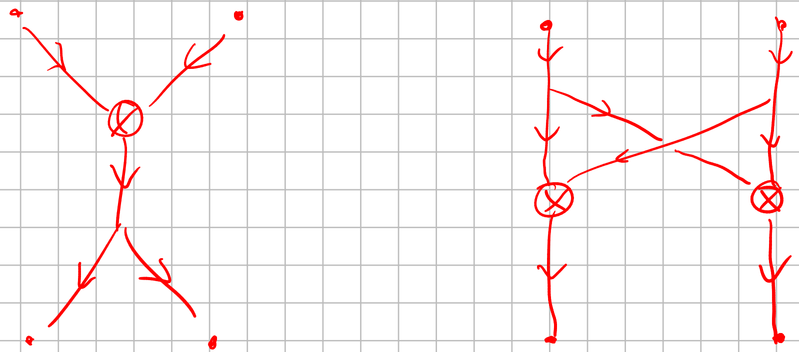
Hopf algebras, equivariant Lagrangian Floer homology and cornered instanton theory



Guillem Cazassus, Oxford

Joint with Paul Kirk, Artem Kotelskiy, Mike Miller, Wai-Kit Yeung

(Work in progress...)

- Equivariant Lagrangian Floer homology

$(M,\omega) \supset L_0, L_1$ (+ assumptions) $\rightsquigarrow HF(L_0,L_1)$

symplectic manifold

pair of Lagrangians

- compact + monotone
- exact + convex at ∞

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Pb: What if M and/or L_0, L_1 are ... singular?

Ex: $M = \tilde{M} /\!/ G = \mu^{-1}(0)/G \qquad G \circlearrowright \tilde{M} \xrightarrow{\mu} \mathfrak{g}^*$

$L_i = \tilde{L}_i / G \qquad$ (ex: Atiyah-Floer conjecture, $M = \mathcal{M}(\Sigma)$ ...)

- **Equivariant Lagrangian Floer homology**

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$L_i = \tilde{L}_i/G$ (ex: Atiyah-Floer conjecture, $M = \mathcal{M}(\Sigma)$ ...)

$\Rightarrow$ try to substitute $HF(M; L_0, L_1)$ by $HF_G(\tilde{M}; \tilde{L}_0, \tilde{L}_1)$

"equivariant Lagrangian Floer homology"

Approaches for defining $HF_G(M; L_0, L_1)$:

- Symplectic vortex equation: Cieliebak-Gaio-Salamon, Mundet i Riera, Frauenfelder, Tian-Xu, ...
- ∞-categories: Hendricks-Lipshitz-Sarkar
- Morse-Bott: Austin-Braam, Fukaya-Daemi
- Borel construction: $EG = \varinjlim_N EG_N$, T^*EG_N

Vitebo, Bourgeois-Oancea, Kim-Lau-Zheng, C. (Work in progress...)

Another approach:

$G \circlearrowright X$

Compact Lie group

compact smooth manifold

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Th (Gugenheim-May)

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$\Rightarrow$ need a G-action at the chain level.

Th: [C., Kirk, Kotelskiy, Miller, Yeung] Work over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Let: • G compact Lie group, $g : G \to \mathbb{R}$ Morse fct°

• (M,ω) : symplectic, with Hamiltonian action $G \circlearrowright M \xrightarrow{\mu} \mathfrak{g}^*$

moment map.

• $L_0, L_1 \subset M$ pair of Lagrangians

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be such that: • $e \in G$ is a local minimum for g

• (M, L_0, L_1) satisfy either:

Weinstein correspondence $\subset T^*G \times M^- \times M$

$$\begin{cases} \times\ M, L_0, L_1 \text{ exact} \\ \times\ M \text{ convex at } \infty \\ \times\ L_0, L_1 \text{ compact} \\ \times\ \mu : M \to \mathfrak{g}^* \text{ proper} \end{cases}$$

or

$$\begin{cases} \times\ M, L_0, L_1 \text{ compact} \\ \times\ M, L_0, L_1, \Lambda_G(M) \text{ positively monotone} \\ \times\ \text{minimal Maslov number of } L_0, L_1, \Lambda_G(M) \\ \quad \text{is proportional to an integer } N \geq 3. \end{cases}$$

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Then × The Morse complex $CM_*(G, g)$ is an A_∞-algebra, e is a strict unit: $\mu^1(e) = 0$, $\mu^2(x, e) = \mu^2(e, x) = x$, $\mu^k(\ldots, e, \ldots) = 0$, $k \geq 3$.

× The Floer complex $CF(L_0, L_1)$ is an A_∞-module over $CM_*(G, g)$

→ Equivariant Lagrangian Floer homologies

(use Mike Miller's thesis appendix)

$A = CM_*(G, g)$, $\bar{A} = A/\langle e \rangle$

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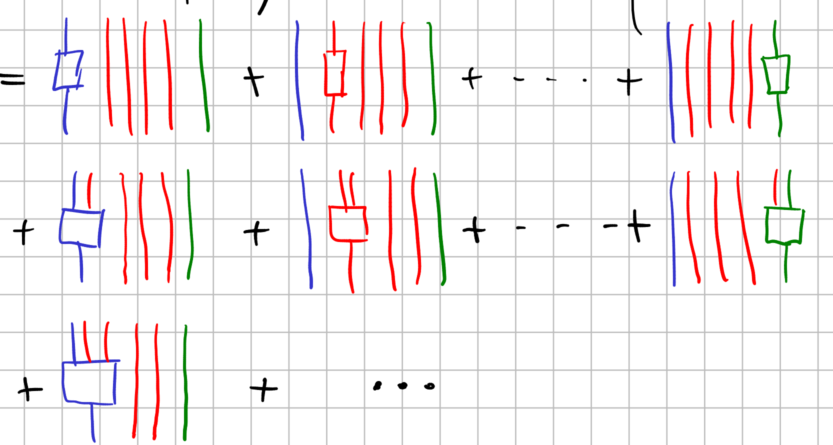
strictly unital A_∞-algebra

left/right A_∞-modules

Bar construction: $B(M, A, N) = M \otimes \left(\bigoplus \bar{A}^{\otimes k} \right) \otimes N$

with differential $\partial =$



μ^1 maps

μ^2 maps

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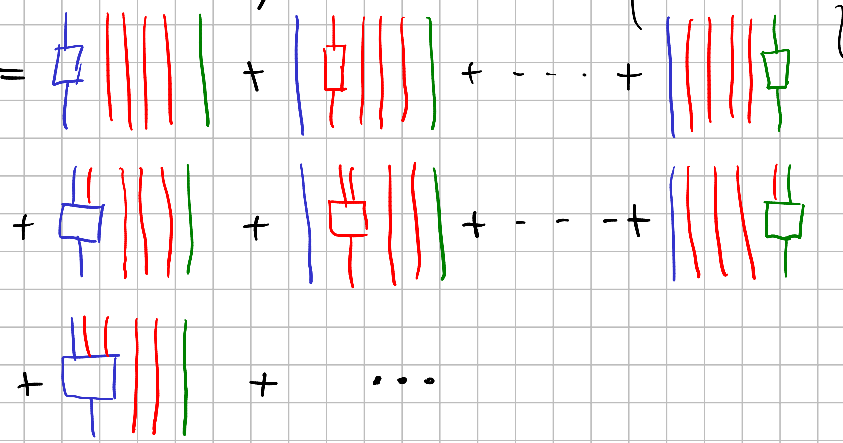

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co-Bar construction: $cB(N,A,M) = \mathrm{Hom}_A(B(N,A,N), M)$

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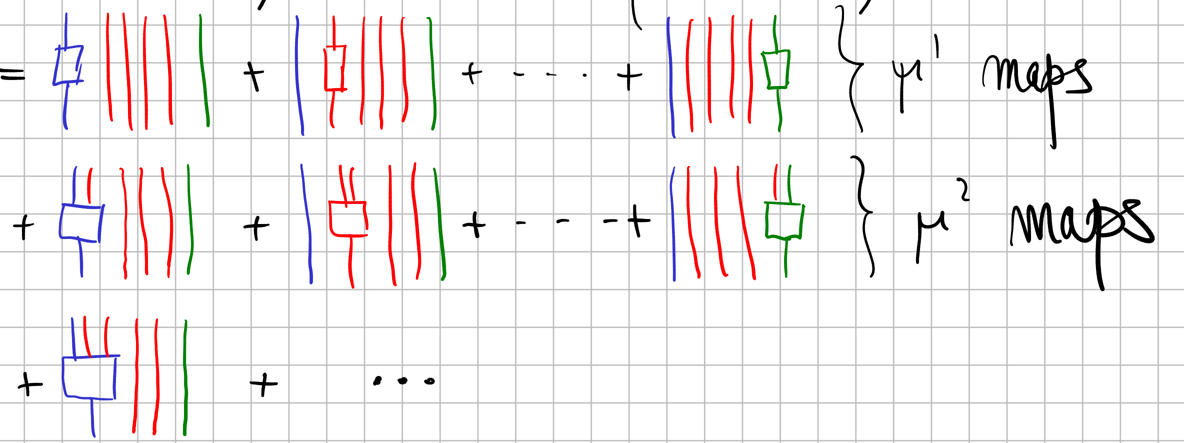

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Borel complex: $C_A^+(M) = B(M,A,N)$

co-Borel complex: $C_A^-(M) = cB(N,A,M)$

Tate complex: $C_A^\infty(M) = \mathrm{Cone}\left(N_M : B(M,A,D_A) \longrightarrow C_A^-(M) \right)$

$D_A = cB(R,A,A)$ "dualizing complex"

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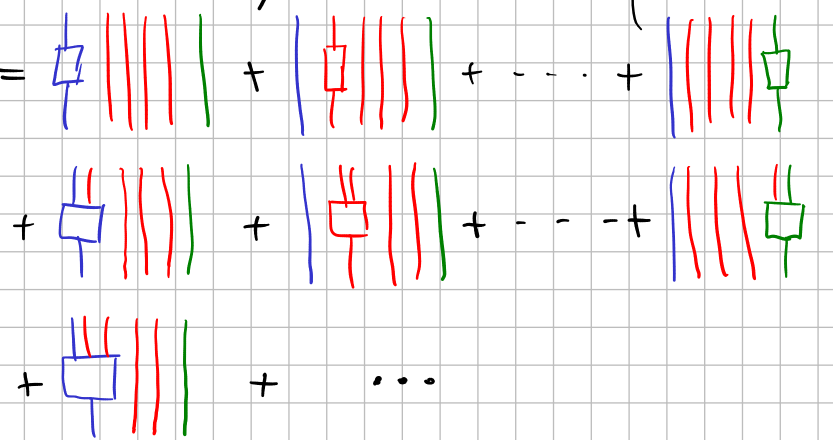

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⇝ Get homology groups $HF_G^+(L_0,L_1)$, $HF_G^-(L_0,L_1)$ and $HF_G^\infty(L_0,L_1)$.

