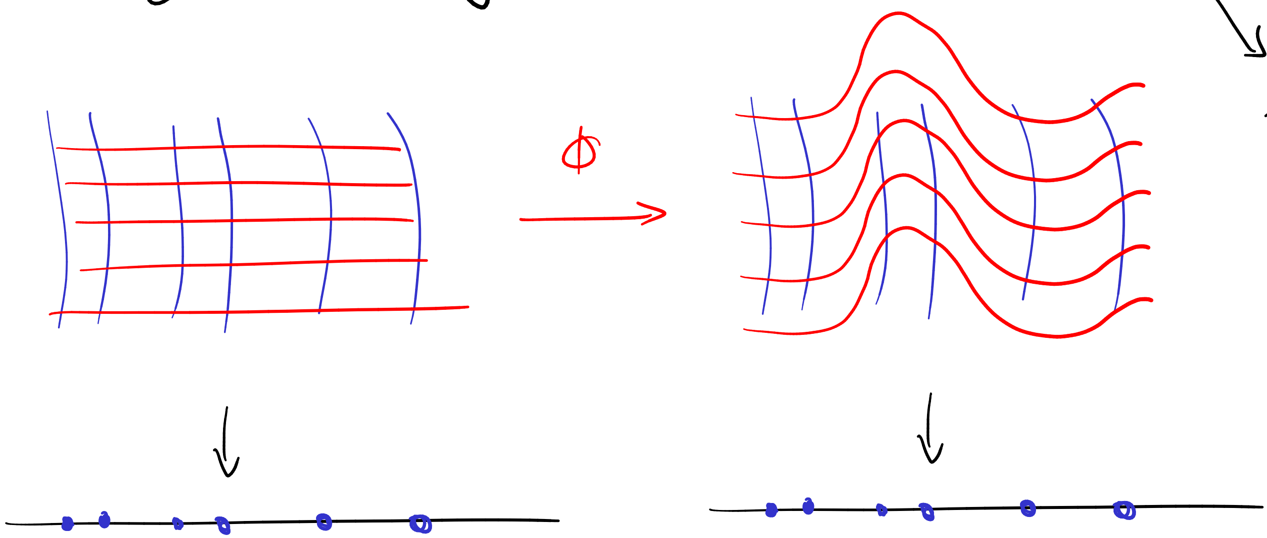
→ **Gauge theory** X^4 Riem. metric g, $G = SU(2)$, $\mathfrak{g} = \mathfrak{su}(2)$

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \pi \\ X \end{array} : \text{principal bundle}$$



$$\to \quad \begin{array}{c} V = P \times_G \mathbb{C}^2 \\ \downarrow \\ X \end{array} : \text{rank 2 } \mathbb{C}\text{-vector bundle}$$

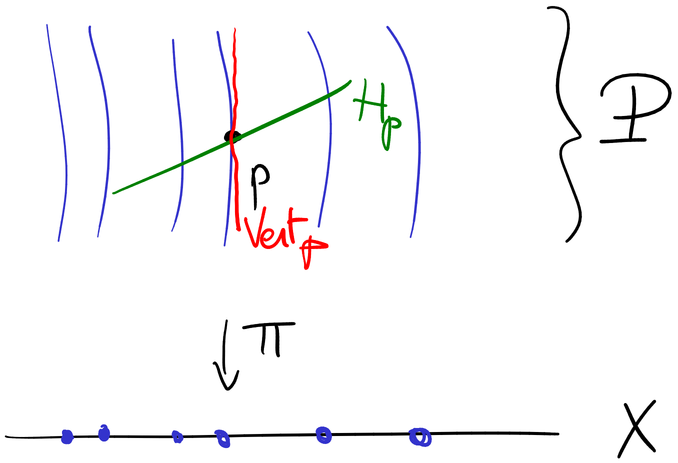
• **Gauge transformations** : $\phi : P \to P$ G-equivariant, over X



Gauge group: $\mathcal{G} = \mathrm{Aut}\, P = \{\text{such } \phi\}$

$\cong C^\infty(X, G)$ if P trivial

A : connexion on P : G-invariant horizontal distribution H_p :

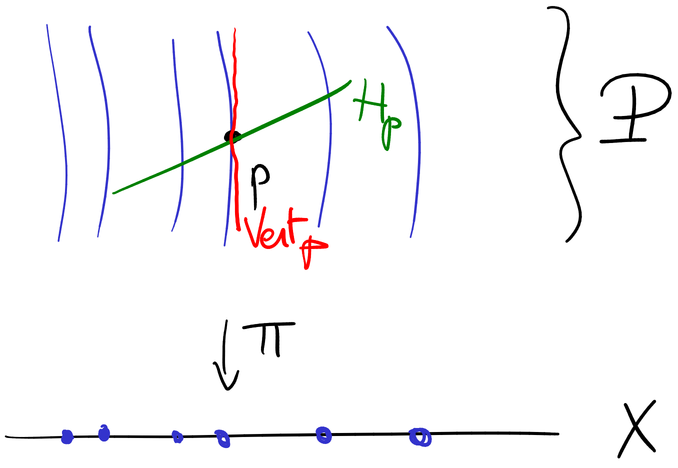


$$T_p P = H_p \oplus \underbrace{\mathrm{Vert}_p}_{= \ker d\pi_p}$$

$\Longleftrightarrow$ $\mathfrak{g}_P$- valued 1-form $A \in \Omega^1(X; \mathfrak{g}_P)$,

$$\mathfrak{g}_P = P \times_{\mathrm{Ad}\,G} \mathfrak{g}$$

A : connexion on P : G-invariant horizontal distribution H_p:



