Symplectic Instanton homology: connected sum, Dehn surgery, and maps from cobordisms

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Soutenance de thèse

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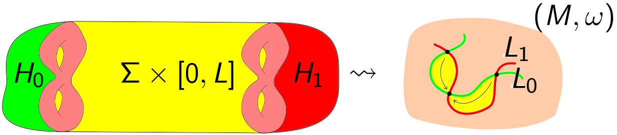

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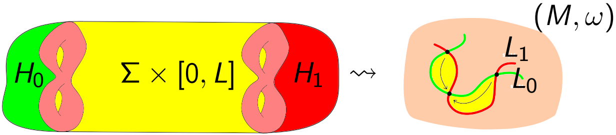

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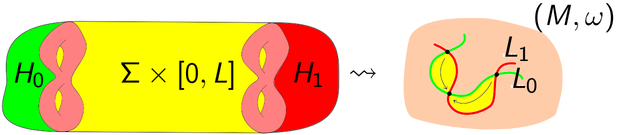

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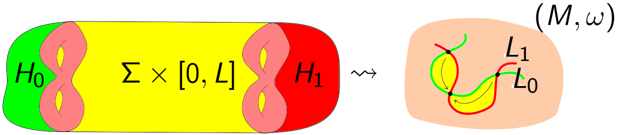

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Atiyah-Floer conjecture: $I_*(Y) \simeq HF(L_0, L_1)$.

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- **Wehrheim-Woodward, '07:** "Floer Field theory", use moduli spaces of twisted $U(r)$-bundles, provide a general framework for such kind of constructions.

Questions addressed in this thesis

Question: How does HSI behaves under:

- Connected sums?
- Dehn surgery?
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$\rightarrow$ Need to define a twisted version.

Twisted version

Let Y be a closed oriented 3-manifold, and $c \in H_1(Y; \mathbb{Z}/2\mathbb{Z})$. We will define a $\mathbb{Z}/8\mathbb{Z}$-relatively graded abelian group $HSI(Y, c)$ such that $HSI(Y, 0) = HSI(Y)$.

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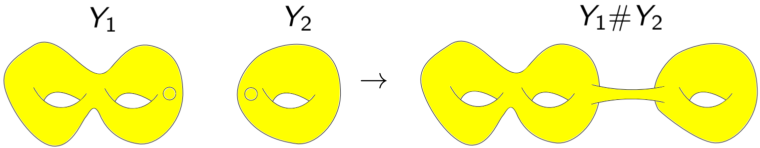
Remark: The class c can be seen as an isomorphism class of an $SO(3)$-bundle (whose second Stiefel-Whitney class is dual to c).

Connected sum



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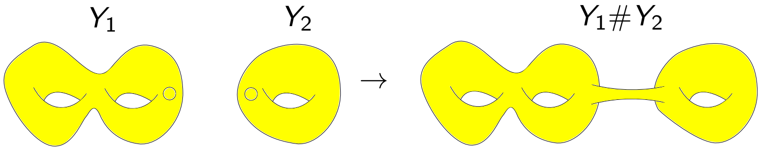


Theorem (Connected sum formula, C.)

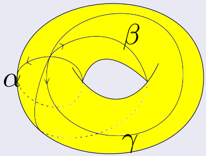
Let (Y_1, c_1), (Y_2, c_2) be 3-manifolds with homology classes as previously, then:

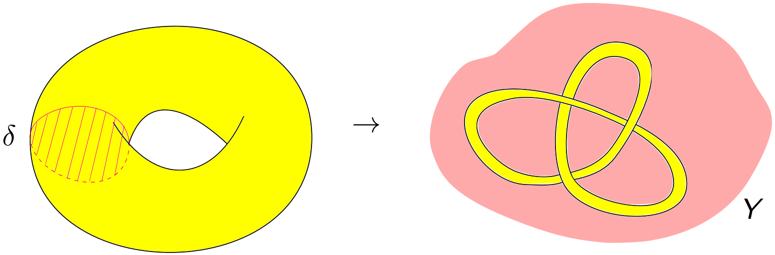
$$HSI(Y_1 \# Y_2, c_1 + c_2) \simeq HSI(Y_1, c_1) \otimes HSI(Y_2, c_2) \oplus Tor(HSI(Y_1, c_1), HSI(Y_2, c_2))[-1].$$

Dehn surgery

Definition (Surgery triad)

Y^3: compact oriented, with $\partial Y \simeq T^2$,
α, β, $\gamma \subset \partial Y$ simple curves such that
$\alpha.\beta = \beta.\gamma = \gamma.\alpha = -1$,
Y_δ: Dehn fillings, obtained by gluing a solid torus with meridian mapped to $\delta \in \{\alpha, \beta, \gamma\}$.