Th: [C., Kirk, Kotelskiy, Miller, Yeung] Morse theoretic analog

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compact Lie group

smooth compact mfd

$g : G \to \mathbb{R}$
$f : X \to \mathbb{R}$
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compact Lie group

smooth compact mfd

1. $CM_*(G,g)$: A_∞-algebra
2. $CM_*(X,f)$: A_∞-module / $CM_*(G,g)$

such that get algebra & module structure on homology induced by:

$$m_G : G \times G \to G \quad \rightsquigarrow \quad (m_G)_* : H_*(G) \otimes H_*(G) \to H_*(G)$$

$$m_X : G \times X \to X \quad\quad (m_X)_* : H_*(G) \otimes H_*(X) \to H_*(X)$$

Pushforwards on Morse homology (→ Kronheimer-Mrowka's book, sec 2.8)

$$\Phi : \underset{f \searrow \mathbb{R}}{M} \longrightarrow \underset{g \searrow \mathbb{R}}{N} \quad \text{smooth map}$$

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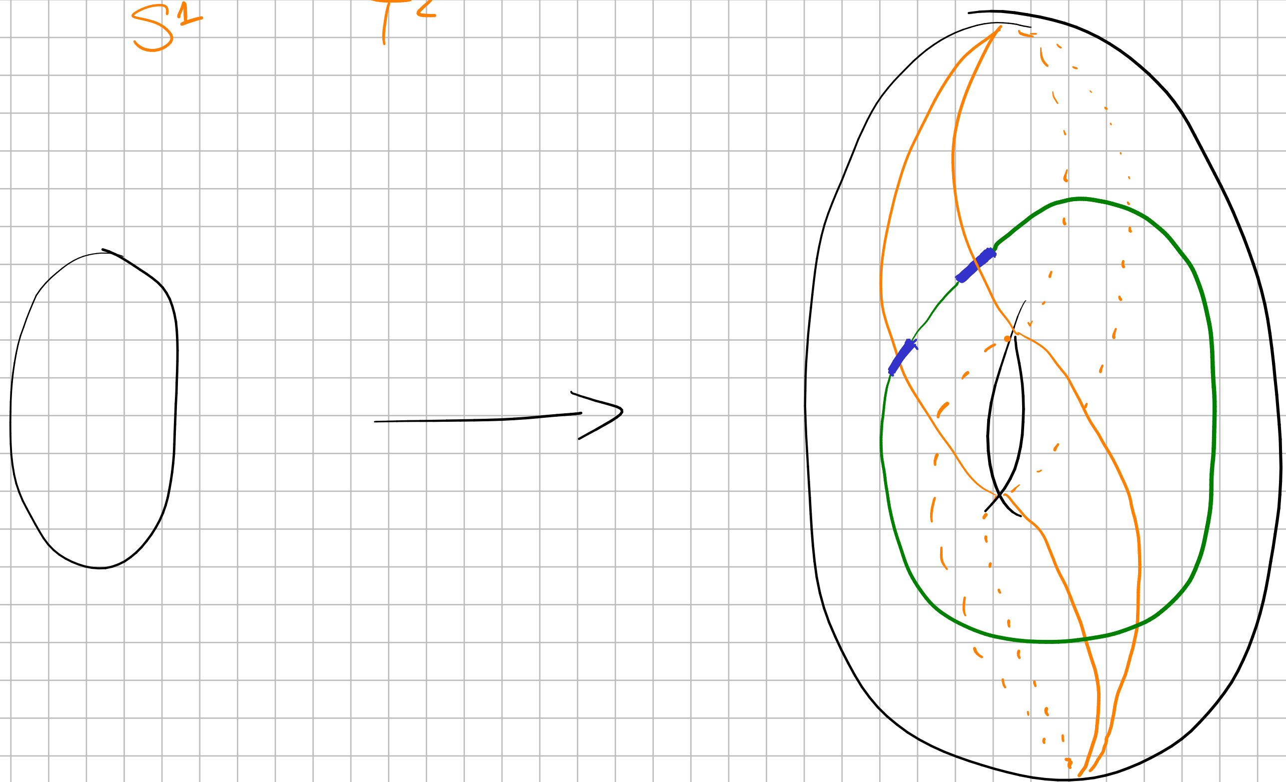
$f: M \to \mathbb{R}$, $g: N \to \mathbb{R}$



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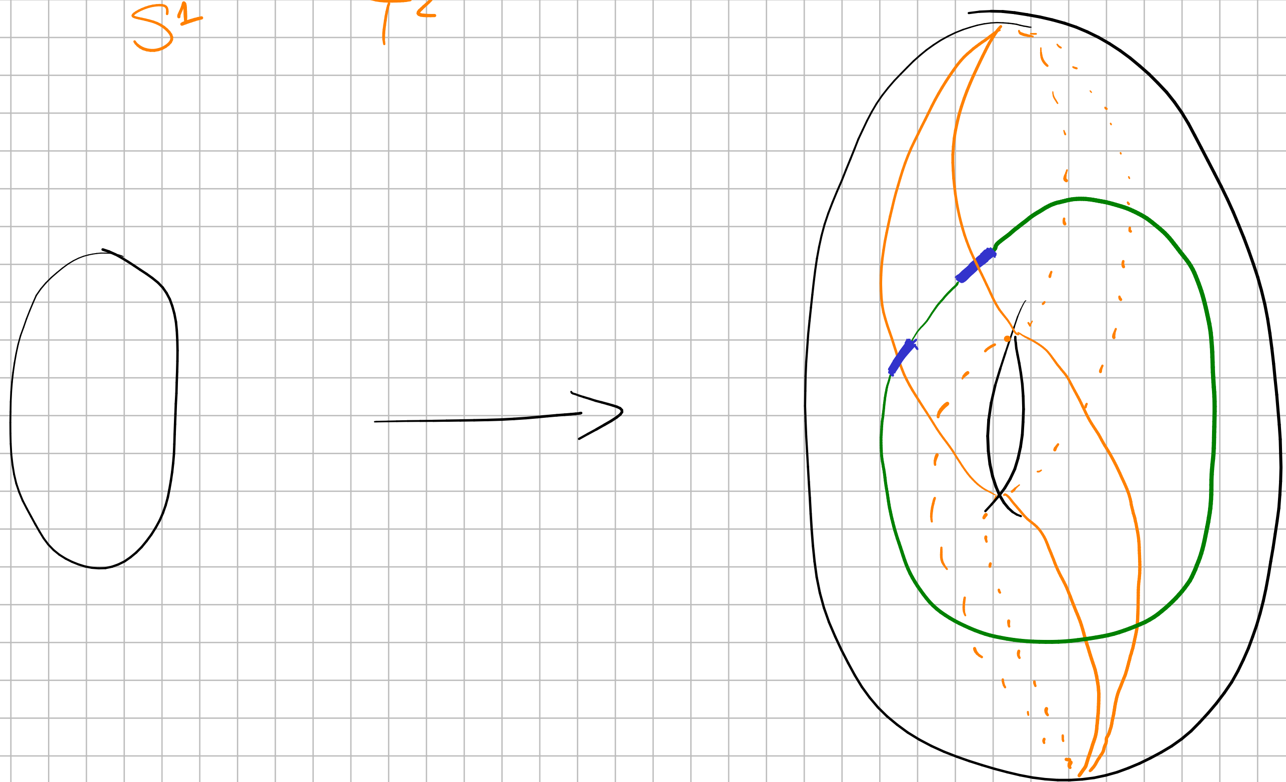
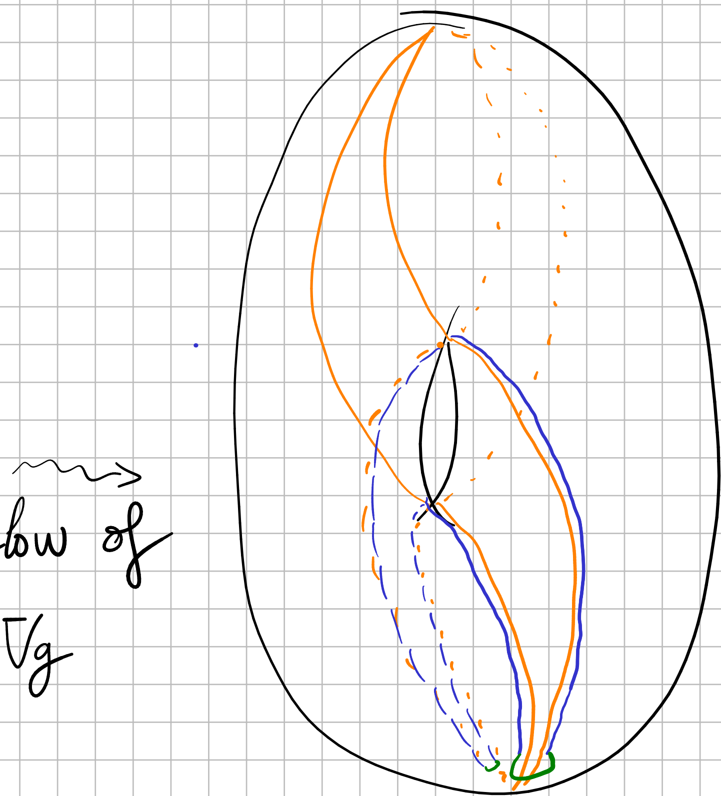