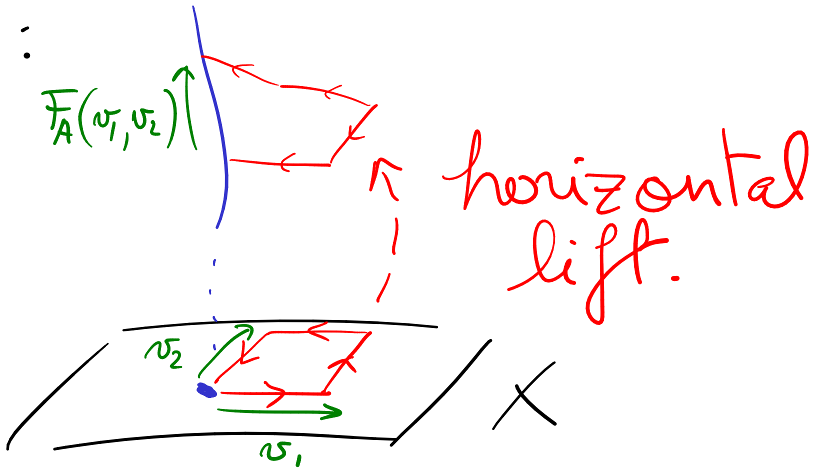
$$T_pP = H_p \oplus \underbrace{\mathrm{Vert}_p}_{=\ker d\pi_p}$$

$\Leftrightarrow$ $\mathfrak{g}_P$-valued 1-form $A \in \Omega^1(X; \mathfrak{g}_P)$,

$$\mathfrak{g}_P = P \times_{\mathrm{Ad}\,G} \mathfrak{g}$$

$\rightsquigarrow$ curvature $F_A \in \Omega^2(X; \mathfrak{g}_P)$ ($F_A = dA + \frac{1}{2}[A \wedge A]$ in a loc. trivialization)

measures local holonomy :



$\mathcal{G} \circlearrowright \mathcal{A}$: space of connexions

$\cup$

$SU(2)$: constant gauge transformations

$\cup$

$\mathbb{Z}_2$: center of $SU(2)$: acts trivially on $\mathcal{A}$

$\Rightarrow \widetilde{\mathcal{G}} := \mathcal{G}/\mathbb{Z}_2 \circlearrowright \mathcal{A}$

reduced gauge group.

Two integrals associated with a connexion A:

- the Yang-Mills functional

$$\mathcal{YM}(A) = \int_X |F_A|_g^2 \, d\mathrm{vol}_X = \int_X \langle F_A \wedge * F_A \rangle$$

Two integrals associated with a connexion A:

- The Yang-Mills functional

$$\mathcal{YM}(A) = \int_X |F_A|_g^2 \, d\mathrm{vol}_X = \int_X \langle F_A \wedge * F_A \rangle$$

- The "topological charge"

$$k = -c_2(P)[X] = \frac{-1}{8\pi^2} \int_X \langle F_A \wedge F_A \rangle \in \mathbb{Z}$$

→ indep. on A

→ classifies P up to isomorphism

Two integrals associated with a connexion A:

- The Yang-Mills functional

$$\mathcal{YM}(A) = \int_X |F_A|_g^2 \, dvol_X = \int_X \langle F_A \wedge *F_A \rangle$$

- The "topological charge"

$$k = -c_2(P)[X] = \frac{-1}{8\pi^2} \int_X \langle F_A \wedge F_A \rangle \in \mathbb{Z}$$

→ indep. on A
→ classifies P up to isomorphism

Rk: $* : \Omega^2(X) \to \Omega^2(X)$ satisfies $*^2 = 1$

$$\rightsquigarrow \quad \Omega^2(X) = \underbrace{\Omega^2_+(X)}_{*=1} \oplus \underbrace{\Omega^2_-(X)}_{*=-1}$$

$$F_A = F_A^+ + F_A^-$$

Two integrals associated with a connexion A:

- The Yang-Mills functional

$$\mathcal{YM}(A) = \int_X |F_A|_g^2 \, d\mathrm{vol}_X = \int_X \langle F_A \wedge * F_A \rangle$$

- The "topological charge"

$$k = -c_2(P)[X] = \frac{-1}{8\pi^2} \int_X \langle F_A \wedge F_A \rangle \in \mathbb{Z}$$

→ indep. on A
→ classifies P up to isomorphism

Rk: $* : \Omega^2(X) \to \Omega^2(X)$ satisfies $*^2 = 1$

$$\leadsto \quad \Omega^2(X) = \underbrace{\Omega^2_+(X)}_{*=1} \oplus \underbrace{\Omega^2_-(X)}_{*=-1}$$

$$F_A = F_A^+ + F_A^-$$

$$\mathcal{YM}(A) = \|F_A^+\|^2 + \|F_A^-\|^2$$

$$8\pi^2 k = \|F_A^+\|^2 - \|F_A^-\|^2$$

$$YM(A) = \|F_A^+\|^2 + \|F_A^-\|^2 \qquad 8\pi^2 k = \|F_A^+\|^2 - \|F_A^-\|^2$$

Def: A is self dual (SD) if $F_A^- = 0$ (i.e. $*F_A = F_A$)

A is anti-self dual (ASD) if $F_A^+ = 0$ (i.e. $*F_A = -F_A$)

$$\mathcal{YM}(A) = \| F_A^+ \|^2 + \| F_A^- \|^2 \qquad 8\pi^2 k = \| F_A^+ \|^2 - \| F_A^- \|^2$$

<u>Def</u>: A is <u>self dual</u> (SD) if $F_A^- = 0$ (i.e. $* F_A = F_A$)

A is <u>anti-self dual</u> (ASD) if $F_A^+ = 0$ (i.e. $* F_A = -F_A$)

$\Rightarrow$ min of $\mathcal{YM}$.