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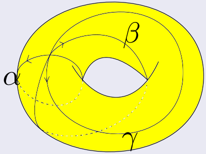

Theorem (Dehn surgery exact sequence, C.)

$c \in H_1(Y; \mathbb{Z}/2\mathbb{Z})$, $c_\delta \in H_1(Y_\delta; \mathbb{Z}/2\mathbb{Z})$ induced classes, $\delta \in \{\alpha, \beta, \gamma\}$
$k_\alpha \in H_1(Y_\alpha; \mathbb{Z}/2\mathbb{Z})$ core of the solid torus.
There exists a long exact sequence:

$$HSI(Y_\beta, c_\beta) \longrightarrow HSI(Y_\gamma, c_\gamma) \longrightarrow HSI(Y_\alpha, c_\alpha + k_\alpha) \longrightarrow HSI(Y_\beta, c_\beta)$$

4-dimensional cobordisms

Theorem (Maps from cobordisms, C.)

1. *Let W be a smooth connected oriented 4-cobordism from Y_1 to Y_2, and $c \in H^2(W; \mathbb{Z}/2\mathbb{Z})$. Then there exists an associated morphism*
$$F_{W,c}\colon HSI(Y_1, c_1) \to HSI(Y_2, c_2),$$
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where $c_i = PD(c_{|Y_i})$.
2. *Moreover, two arrows in the previous long exact sequence are such morphisms.*
3. *If W contains an embedded 2-sphere S with either*
   - $S.S = 1$,
   - $S.S = -1$ *and* $c_{|S} \neq 0$,

   then, $F_{W,c} = 0$.

Some computations

The rank of $HSI(Y, c)$ is minimal (i.e. equal to $\mathrm{Card} H_1(Y, \mathbb{Z})$) for families of rational homology spheres including:

- branched double covers of quasi-alternating links,
- boundaries of plumbings associated to a weighted tree (G, m) such that $m(v) \geq d(v)$ for every vertex v, ($d(v)$: incidence number, $m(v)$: Euler class)
- n-surgery along a torus knot $T(p, q)$, with $n \geq pq - 1$.

Floer Field Theory

Cob_{n+1}: Category of compact connected $(n+1)$-dimensional cobordisms between closed connected n-dimensional manifolds.

Definition (Wehrheim-Woodward, '07)

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- **Objects:** symplectic manifolds,
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Problem

Composition is not always defined

Definition (Geometric composition)

Let $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$,

$$L_{01} \circ L_{12} = \{(x_0, x_2) \mid \exists x_1 \in M_1 : (x_0, x_1) \in L_{01}; (x_1, x_2) \in L_{12}\}$$

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Definition (Embedded geometric composition)

- $L_{01} \times M_2 \pitchfork M_0 \times L_{12}$ transverse intersection.
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Definition (Category *Symp*, Wehrheim-Woodward '10)

- **Objects:** symplectic manifolds,
- **Morphisms:** sequences $\underline{L} = M_0 \xrightarrow{L_{01}} \cdots \xrightarrow{L_{(k-1)k}} M_k$, identifying $(\cdots, L_{(i-1)i}, L_{i(i+1)}, \cdots)$ with $(\cdots, L_{(i-1)i} \circ L_{i(i+1)}, \cdots)$ provided the composition is embedded.

Quilted Floer Homology

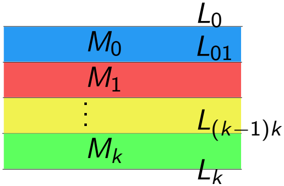
Definition (Quilted Floer Homology)

Let $\underline{L} = (L_0, L_{01}, \cdots) \in \mathrm{Hom}_{Symp}(pt, pt)$ (with extra assumptions), define inside $M_0^- \times M_1 \times M_2^- \cdots$:

$$HF(\underline{L}) = HF(L_0 \times L_{12} \times L_{34} \cdots,\ L_{01} \times L_{23} \cdots).$$

Generators of the chain complex:
$\mathcal{I}(\underline{L}) = \{(x_0, \cdots, x_k) \mid (x_i, x_{i+1}) \in L_{i(i+1)}\}$,

Differential: counts pseudo-holomorphic "quilts".