flow of $-\nabla g$



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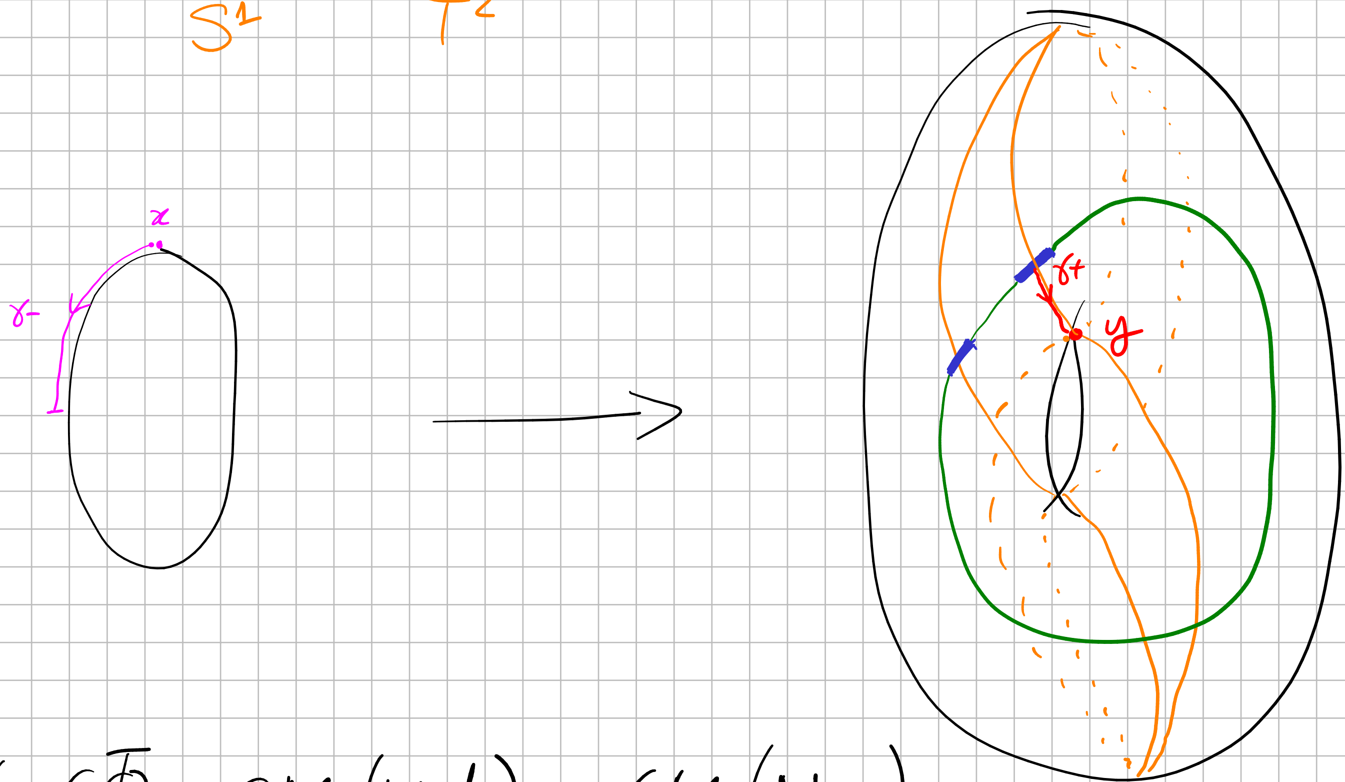

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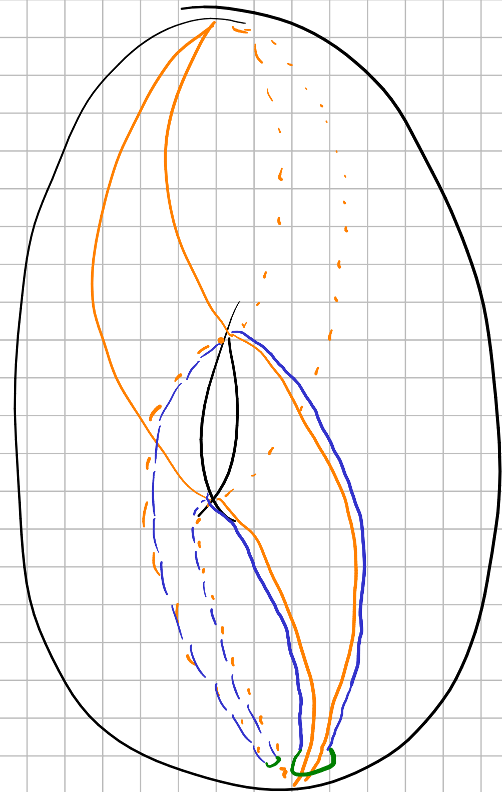

$\rightsquigarrow C\Phi_* : CM_*(M,f) \to CM_*(N,g)$

$$C\Phi_*(x) = \sum_y \# \left\{ x \longrightarrow \Phi \longrightarrow y \right\} \cdot y$$

$\gamma_- : \mathbb{R}_{\leq 0} \longrightarrow M$

$\gamma_+ : \mathbb{R}_{\geq 0} \longrightarrow N$

$\Phi(\gamma_-(0)) = \gamma_+(0)$

Pushforwards of multiplication maps

* $m_G : G \times G \longrightarrow G$

$$G \times G \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto g(x)+g(y)$$

$$G \xrightarrow{g} \mathbb{R}$$

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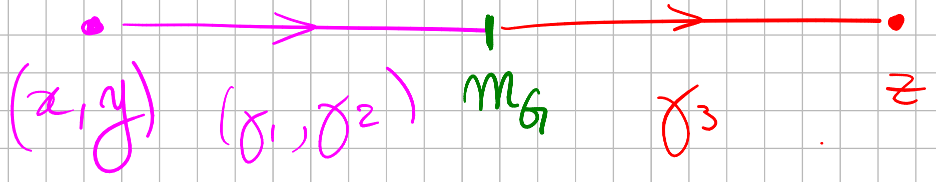
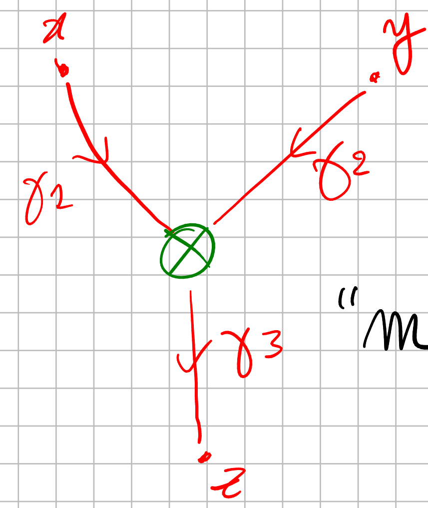


$\Longleftrightarrow$



"grafted line"

"multiplicative Y"

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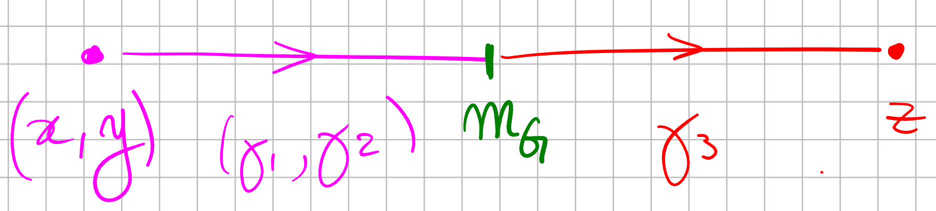

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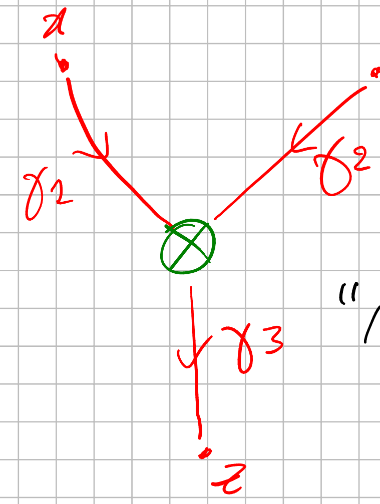

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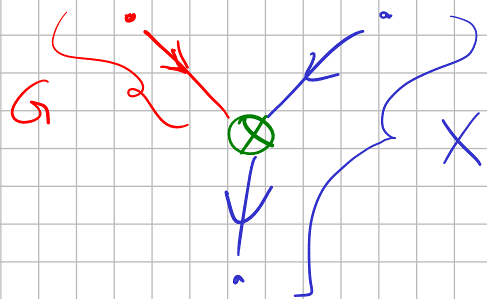

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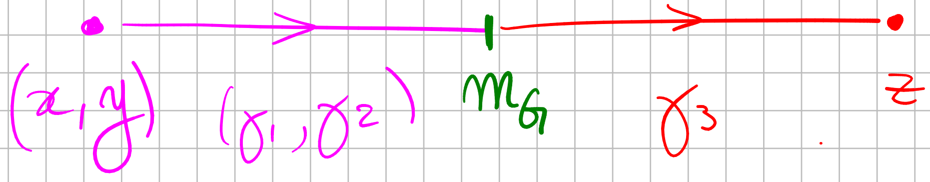