<u>Rk</u>: SD for $X \Leftrightarrow$ ASD for $\overline{X}$ (opposite orientation).

$$YM(A) = \|F_A^+\|^2 + \|F_A^-\|^2 \qquad 8\pi^2 k = \|F_A^+\|^2 - \|F_A^-\|^2$$

Def: A is self dual (SD) if $F_A^- = 0$ (i.e. $*F_A = F_A$)

A is anti-self dual (ASD) if $F_A^+ = 0$ (i.e. $*F_A = -F_A$)

$\Rightarrow$ min of YM.

Rk: SD for $X \Leftrightarrow$ ASD for $\overline{X}$ (opposite orientation).

$$\mathcal{M}_{SD}(P,g) = \{\text{SD conn's}\}/\mathcal{G} = \mathcal{M}_{SD}(X,k)$$

$$YM(A) = \|F_A^+\|^2 + \|F_A^-\|^2 \qquad 8\pi^2 k = \|F_A^+\|^2 - \|F_A^-\|^2$$

<u>Def:</u> A is <u>self dual</u> (SD) if $F_A^- = 0$ (i.e. $*F_A = F_A$)

A is <u>anti-self dual</u> (ASD) if $F_A^+ = 0$ (i.e. $*F_A = -F_A$)

$\Rightarrow$ min of YM.

<u>Rk:</u> SD for $X \Leftrightarrow$ ASD for $\overline{X}$ (opposite orientation).

$$\mathcal{M}_{SD}(P, g) = \{\text{SD conn's}\}/\mathcal{G} = \mathcal{M}_{SD}(X, k)$$

<u>Rk:</u>
- $k < 0$: $\mathcal{M}_{SD}(P) = \emptyset$
- $k = 0$: SD $\Leftrightarrow$ Flat ($F_A = 0$)

$$\mathcal{M}_{SD} = \mathrm{Hom}(\pi_1 X, SU(2))/SU(2) = \{pt\} \text{ if } \pi_1 X = \{1\}.$$

Thm (Donaldson) Assume • X^4 simply connected $(\pi_1 = 0)$ (pos. def.)

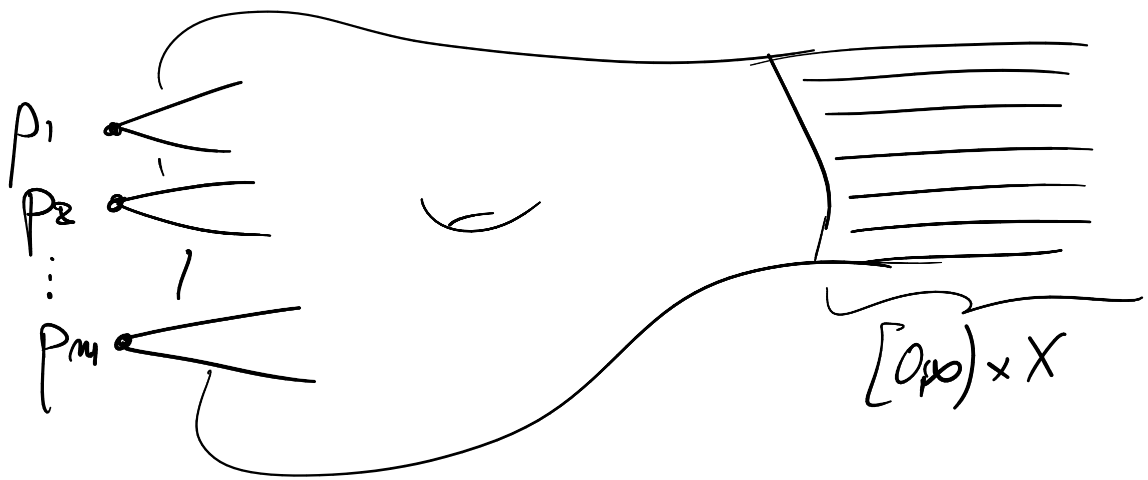
• $P \to X$ with $k = 1$

Then, for generic g:

* $\mathcal{M}_{SD}(P, g)$ is smooth, of dimension 5, away from m points $p_1, \dots, p_m$, where $m = \frac{1}{2} \# \{ \alpha \in H^2(X, \mathbb{Z}) \mid q_X(\alpha, \alpha) = 1 \}$

$= 8\underset{1}{k} - 3(1 - \underset{0}{b_1} + \underset{0}{b_2^-})$ (Atiyah-Hitchin-Singer)

* $\mathcal{M}_{SD}(P, g)$ is oriented.
* each p_i has a neighborhood $N_i \simeq \text{Cone } \mathbb{CP}^2$ (or $\overline{\mathbb{CP}}^2$)
* $\mathcal{M}_{SD}(P, g)$ has an "end" $\simeq [0, \infty) \times X$ (i.e. $\mathcal{M}_{SD} \setminus (0, \infty) \times X$ is compact)



• Th2 $\Rightarrow$ Th1   Goal: apply the following lemma:

lemma: q: quadratic form on $\mathbb{Z}^r$ positive, definite, unimodular

then $\frac{1}{2}\#\{\alpha \in \mathbb{Z}^k \mid q(\alpha,\alpha) = 1\} \leq r$, with equality iff $q \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

• <u>Th2 $\Rightarrow$ Th1</u>   Goal: apply the following lemma:

<u>lemma</u>: q: quadratic form on $\mathbb{Z}^r$ positive, definite, unimodular

then $\frac{1}{2}\#\{\alpha \in \mathbb{Z}^k \mid q(\alpha,\alpha) = 1\} \leq r$, with equality iff $q \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

<u>proof of lemma</u>: If α, α' satisfy , they must either be proportional, or orthogonal. $\square$

