UNIVERSITY OF OXFORD Mathematical Institute

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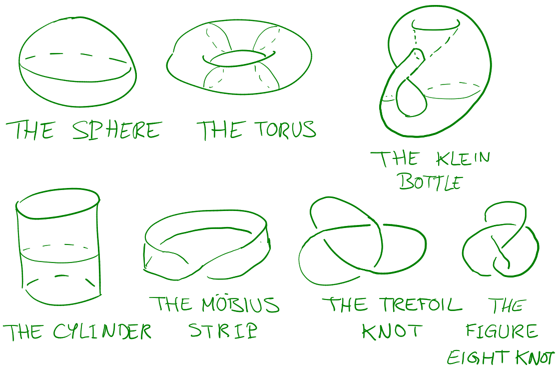
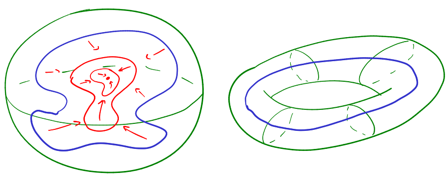
Research Outline

My research interests lie at the crossroad of *low dimensional topology* and *symplectic geometry*.

Low dimensional topology, roughly speaking, is about studying shapes like the sphere, the torus, the cylinder, the Möbius strip, and the Klein bottle. These are all surfaces, i.e. two-dimensional examples, but we also study higher dimensional analogues, of dimension 3 and 4 (impossible to draw), as well as one-dimensional ones, such as knots.



In topology we like to think about these spaces as being made of some abstract elastic material that is allowed to bend and stretch as much as we want. For example the surface of a tea mug can be regarded as equivalent to a torus, since we can deform one to the other. In contrast, the two-sphere is not equivalent to a torus. It turns out that it is generally much more difficult to prove that two object are not equivalent, than proving that they are: to prove equivalence it is enough to exhibit one transformation, but proving the opposite amounts to show that for *all* possible transformation of the first, the outcome will never be the second. In order to prove such statements we usually use *topological invariants*: algebraic objects that we associate to spaces and that are invariant by deformation. If the invariants of two spaces are different, then these spaces are not equivalent.

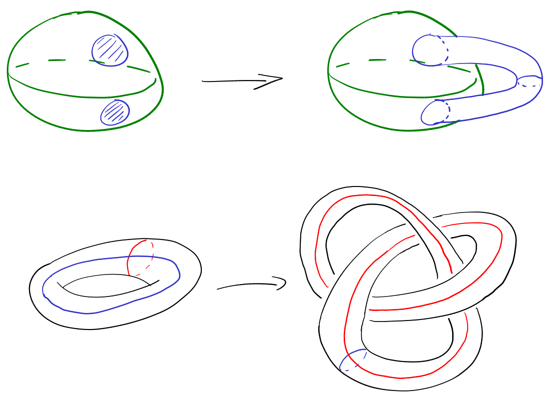


For instance, a very important one is the *fundamental group*, an invariant that measures whether closed loops in a space can be shrinked to a point. For example the blue loop in the sphere, as well as any other possible loop, can be shrinked to a point, as opposed to the one drawn on the torus (which goes around a hole): the sphere and the torus have different fundamental group, therefore they are not equivalent.

Symplectic geometry has its origins in the study of celestial mechanics: it provides a nice framework to formulate and study Hamilton's equations of motion. These are the kind of equations that one needs to solve in order to send a shuttle to the moon, or to predict the trajectories of the planets of the solar system.

Symplectic geometry also plays an important role in *string theory*, a branch of theoretical physics aimed at unifying general relativity (a theory that describes well large scales) and quantum mechanics (for small scales), two theories that are effective in their application range, but notoriously incompatible. One of Einstein's big insights was to re-think our physics in terms of living in a four-dimensional space-time rather than a three-dimensional space. String theory posits that we might re-think it even more. Among other possibilities, it suggests that maybe we actually live in a 10-dimensional space: we should think of points in our space-time as very small six-dimensional spaces called *symplectic manifolds*, in which some mathematical objects called *pseudo-holomorphic curves* are living, and which we can think of as little vibrating strings.

My research focuses on the study of these symplectic manifolds and pseudo-holomorphic curves from a mathematical prospective, as well as other objects and equations coming from theoretical physics (Yang-Mills theory). These can be used to define and compute topological invariants associated to spaces of dimension 3 and 4, as well as knotted objects inside these. The main contribution of my PhD thesis was to describe how *symplectic instanton homology*, a powerful but complicated invariant that has been introduced by Manolescu and Woodward, undergoes *Dehn surgery* (described below), an important operation since it permits to construct any three-dimensional space out of our 3-space.



In general, surgery is the operation of removing a piece of a space, and then re-gluing to it a different piece. For example, one can remove two blue discs on a sphere, re-glue a blue cylinder and obtain a torus. Dehn surgery is the operation of removing a knotted shape in our 3-dimensional space (the interior of the picture on the right), and then re-glue the interior of the picture on the left, in a way that identifies the blue curve with the blue curve, and the red curve with the red curve. This seems impossible in our three space, precisely because the topology of the space will change and will not be our three space anymore, but abstractly it is possible.

It is my hope that my work will add some contributions to the following research directions:

- prove theorems in low dimensional topology, such as giving constraints for a space to be turned into another, for some given topological operations.
- provide a rigorous mathematical ground for a unified particle physics theory. Such a ground is currently still missing and is the subject of intense research from many different directions. For example, physicists predict the existence of some *extended topological quantum field theories*, which are structures that assign topological invariants to several spaces of several dimensions and satisfy some gluing properties. Such structures were introduced to formalize intuitive but ill-defined physics ideas such as *Feynman diagrams* and *path integrals*, and are studied extensively from an algebraic prospective. I am currently working to construct such a theory geometrically.
- import tools from theoretical physics in order to apply them to seemingly unrelated branches of mathematics such as knot theory and low dimensional topology, but also in other sciences as in graph theory, machine learning and image recognition, or quantum computing.