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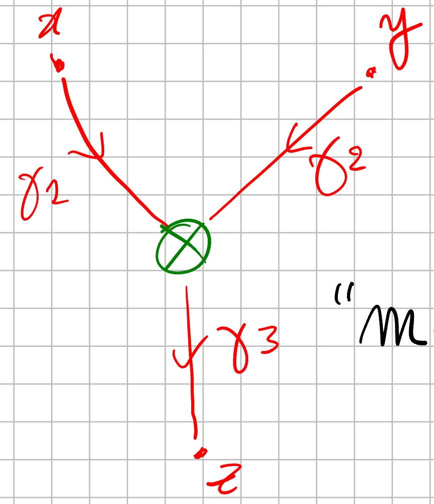

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$\to$ Get chain-level products:

$$\begin{cases} \mu_G : CM(G,g) \otimes CM(G,g) \longrightarrow CM(G,g) \\ \mu_X : CM(G,g) \otimes CM(X,f) \longrightarrow CM(X,f) \end{cases}$$

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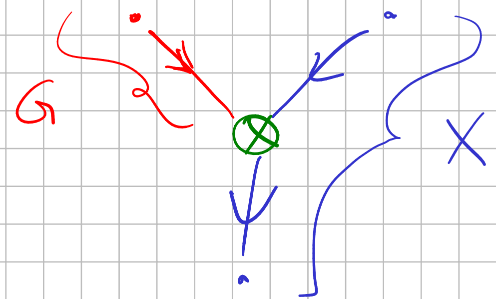

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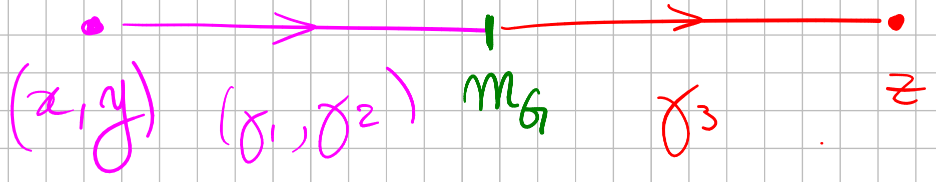

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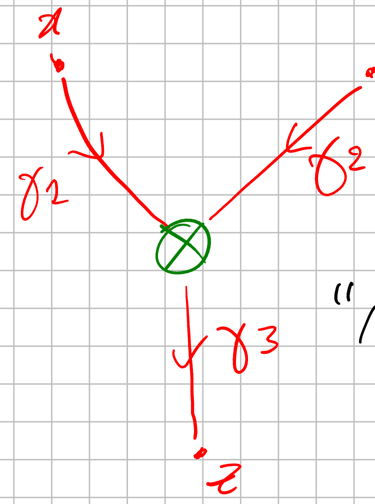

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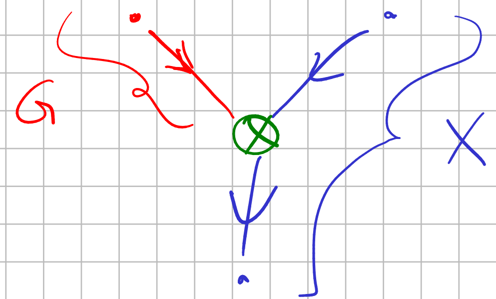

Q: Associative?

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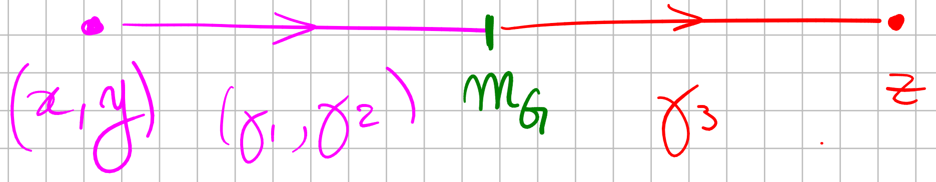

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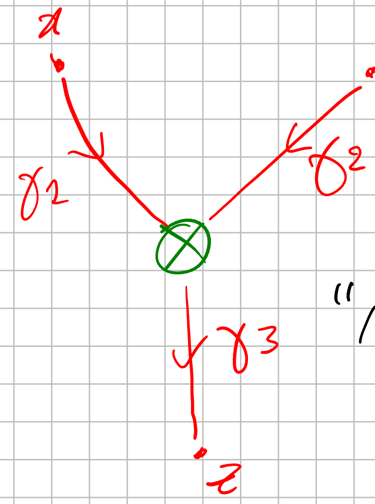

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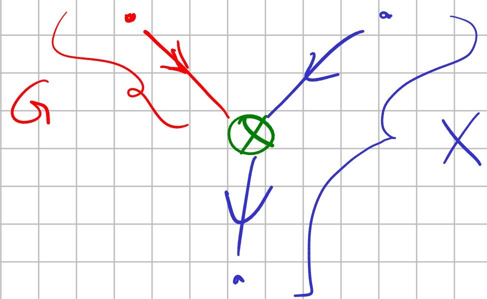

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Q: Associative?

A: Yes, up to homotopy . . .

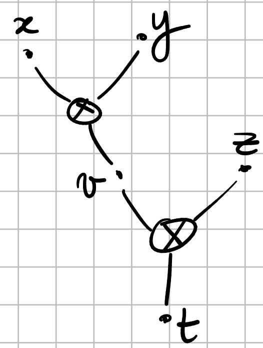
Associativity $x, y, z, t \in \mathrm{Cit}(G, g)$

$$\left\langle \mu_G(\mu_G(x,y), z) - \mu_G(x, \mu_G(y,z)), t \right\rangle$$

Associativity $x, y, z, t \in \mathrm{Crit}(G, g)$

$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

$$= \# \bigcup_{v} \Bigg\{$$



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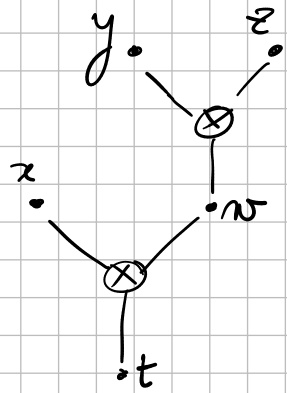

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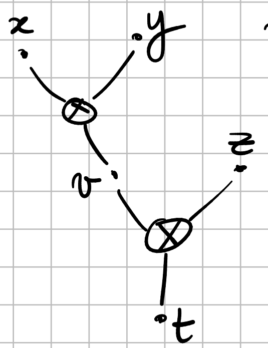

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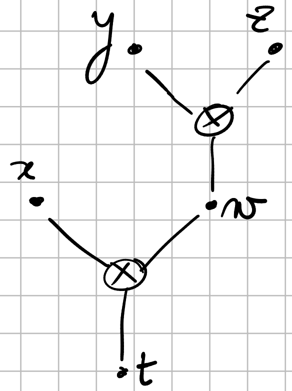
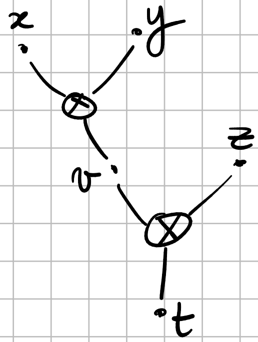
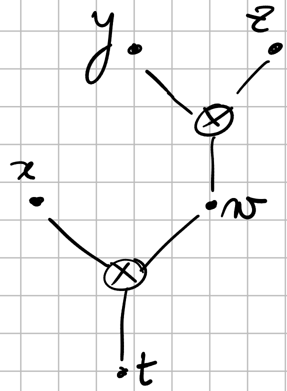


$$\Big\}$$

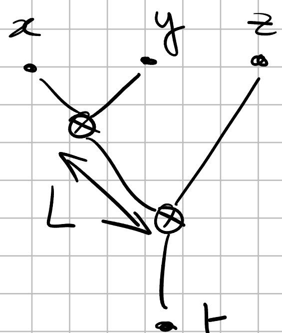
$\subset \partial \overline{A}$ $\qquad\qquad$ $\subset \partial \overline{B}$

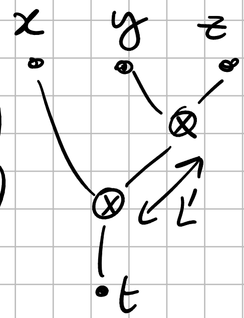
Associativity $x, y, z, t \in \mathrm{Crit}(G, g)$

$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$

$= \# \bigcup_v \{$  $\} - \# \bigcup_w \{$  $\}$

$\subset \partial\overline{\mathcal{A}}$ $\qquad$ $\subset \partial\overline{\mathcal{B}}$

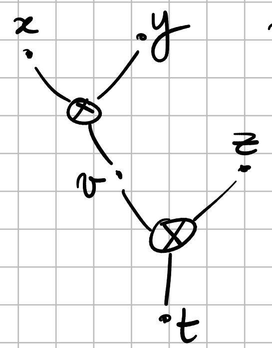
$\mathcal{A} = \bigcup_{L \geqslant 0} \mathcal{A}_L$, $\mathcal{A}_L = \{$  $\}$

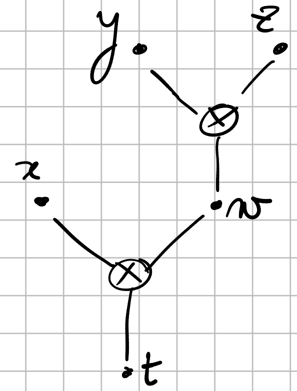
$\mathcal{B} = \bigcup_{L' \geqslant 0} \mathcal{B}_{L'}$, $\mathcal{B}_{L'} = \{$  $\}$