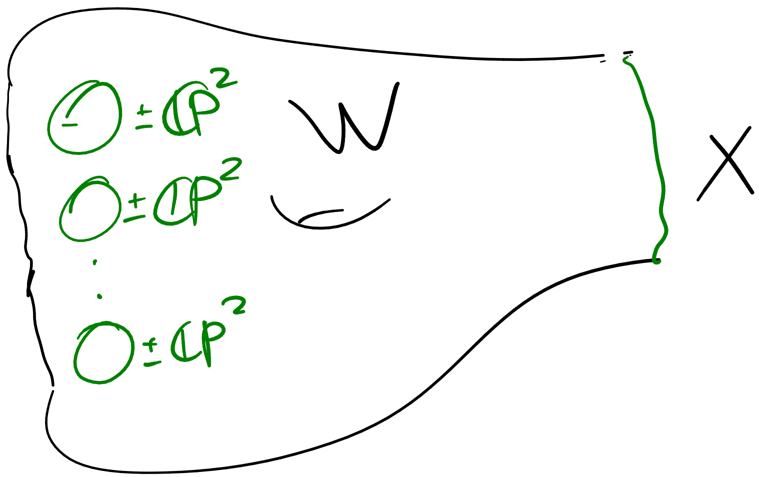
• Th 2 $\Rightarrow$ Th 1   Goal: apply the following lemma:

lemma: q: quadratic form on $\mathbb{Z}^r$ positive, definite, unimodular

then $\frac{1}{2}\#\{\alpha \in \mathbb{Z}^k \mid q(\alpha,\alpha) = 1\} \leq r$, with equality iff $q \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

proof of lemma: If α, α' satisfy, they must either be proportional or orthogonal.

Consider $W = \mathcal{M}_{SD}(P, g) \setminus \left((0,\varepsilon) \times X \cup \bigcup_i N_i \right)$



W: cobordism from $\underbrace{\mathbb{CP}^2 \cup \cdots \cup \mathbb{CP}^2}_{m_1} \cup \underbrace{\overline{\mathbb{CP}}^2 \cup \cdots \cup \overline{\mathbb{CP}}^2}_{m_2}$

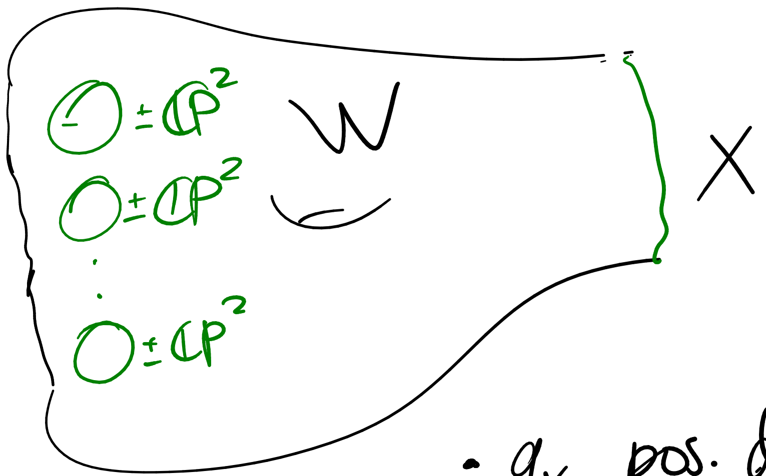
to X.

• Th2 $\Rightarrow$ Th1   Goal: apply the following lemma:

lemma: q: quadratic form on $\mathbb{Z}^r$ positive, definite, unimodular
then $\frac{1}{2}\#\{\alpha \in \mathbb{Z}^k \mid q(\alpha,\alpha)=1\} \leq r$, with equality iff $q \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

proof of lemma: If α, α' satisfy $q(\alpha,\alpha)=1$, they must either be proportional or orthogonal. $\square$

Consider $W = \mathcal{M}_{SD}(P,g) \setminus \left((0,\varepsilon) \times X \cup \bigcup_i N_i \right)$



W: cobordism from $\underbrace{\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2}_{m_1} \sqcup \underbrace{\overline{\mathbb{CP}}^2 \sqcup \cdots \sqcup \overline{\mathbb{CP}}^2}_{m_2}$

to X.

$\Rightarrow \sigma(X) = \sigma(\mathbb{CP}^2 \sqcup \cdots \overline{\mathbb{CP}}^2) = m_1 - m_2$

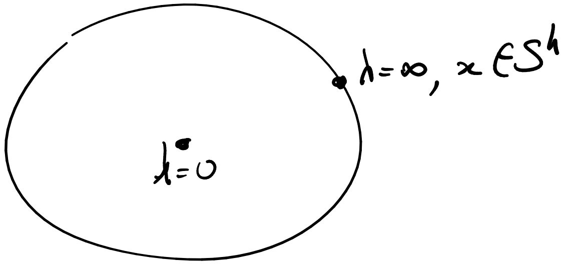
• q_X pos. definite $\Rightarrow \sigma(X) = rk(H_2 X) = r$

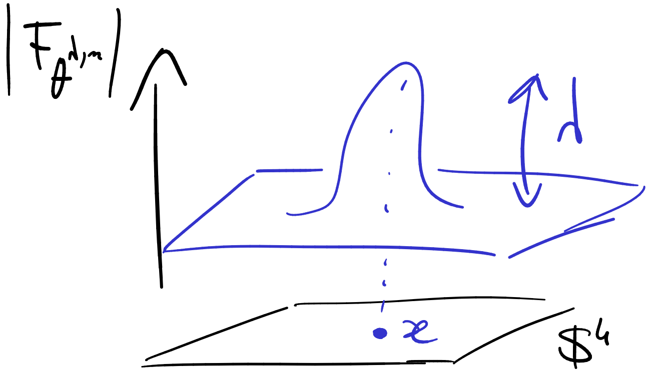
$\Rightarrow r = m_1 - m_2 \leq m_1 + m_2 = m \underset{\text{lemma}}{\Longrightarrow} q_X \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ $\square$