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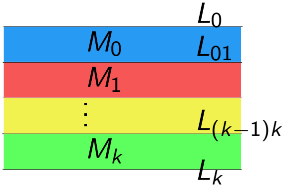


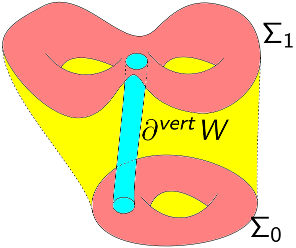
Theorem (Wehrheim-Woodward, Lekili-Lipyanskiy)

If the composition $L_{(i-1)i} \circ L_{i(i+1)}$ is embedded, then
$HF(\cdots, L_{(i-1)i}, L_{i(i+1)}, \cdots) \simeq HF(\cdots, L_{(i-1)i} \circ L_{i(i+1)}, \cdots)$.

A Floer Field theory with boundary

Definition (Category $\widetilde{Cob}_{2+1}$, cobordisms with vertical boundaries)

- **Objects:** connected surfaces with one parametrized boundary,
- **Morphisms:** (diffeomorphism classes of) tuples (W, c), where
  - W: compact connected oriented 3-manifold, with $\partial W = \Sigma_0 \cup \partial^{vert} W \cup \Sigma_1$, $\partial^{vert} W$ parametrized tube,
  - $c \in H_1(W, \mathbb{Z}/2\mathbb{Z})$.
- **Composition:** gluing cobordisms and adding classes.



Definition of a functor $\widetilde{Cob_{2+1}} \to Symp$

- **Surfaces:** $\Sigma \mapsto \mathscr{N}(\Sigma)$ = moduli space of flat connexions on the trivial $SU(2)$-bundle over Σ, $A \in \Omega^1(\Sigma) \otimes \mathfrak{su}(2)$, such that $A_{|\partial\Sigma} = \theta ds$, with $|\theta| < \pi\sqrt{2}$ and $s \in \partial\Sigma$ parameter, modulo gauge transformations fixing $\partial\Sigma$. (Huebschmann-Jeffrey, '93)

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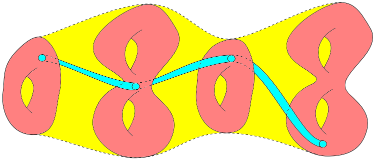
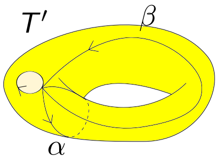
    - If W is not elementary: decompose it into elementary pieces, and take the corresponding sequence of correspondences.



Example of an explicit description

Both moduli spaces and correspondences admit explicit representation-theoretic descriptions.

- $\Sigma = T'$ **punctured torus,**



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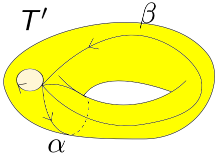

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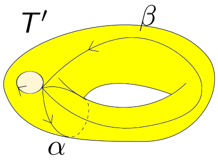

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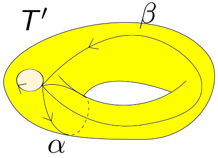

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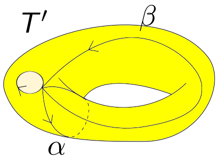

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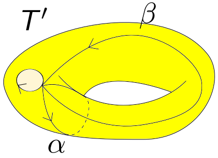

- W **solid torus,** α **bounds a disk,**

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$$\pi_1(T', *) = \langle \alpha, \beta \rangle, \partial T' = [\alpha, \beta],$$
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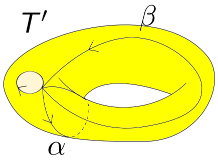
$$\mathscr{N}(T') = \left\{ (\theta, A, B) \mid e^\theta = [A, B] \right\},$$
$$= \{(A, B) \mid [A, B] \neq -I\}.$$



- **W solid torus, α bounds a disk,**
  - $c = 0$, $L(W, c) = \{(I, B)\}$,
  - $c \neq 0$, $L(W, c) = \{(-I, B)\}$.

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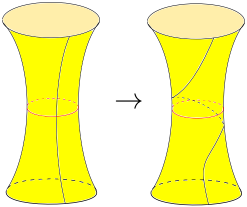
Definition

Y^3 compact, $c \in H_1(Y, \mathbb{Z}/2\mathbb{Z})$ class, $W = Y \setminus \left(D^2 \times [0,1]\right)$, viewed as a cobordism from D^2 to D^2.
Take then $HSI(Y, c) = HF(\underline{L}(W, c))$.