Associativity $\quad x,y,z,t \in \mathrm{Crit}(G,g)$

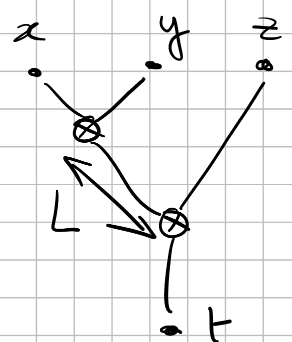
$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

$$= \# \underbrace{\bigcup_v \left\{ \quad \right\}}_{\subset \partial \overline{\mathcal{A}}} - \# \underbrace{\bigcup_w \left\{ \quad \right\}}_{\subset \partial \overline{\mathcal{B}}}$$

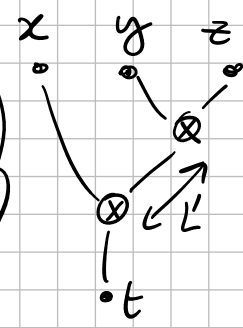




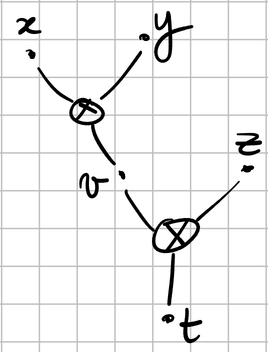
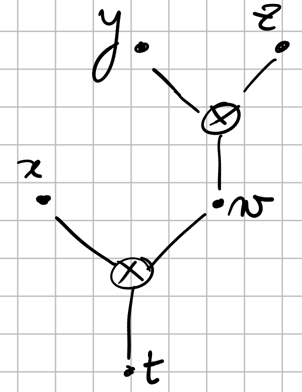
$\mathcal{A} = \bigcup_{L \geq 0} \mathcal{A}_L \quad , \quad \mathcal{A}_L = \left\{ \quad \right\}$



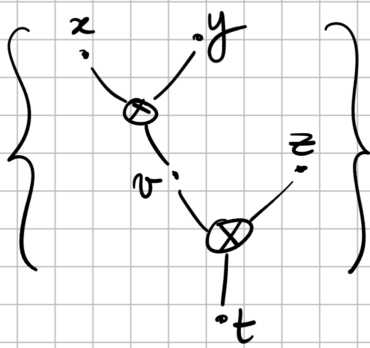
$\mathcal{B} = \bigcup_{L' \geq 0} \mathcal{B}_{L'} \quad , \quad \mathcal{B}_{L'} = \left\{ \quad \right\}$



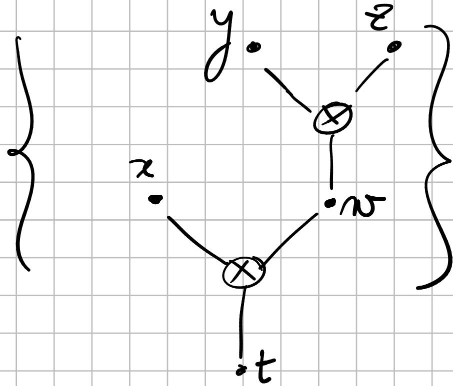
Let $\mathcal{M}(x,y,z;t) = \mathcal{A} \underset{\mathcal{A}_0 = \mathcal{B}_0}{\cup} \mathcal{B}$

$$\partial \overline{\mathcal{M}}(x, y, z; t) = \bigcup_{v} \left\{ \right.$$  $$\left. \right\} \cup \bigcup_{w} \left\{ \right.$$  $$\left. \right\}$$

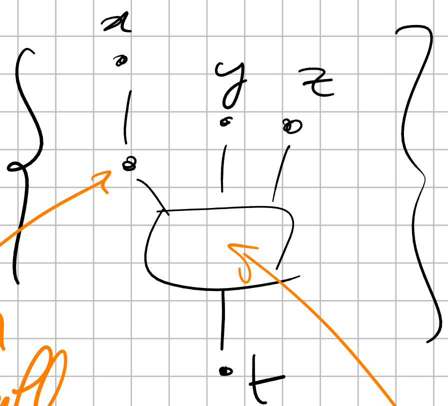
$$\partial \overline{\mathcal{M}}(x, y, z; t) = \bigcup_v$$



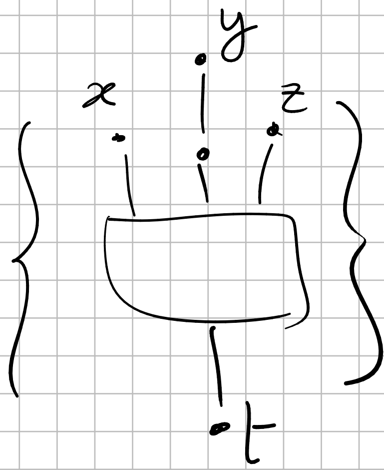
$$\cup \bigcup_w$$



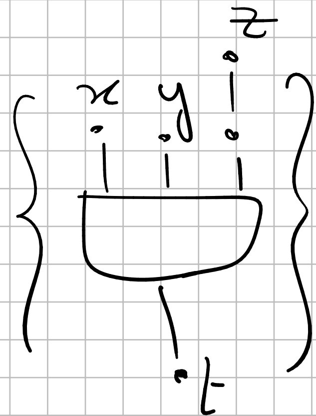
$\cup$



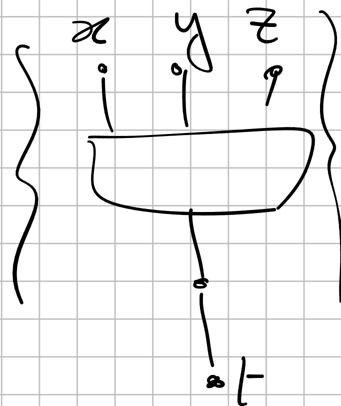
$\cup$



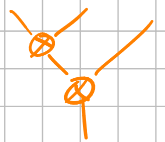
$\cup$



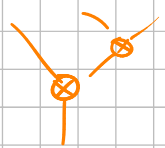
$\cup$

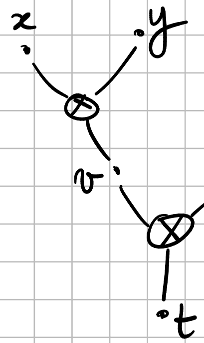


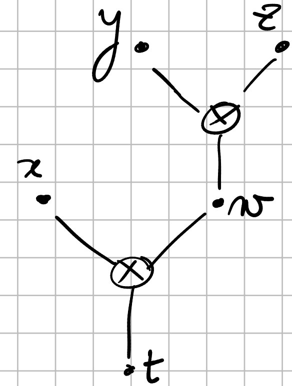
union over all possible breakings



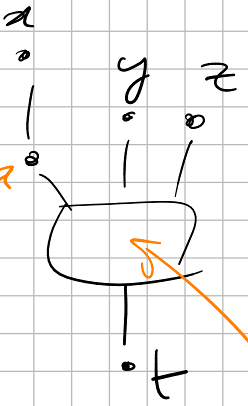
or

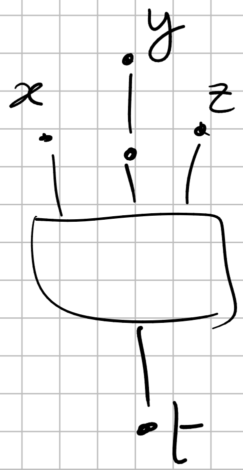


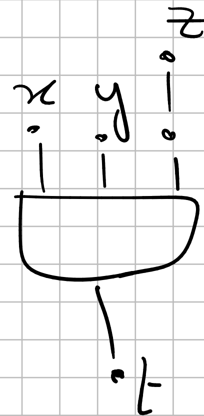
$\partial\overline{\mathcal{M}}(x,y,z;t) = \bigcup_v \Big\{$ 

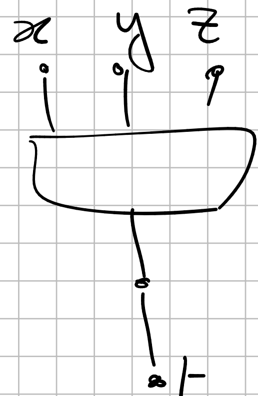
$\Big\} \cup \bigcup_w \Big\{$ 

$\Big\}$

$\cup \Big\{$ 

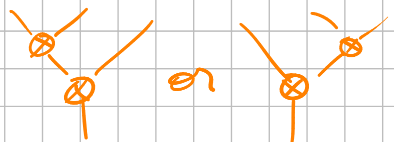
$\Big\} \cup \Big\{$ 

$\Big\} \cup \Big\{$ 

$\Big\} \cup \Big\{$ 

$\Big\}$

union over all possible breakings



$$\Longrightarrow \quad \mu_G(\mu_G(\cdot,\cdot),\cdot) - \mu_G(\cdot,\mu_G(\cdot,\cdot)) = \mu^3 \circ (\mu' \otimes id \otimes id + id \otimes \mu' \otimes id + id \otimes id \otimes \mu') + \mu' \circ \mu^3$$

(third A_∞-relation)

The A_∞-structure on $CM_*(G,\underline{g})$:

- μ^1: Morse differential.