Ex: $X = S^4$ + round metric

Th: [Atiyah-Drinfeld-Hitchin-Manin]

$$\mathcal{M}_{SD}(S^4_{round}, k=1) \simeq \underset{\text{(open ball)}}{B^5} \simeq \left\{ \Theta^{\lambda,x} \;\middle|\; \underset{\substack{\uparrow \\ \text{"scale"}}}{\lambda} \geq 0 \;,\; \underset{\substack{\uparrow \\ \text{"center"}}}{x} \in S^4 \right\}$$

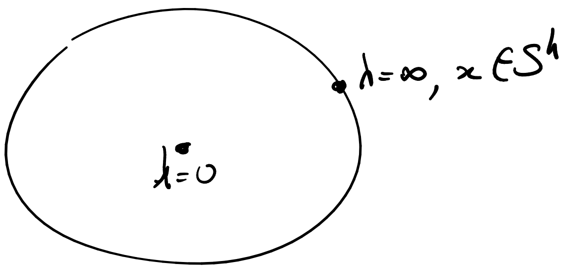


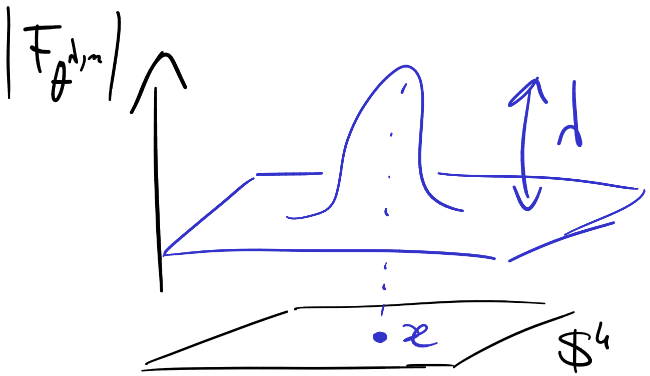


Ex: $X = S^4$ + round metric

Th: [Atiyah - Drinfeld - Hitchin - Manin]

$$\mathcal{M}_{SD}(S^4_{round}, k=1) \simeq \underset{\text{(open ball)}}{B^5} \simeq \{ \Theta^{\lambda,x} \mid \underset{\text{"scale"}}{\lambda} > 0 , \ \underset{\text{"center"}}{x} \in S^4 \}$$





· Ingredients in proof of Th2:

→ Compactness

→ Smoothness

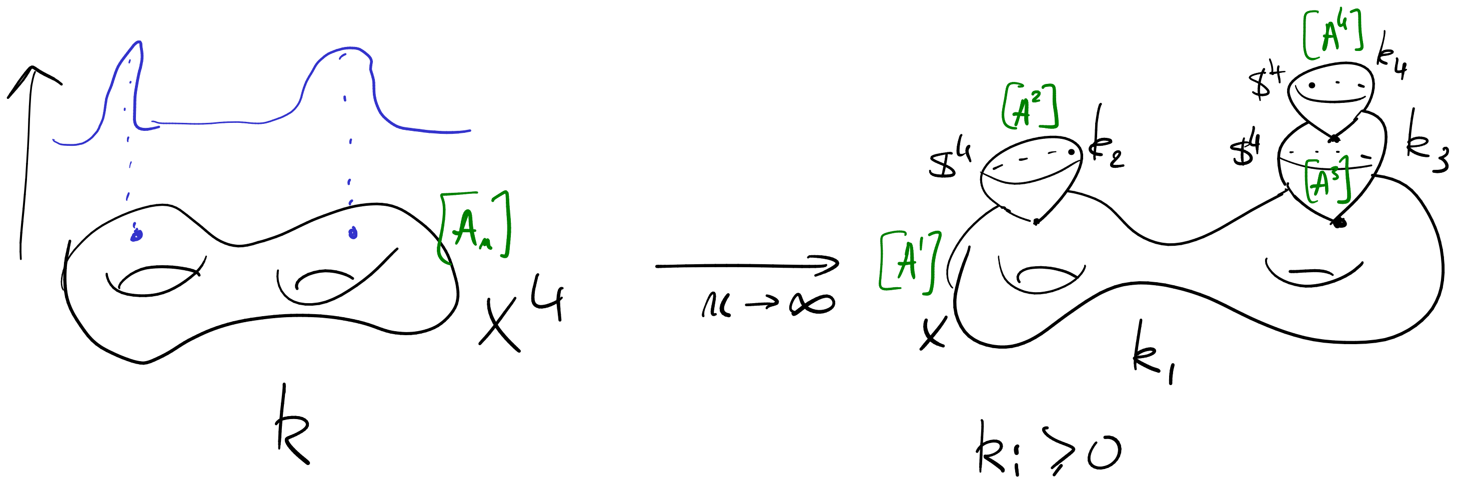
(→ Orientability)

Compactness

- If $|F_A|$ bounded $\rightsquigarrow$ Uhlenbeck compactness.