Functoriality and the geometric composition theorem: the isomorphism type of $HSI(Y, c)$ is well-defined.

Dehn twists

- Dehn twists on a surface:



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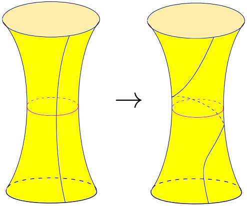
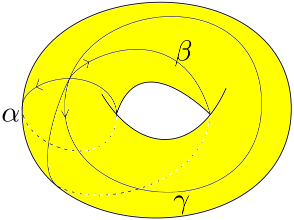


Remark

Let $\alpha, \beta, \gamma \subset T^2$ three curves such that $\alpha.\beta = \beta.\gamma = \gamma.\alpha = -1$.
Then, $\tau_\alpha(\gamma) = \beta^{-1}$



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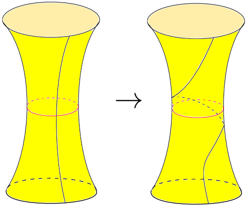

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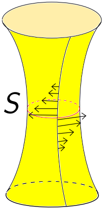
- Generalization to symplectic manifolds:



$S \subset (M, \omega)$ Lagrangian sphere, $\nu S \simeq D^*_\epsilon S$
$R\colon \mathbb{R} \to \mathbb{R}$ with $R(t) = 0$ for $t \geq \epsilon$, and
$R(-t) = R(t) - t$,
$H\colon D^*_\epsilon S \to \mathbb{R}$ defined by $H(q, p) = R(|p|)$
$\Rightarrow \tau_S$ time 2π Hamiltonian flow extends to S.
"generalized Dehn twist around S"

Outline of the proof of the exact sequence

Two steps:

1. Understand the effect of a Dehn twist of the punctured torus T' on the moduli space $\mathscr{N}(T')$.

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1. Understand the effect of a Dehn twist of the punctured torus T' on the moduli space $\mathscr{N}(T')$.
2. Apply the following theorem:

Theorem (Seidel, Wehrheim-Woodward, C.)

Let $L_0 \subset M_0$, $\underline{L} \in Hom_{Symp}(M_0, pt)$, $S \subset M_0$ Lagrangian sphere, $\tau_S \in Symp(M_0)$: generalized Dehn twist around S.
Then, there exists a long exact sequence:

$$\begin{array}{ccc} HF(\tau_S L_0, \underline{L}) & \longrightarrow & HF(L_0, \underline{L}) \\ & \nwarrow \quad \swarrow & \\ & HF(L_0, S, S^T, \underline{L}) & \end{array}$$

Step 1

Theorem (Callahan, Wehrheim-Woodward, C.)

A Dehn twist along a non-separating curve α on the punctured torus T' induces (almost) a generalized Dehn twist around $S = \{[A] \mid Hol_\alpha A = -I\} \subset \mathscr{N}(T')$.

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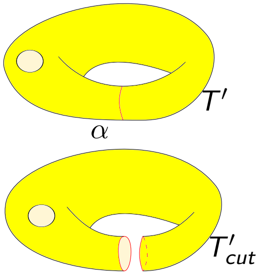

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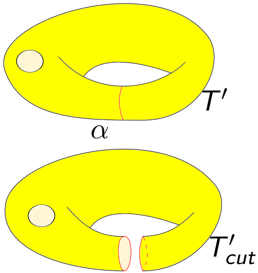

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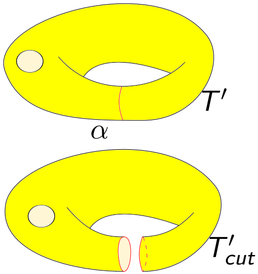
3. Return to $\mathscr{N}(T')$ using reduction:

$$\mathscr{N}(T') \setminus S = \mathscr{N}(T'_{cut}) /\!\!/ SU(2).$$



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Define the following Lagrangian spheres of $\mathscr{N}(T')$:

- $L_\alpha^- = \{[A] | \mathrm{Hol}_\alpha A = -I\}$,
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Apply the Dehn twist exact sequence with:

- $M_0 = \mathscr{N}(T')$,
- $S = L_\alpha^-$,
- $L_0 = L_\gamma$,
- $\underline{L} = L(Y, c)$.

□

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1-manifold	$\to$	lie group
2-manifold with boundary	$\to$	Hamiltonian manifold
3-manifold with corners	$\to$	equivariant Lagrangian correspondence
4-manifold with corners	$\to$	equivariant Floer cochain.