The A_∞-structure on $CM_*(G,g)$:

- μ^1: Morse differential.
- Let $k \geq 2$, $T_k := \{(k+1)\text{-leafed metric rooted ribbon trees}\}$

finite edges have a length

$x_1, \dots, x_k, y \in \mathrm{Crit}(G,g)$

$\rightsquigarrow$ define $\mathcal{M}(x_1, \dots, x_k; y) = \{(\Gamma, \gamma) \mid \Gamma \in T_k,\ \gamma : \Gamma \to G,\ (*)\}$

$$\dim \mathcal{M}(x_1, \dots, x_k; y) = \underbrace{(k-2)}_{\dim T_k} + \underbrace{i(x_1) + \dots + i(x_k) - i(y)}_{\text{Morse indices}}$$

The A_∞-structure on $CM_*(G,g)$:

- μ^1 : Morse differential.
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$$\dim \mathcal{M}(x_1, \dots, x_k; y) = \underbrace{(k-2)}_{\dim T_k} + \underbrace{i(x_1) + \dots + i(x_k) - i(y)}_{\text{Morse indices}}$$

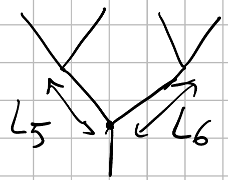
Define $\mu^k : CM_*(G,g)^{\otimes k} \to CM_*(G,g)$

by $\mu^k(x_1, \dots, x_k) = \sum_y{}' \# \mathcal{M}(x_1, \dots, x_k; y) \cdot y$

when $\dim = 0$,
$= 0$ otherwise

$$\mathcal{M}(x_1, \dots, x_k ; y) = \left\{ (\Gamma, \gamma) \,\middle|\, \Gamma \in \mathcal{T}_k \,,\ \gamma : \Gamma \to G ,\ \underline{\underline{(*)}} \right\}$$

$$\mathcal{M}(x_1, \dots, x_k ; y) = \{ (\Gamma, \gamma) \mid \Gamma \in T_k ,\ \gamma : \Gamma \to G ,\ (*) \}$$

ex: $\Gamma =$ 

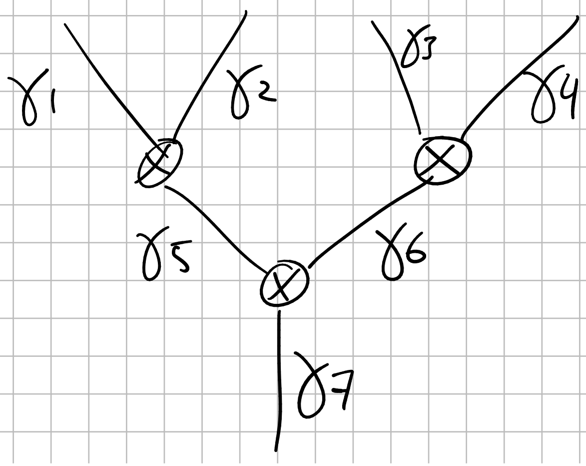
, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$ with:

$\gamma_1, \gamma_2, \gamma_3, \gamma_4 : \mathbb{R}_{\leq 0} \to G$

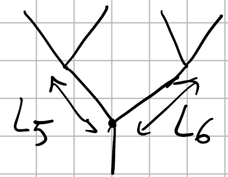
$\gamma_5 : [0, L_5] \to G$

$\gamma_6 : [0, L_6] \to G$

$\gamma_7 : \mathbb{R}_{\geq 0} \to G$



$$\mathcal{M}(x_1,\dots,x_k;y) = \{(\Gamma,\gamma) \mid \Gamma\in T_k,\ \gamma:\Gamma\to G,\ (*)\}$$

ex: $\Gamma =$ 

, $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7)$ with:

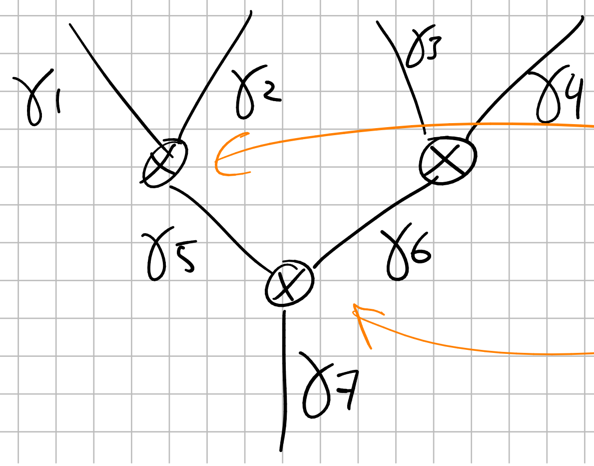
$\gamma_1,\gamma_2,\gamma_3,\gamma_4 : \mathbb{R}_{\leq 0} \to G$ $\quad$ $\gamma_5 : [0,L_5] \to G$ $\quad$ $\gamma_7 : \mathbb{R}_{\geq 0} \to G$

$\gamma_6 : [0,L_6] \to G$

(*)
- flow lines for a domain-dependent pseudo-gradient of g
- limits to $x_1,\dots x_k, y$ at the ends,
- satisfy multiplicative relations at vertices

ex: $\gamma_1(0)\gamma_2(0) = \gamma_5(0)$

$\gamma_5(L_5)\cdot\gamma_6(L_6) = \gamma_7(0)$



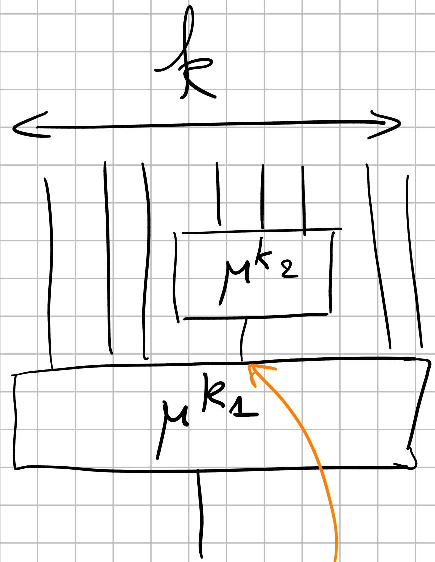
→ These $\mu^k : CM_*(G,g)^{\otimes k} \longrightarrow CM_*(G,g)$, defined by:

$$\mu^k(x_1,\dots,x_k) = \sum_y{}' \,\#\mathcal{M}(x_1,\dots,x_k;y)\cdot y$$

Satisfy the A_∞-relations:

* $\mu^1\circ\mu^1 = 0$ (μ^1 differential)
* $\mu^1\circ\mu^2 + \mu^2\circ(\mu^1\otimes id + id\otimes\mu^1) = 0$ (μ^2 chain map)
* ...

$$\forall k\geq 1,\ \sum_{\substack{k_1+k_2=k+1\\ 1\leq l\leq k_1}}{}'$$

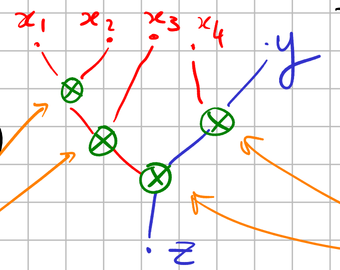


$$= 0$$

A_∞-module structure on $CM_*(X,f)$

Define $\mu_X^{k+1} : CM_*(G,g)^{\otimes k} \otimes CM_*(X,f) \to CM_*(X,f)$ analogously by:

$$\mu_X^{k+1}(x_1, \dots, x_k, y) = \sum_z{}' \; \# \Big\{$$



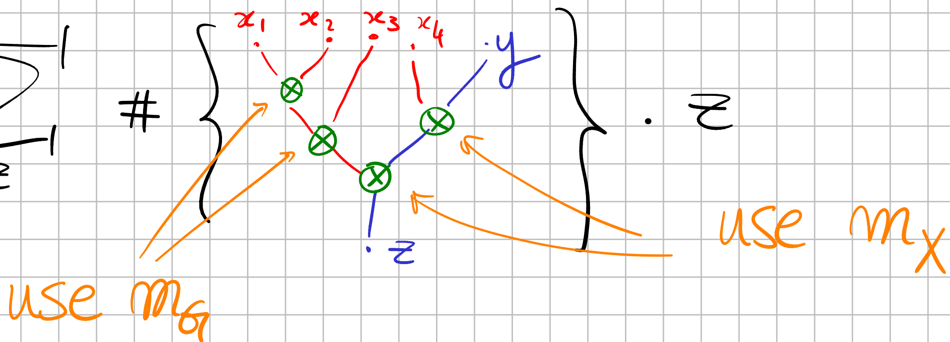
$$\Big\} \cdot z$$

use m_G

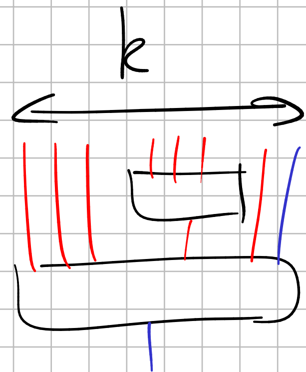
use m_X

A_∞ – module structure on $CM_*(X,f)$

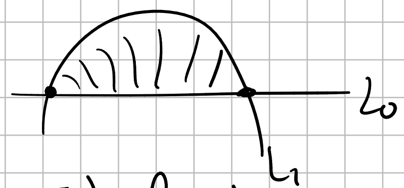
Define $\mu_X^{k+1} : CM_*(G,g)^{\otimes k} \otimes CM_*(X,f) \to CM_*(X,f)$ analogously by:

$$\mu_X^{k+1}(x_1, \dots, x_k, y) = \sum_z{}' \ \# \Big\{$$  $$\Big\} \cdot z$$

→ Satisfy the A_∞ - relations for A_∞-modules:

$$\forall k \geq 1, \ \sum{}'$$  $$= 0$$

From Morse to Floer

X	T^*X
$f : X \to \mathbb{R}$	$L_0 = O_X$ zero-section $L_1 = \Gamma(df)$
$\mathrm{Crit}\, f$	$L_0 \cap L_1$
$x \xrightarrow{\ \gamma\ } y$ flow line	 J-hol disc