Compactness

- If $|F_A|$ bounded $\rightsquigarrow$ Uhlenbeck compactness.
- If $|F_A|$ concentrates near points: $[A_n]$: seq. of points in $\mathcal{M}_{SD}(X, k=1)$

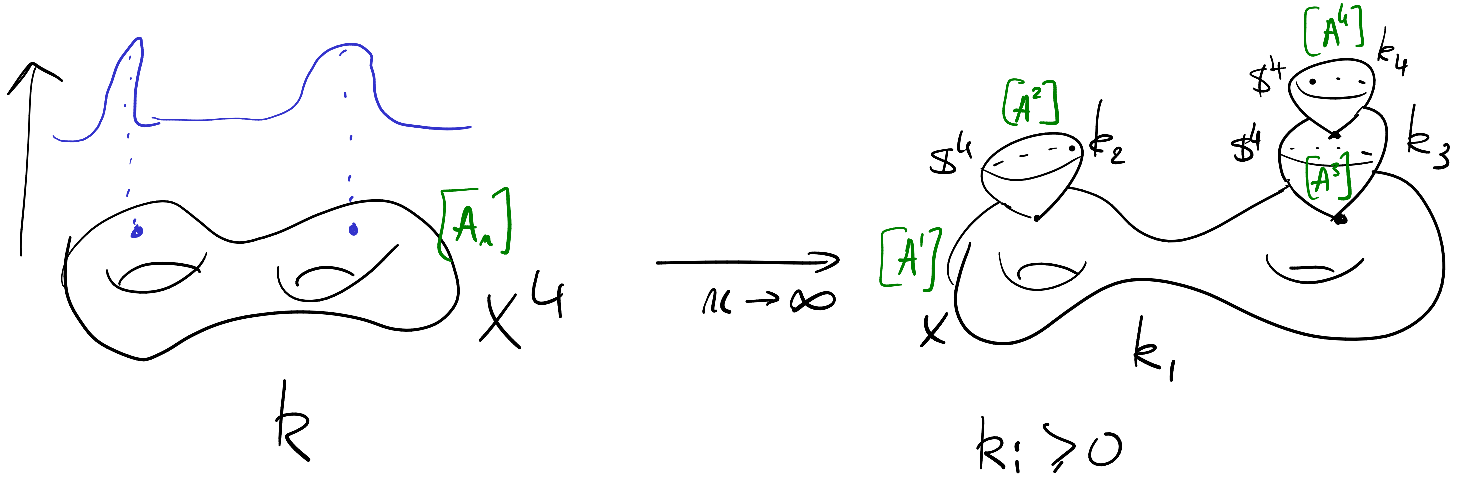


$k_i > 0$

$k = k_1 + \dots + k_4$

Compactness

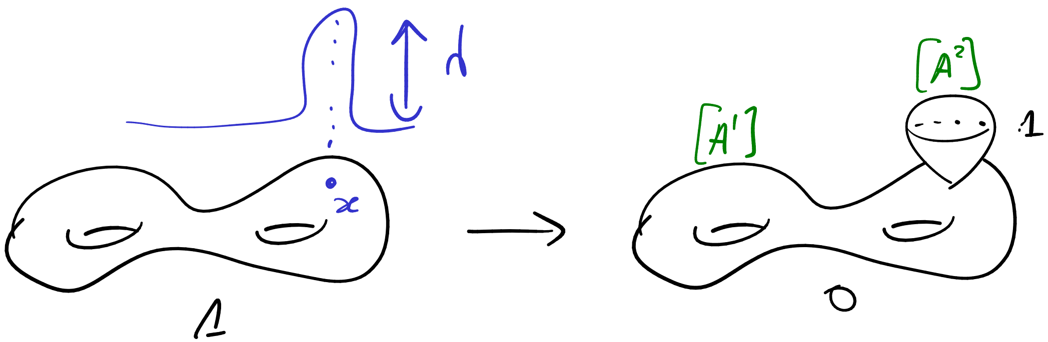
- If $|F_A|$ bounded $\rightsquigarrow$ Uhlenbeck compactness.
- If $|F_A|$ concentrates near points: $[A_n]$: seq. of points in $\mathcal{M}_{SD}(X, k=1)$



$k_i \geq 0$

$k = k_1 + \dots + k_4$

k=1: can only have



$$\Rightarrow \mathrm{End}\big(\mathcal{M}_{SD}(X, k=1)\big) \cong \underset{d}{\underset{\cup\!\!\!\shortmid}{[d_0, \infty)}} \times \underset{x}{\underset{\cup\!\!\!\shortmid}{X}} \times \underbrace{\overset{[A^1]}{\overset{\cap\!\!\!\shortmid}{\mathcal{M}_{SD}(X, k=0)}}}_{=\mathrm{pt.}} \times \underbrace{\overset{[A^2]}{\overset{\cap\!\!\!\shortmid}{\mathcal{M}_{SD}(S^4, k=1)}} / \mathrm{Conf}\, S^4}_{=\mathrm{pt.}}$$

Smoothness / singularities

some Banach bundle

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right] \\ \mathcal{A} \times \mathcal{M} \arrow[u, shift right, bend right, "\mathcal{F}"'] \end{tikzcd}$$

reduced gauge group $\mathrm{Aut}\,P / \mathbb{Z}_2$ $\longrightarrow$ $\tilde{\mathcal{G}} \circlearrowright \mathcal{A} \times \mathcal{M}$

space of connexions on P

space of metrics on X^4.

$$\mathcal{F}(A, g) = F_A^-$$

$$\to \mathcal{M}^g_{SD}(X, k=1) = \left(\mathcal{F}^{-1}(0) \cap \mathcal{A} \times \{g\} \right) / \tilde{\mathcal{G}}$$

Smoothness / singularities

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift left] \\ \mathcal{A} \times \mathcal{M} \arrow[u, bend right, "\mathcal{F}"'] \end{tikzcd}$$

$\mathcal{E}$ ← some Banach bundle

reduced gauge group $\operatorname{Aut} P / \mathbb{Z}_2$ → $\tilde{\mathcal{G}} \circlearrowright \mathcal{A} \times \mathcal{M}$

$\mathcal{A}$: space of connexions on P

$\mathcal{M}$: space of metrics on X^4.

$$\mathcal{F}(A, g) = F_A^-$$

$$\to \mathcal{M}^g_{SD}(X, k=1) = \left(\mathcal{F}^{-1}(0) \cap \mathcal{A} \times \{g\}\right) / \tilde{\mathcal{G}}$$

1) Show $\mathcal{F}^{-1}(0)$ smooth $\left(D\mathcal{F}\big|_{\mathcal{F}^{-1}(0)} \text{ surjective} + \text{IFT} \right)$

Smoothness / singularities

$$\mathcal{F}(A,g) = F_A^-$$

$$\to \mathcal{M}^g_{SD}(X, k=1) = \big(\mathcal{F}^{-1}(0) \cap \mathcal{A} \times \{g\}\big) / \tilde{\mathcal{G}}$$

$\mathcal{E}$ ← some Banach bundle

$$\begin{array}{c} \mathcal{E} \\ \downarrow \uparrow \mathcal{F} \\ \tilde{\mathcal{G}} \circlearrowright \mathcal{A} \times \mathcal{M} \end{array}$$

reduced gauge group $\mathrm{Aut}\,P / \mathbb{Z}_2$ → $\tilde{\mathcal{G}}$

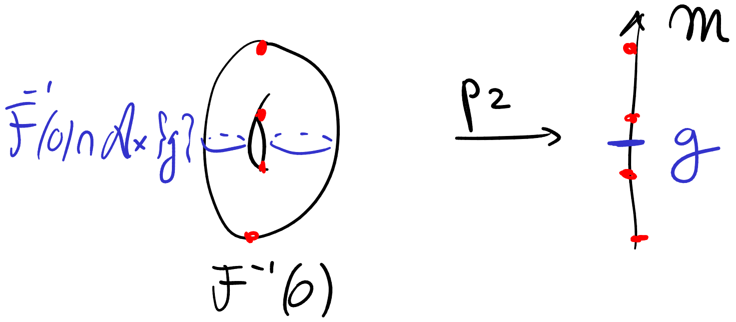
space of connexions on P → $\mathcal{A}$

space of metrics on X^4. → $\mathcal{M}$

1) Show $\mathcal{F}^{-1}(0)$ smooth ($D\mathcal{F}|_{\mathcal{F}^{-1}(0)}$ surjective + IFT)

2) Sard-Smale theorem applied to $\mathcal{F}^{-1}(0) \xrightarrow{p_2} \mathcal{M}$

$$(A,g) \mapsto g$$



$\Rightarrow$ $\mathcal{F}^{-1}(0) \cap \mathcal{A} \times \{g\}$ smooth for generic g.

Smoothness / singularities

$$\mathcal{F}(A,g) = F_A^-$$

$$\to \mathcal{M}^g_{SD}(X, k=1) = \left(\mathcal{F}^{-1}(0)\cap \mathcal{A}\times\{g\}\right)\big/\tilde{\mathcal{G}}$$

$$\begin{array}{c} \mathcal{E} \\ \downarrow\uparrow \mathcal{F} \\ \tilde{\mathcal{G}} \circlearrowright \mathcal{A}\times\mathcal{M} \end{array}$$

$\mathcal{E}$ ← some Banach bundle

reduced gauge group $\mathrm{Aut}P/\mathbb{Z}_2$ → $\tilde{\mathcal{G}}$

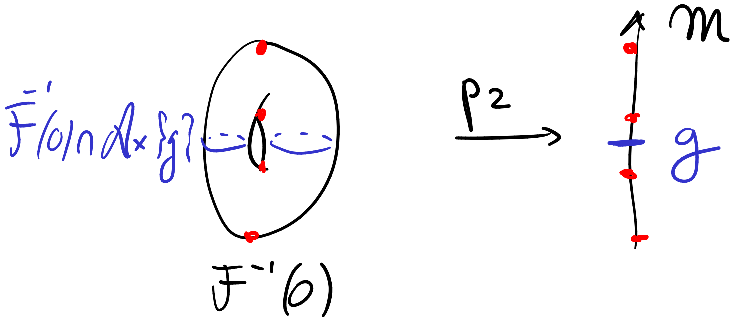
$\mathcal{A}$ ← space of connexions on P

$\mathcal{M}$ ← space of metrics on X^4.

1) Show $\mathcal{F}^{-1}(0)$ smooth $\left(D\mathcal{F}|_{\mathcal{F}^{-1}(0)}\ \text{surjective} + \text{IFT}\right)$

2) Sard-Smale theorem applied to $\mathcal{F}^{-1}(0) \xrightarrow{p_2} \mathcal{M}$

$$(A,g) \mapsto g$$



$\Rightarrow \mathcal{F}^{-1}(0)\cap\mathcal{A}\times\{g\}$ smooth for generic g.

3) quotient by $\tilde{\mathcal{G}}$: two cases
- ↗ A is reducible
- ↘ A is irreducible

$$\begin{array}{ccc} P & \text{SU(2)-bundle} \rightsquigarrow & V = P \times_{SU(2)} \mathbb{C}^2 \\ \downarrow & & \downarrow \\ X & & X \end{array}$$

: rank 2 $\mathbb{C}$-vector bundle

Def: • A is <u>reducible</u> if it splits: $V = \lambda_1 \oplus \lambda_2$, $A = a_1 \oplus a_2$

$\Leftrightarrow \mathrm{Stab}_A = U(1) \subset \tilde{\mathcal{G}}$.