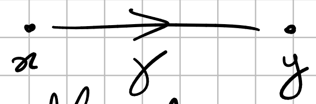
From Morse to Floer

X $\qquad$ T^*X

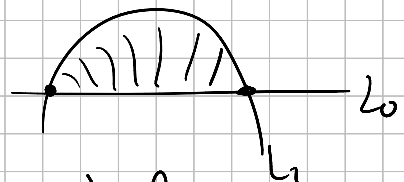
$f : X \to \mathbb{R}$ $\qquad$ $L_0 = 0_X$ zero-section

$L_1 = \Gamma(df)$

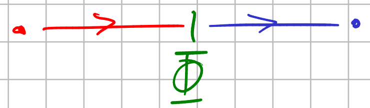
$\mathrm{Crit}\, f$ $\qquad$ $L_0 \cap L_1$



flow line

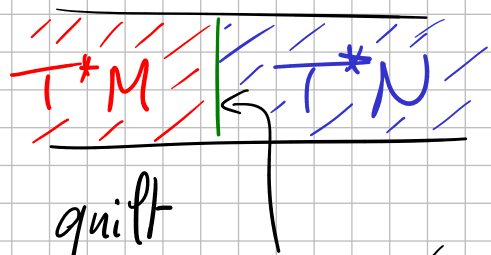


J-hol disc



grafted line

$\Phi : M \to N$



quilt

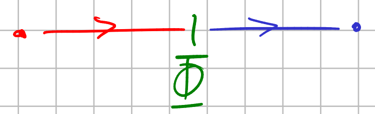
(Wehrheim - Woodward)

$N_{\Gamma(\Phi)} \subset (T^*M)^- \times (T^*N)$

conormal bundle

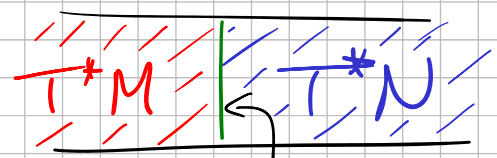
(Lagrangian correspondence)

From Morse to Floer II



grafted line

$\Phi : M \to N$



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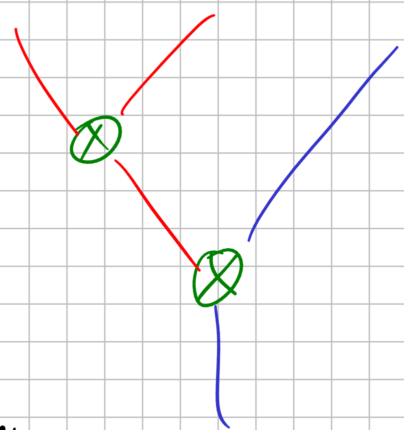
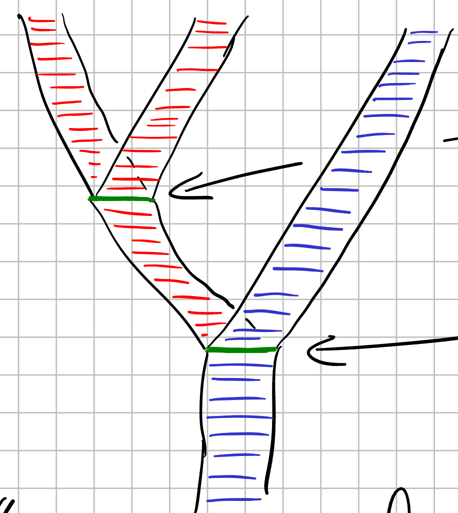


multiplicative tree



$N_{\Gamma(m_G)}$

$N_{\Gamma(m_X)}$

"pseudo-holomorphic foam"

$\underline{\text{Rk}}$: $N_{\Gamma(m_X)} = \Lambda_G(T^*X) \longleftarrow$ "Weinstein correspondence"

$\underline{\text{Def}}$: (Weinstein correspondence)

$G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian manifold

$$\Lambda_G(M) = \left\{ \left((g,p), m, m' \right) \,\middle|\, \begin{array}{l} m' = g\,m \\ R^*_{g^{-1}} p = \mu(m) \end{array} \right\} \subset T^*G \times M^- \times M$$

Lagrangian submanifold

<u>Rk</u>: $N_{\Gamma(m_X)} = \Lambda_G(T^*X)$ ⟵ "Weinstein correspondence"

<u>Def</u>: (Weinstein correspondence)

$G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian manifold

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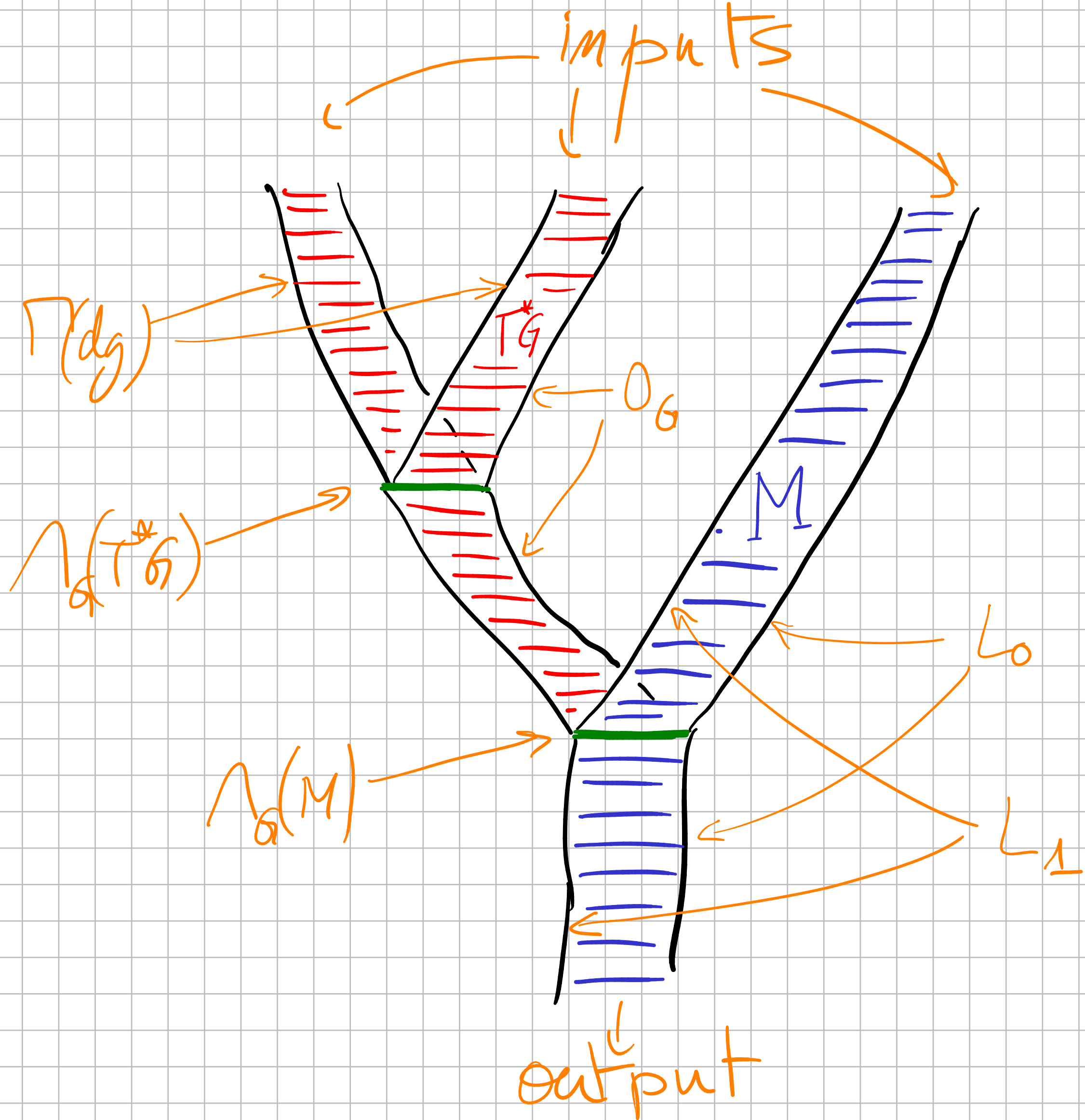
Lagrangian submanifold

$\Rightarrow$ Can transpose our construction to the Floer complex $CF(M; L_0, L_1)$.

Define $\mu^{k+1} : \underbrace{CF(T^*G; 0_G, \Gamma(dg))^{\otimes k}}_{\simeq CM_*(G,g)} \otimes CF(M; L_0, L_1) \longrightarrow CF(M; L_0, L_1)$

by counting foams:



The other A_∞ - structure on $CM_*(X,f)$ (Fukaya)

$\Delta : X \longrightarrow X \times X \rightsquigarrow \Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$ coproduct

$x \mapsto (x,x)$ (~ cup product)

The other A_∞-structure on $CM_*(X,f)$ (Fukaya)

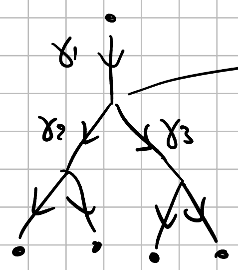
$$\Delta : X \to X \times X \quad \rightsquigarrow \quad \Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$$
$$x \mapsto (x,x)$$

coproduct ($\sim$ cup product)

$\hookrightarrow$ Chain-level version : A_∞-coalgebra structure

$$\delta_k : CM_*(X) \longrightarrow CM_*(X)^{\otimes k}$$

count



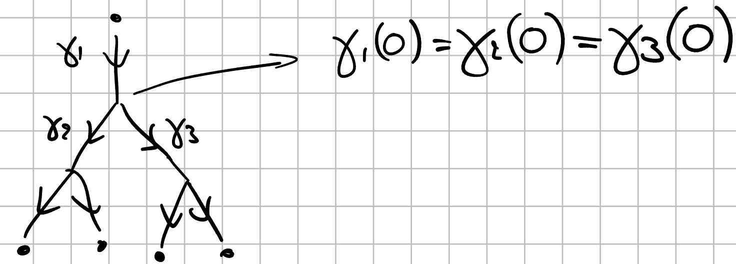
$\gamma_1(0) = \gamma_2(0) = \gamma_3(0)$

The other A_∞-structure on $CM_*(X,f)$ (Fukaya)

$\Delta : X \to X \times X$ $\quad\rightsquigarrow\quad$ $\Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$ coproduct
$\quad x \mapsto (x,x)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (~ cup product)

↳ Chain-level version : A_∞-coalgebra structure

$$\delta_k : CM_*(X) \longrightarrow CM_*(X)^{\otimes k}$$

count 

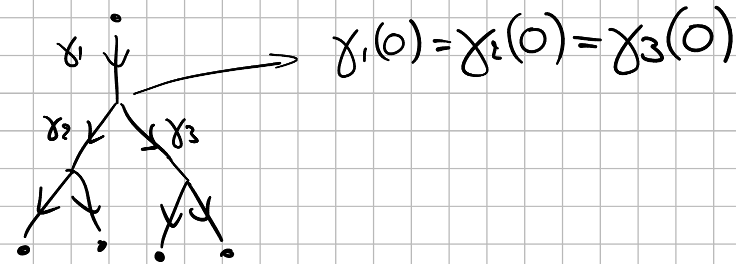
↳ symplectic version : $\mathcal{F}uk(M)$: A_∞- (co-)category ...