λ_1, λ_2: $\mathbb{C}$-line bundles

a_1, a_2: $U(1)$-connexions

• A is <u>irreducible</u> otherwise

$\Leftrightarrow \mathrm{Stab}_A = \{1\} \subset \tilde{\mathcal{G}}$.

$$\begin{array}{c} P \\ \downarrow \\ X \end{array} \quad SU(2)\text{-bundle} \rightsquigarrow \begin{array}{c} V = P \times_{SU(2)} \mathbb{C}^2 \\ \downarrow \\ X \end{array} : \text{rank } 2 \ \mathbb{C}\text{- vector bundle}$$

Def: • A is reducible if it splits: $V = \lambda_1 \oplus \lambda_2$, $A = a_1 \oplus a_2$

$\mathbb{C}$-line bundles $\qquad$ $U(1)$-connexions

$\Leftrightarrow \mathrm{Stab}_A = U(1) \subset \tilde{\mathcal{G}}$.

• A is irreducible otherwise

$\Leftrightarrow \mathrm{Stab}_A = \{1\} \subset \tilde{\mathcal{G}}$.

Reducibles are classified by $\alpha = c_1(\lambda_1) \in H^2(X;\mathbb{Z})$

because V is an $SU(2)$-bundle.

• $0 = c_1(V) = c_1(\lambda_1) + c_2(\lambda_2)$

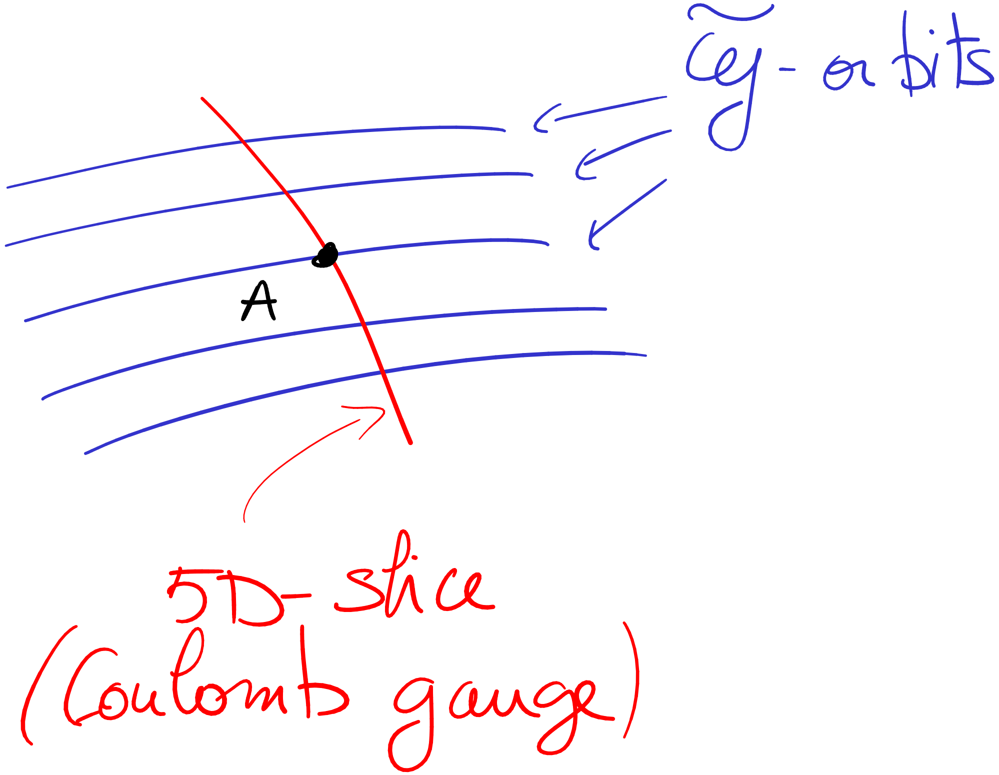
• $1 = -c_2(V)[X] = -(c_1(\lambda_1) \cup c_1(\lambda_2))[X]$

$= (\alpha \cup \alpha)[X]$

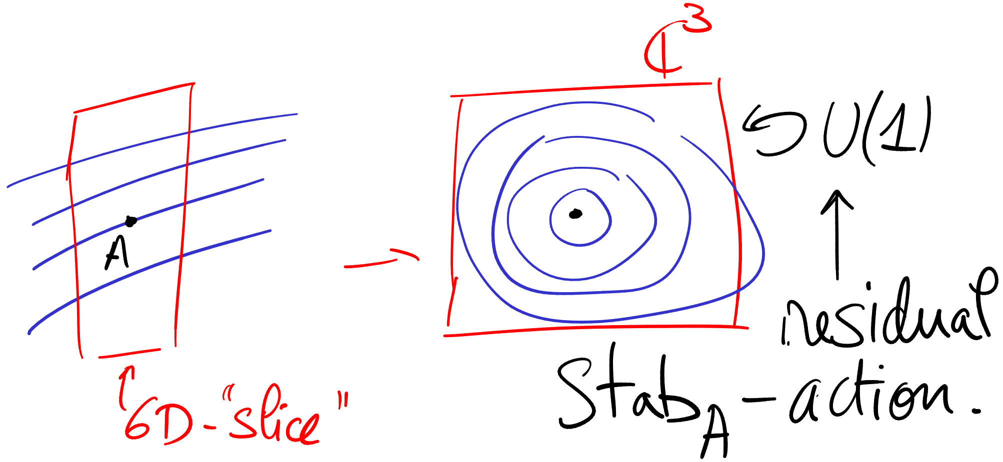
$$\boxed{1 = q_X(\alpha, \alpha)}$$

local picture of $\underset{\tilde{\mathcal{G}}\circlearrowleft}{\{A \,/\, F_A^- = 0\}}$ near A $\left(\mathcal{M}_{SD} = \{F_A^- = 0\}/\tilde{\mathcal{G}}\right)$

<u>A irreducible:</u>



<u>A reducible:</u>



$$\to \mathbb{C}^3/U(1) \simeq \mathrm{Cone}\left(S^5/U(1)\right) = \mathrm{Cone}\left(\mathbb{CP}^2\right)$$