The other A_∞-structure on $CM_*(X,f)$ (Fukaya)

$\Delta : X \to X \times X$, $x \mapsto (x,x)$ $\rightsquigarrow$ $\Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$ coproduct ($\sim$ cup product)

↳ Chain-level version: A_∞-coalgebra structure

$$\delta_k : CM_*(X) \longrightarrow CM_*(X)^{\otimes k}$$

count 

↳ symplectic version: $\mathcal{F}uk(M)$: A_∞- (co-)category ...

<u>Rk:</u> $\mu^1 = \delta_1$ same differentials

→ How are these two A_∞-structures related?

Hopf algebras, Hopf modules

$$G \circlearrowleft X \quad \rightsquigarrow \quad H_* G \circlearrowleft H_* X$$

group   mfd   Hopf algebra   Hopf module

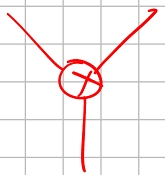
Hopf algebras, Hopf modules

$G \circlearrowright X$ $\rightsquigarrow$ $H_*G \circlearrowright H_*X$

group, mfd — Hopf algebra, Hopf module

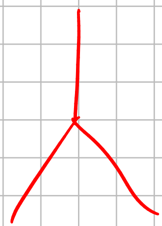
Def: Hopf algebra $(H, m, \Delta, \eta, \epsilon, S)$:

product



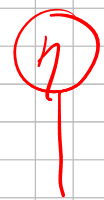
$H\otimes H \xrightarrow{m} H$

coproduct



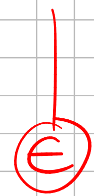
$H \xrightarrow{\Delta} H\otimes H$

unit



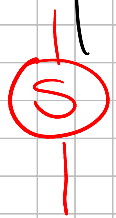
$\mathbb{F}_2 \xrightarrow{\eta} H$

counit



$H \xrightarrow{\epsilon} \mathbb{F}_2$

antipode



$H \xrightarrow{S} H$

(from $G \to G$, $g \mapsto g^{-1}$)

Hopf algebras, Hopf modules

$G \circlearrowright X$ $\rightsquigarrow$ $H_* G \circlearrowright H_* X$

group, mfd — Hopf algebra, Hopf module

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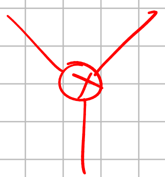
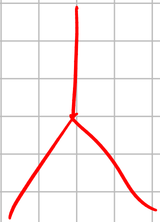


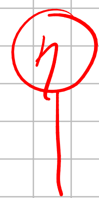
$$H \otimes H \xrightarrow{m} H$$

coproduct



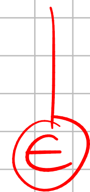
$$H \xrightarrow{\Delta} H \otimes H$$

unit



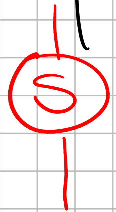
$$\mathbb{F}_2 \xrightarrow{\eta} H$$

counit



$$H \xrightarrow{\epsilon} \mathbb{F}_2$$

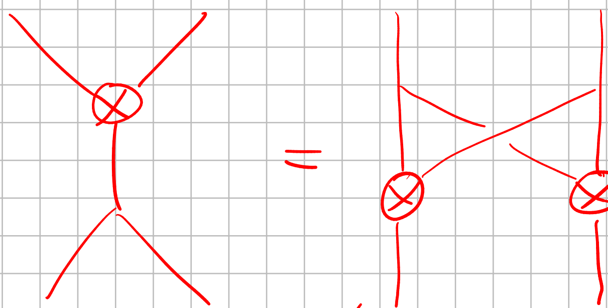
antipode



$$H \xrightarrow{S} H$$

(from $G \to G$, $g \mapsto g^{-1}$)

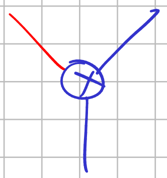
satisfy several relations, including:



$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

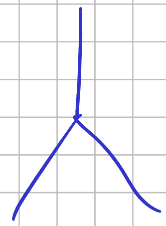
Def: Hopf module $H \circlearrowright (M, m_M, \Delta_M)$

product



$$H \otimes M \xrightarrow{m_M} M$$

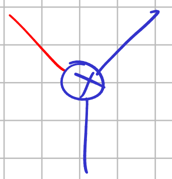
coproduct



$$M \xrightarrow{\Delta_M} M \otimes M$$

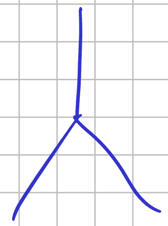
Def: Hopf module $H \circlearrowright (M, m_M, \Delta_M)$

product



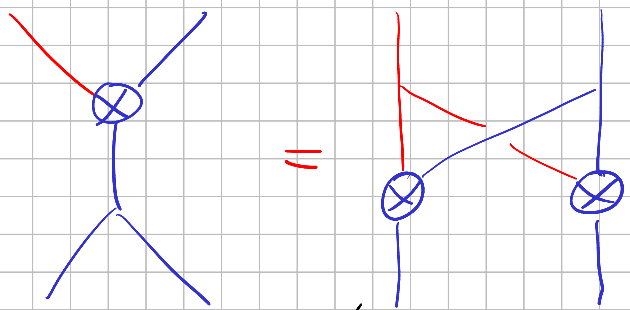
$$H \otimes M \xrightarrow{m_M} M$$

coproduct



$$M \xrightarrow{\Delta_M} M \otimes M$$

satisfy several relations, including:



$$\Delta_M \circ m_M = (m_M \otimes m_M) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta_M)$$

Cornered instanton theory

- For Heegaard-Floer: Douglas-Lipshitz-Manolescu.

Cornered instanton theory

- For Heegaard-Floer: Douglas-Lipshitz-Manolescu.

Geometry (~Moore-Tachikawa category)

Algebra

$$\begin{tikzcd} \mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left, "\text{Cornered instanton theory}"] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \end{tikzcd}$$

Topology

Extended moduli spaces (Huebschmann-Jeffrey, Manolescu-Woodward, C.)

Cornered instanton theory

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Geometry (~ Moore-Tachikawa category)

Algebra

Cornered instanton theory

$$
\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left, "\text{Cornered instanton theory}"] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
& \mathrm{Lie}_{\mathbb{R}} \arrow[u, "T^*"] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}
$$

Homology

Topology

Extended moduli spaces
(Huebschmann-Jeffrey,
Manolescu-Woodward, C.)

Toy model

Cotangent bundles

$$\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
& \mathrm{Lie}_{\mathbb{R}} \arrow[u, hook, "T^*"] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] \arrow[from=U, to=D, Rightarrow, "C"] & H
\end{tikzcd}
\quad \overset{T^*}{\longmapsto} \quad
\begin{tikzcd}
G \arrow[r, bend left=50, "T^*X"{name=U}] \arrow[r, bend right=50, "T^*Y"'{name=D}] \arrow[from=U, to=D, Rightarrow, "N^*_C"] & H
\end{tikzcd}$$

cotangent bundles

conormal bundle $\subset (T^*X)^- \times T^*Y$

- G, H : Lie groups
- $(G\times H) \circlearrowright X, Y$ smooth manifolds
- $C \subset X \times Y$ $(G\times H)$-invariant correspondence

- $(G\times H) \circlearrowright T^*X, T^*Y$ Hamiltonian manifolds
- $N^*_C \subset (T^*X)^- \times T^*Y$ $(G\times H)$-Lagrangian correspondence

composition of 1-morphisms:

$$\begin{tikzcd}
G \arrow[r, "X"] \arrow[rr, bend right, "(X\times Y)/H"'] & H \arrow[r, "Y"] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "M"] \arrow[rr, bend right, "(M\times N)/\!\!/H"'] & H \arrow[r, "N"] & K
\end{tikzcd}$$

symplectic quotient

$$
\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
 & \mathrm{Lie}_{\mathbb{R}} \arrow[u, hook, "T^*"'] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}
$$

$$
\begin{tikzcd}
G \arrow[r, bend left=50, "X"] \arrow[r, bend right=50, "Y"'] \arrow[r, phantom, "\Downarrow C"] & H
\end{tikzcd}
\longmapsto
\begin{tikzcd}
H_*(G) \arrow[r, bend left=50, "H_*(X)"] \arrow[r, bend right=50, "H_*(Y)"'] \arrow[r, phantom, "\Downarrow H_*(C)"] & H_*(H)
\end{tikzcd}
$$

- G, H : Lie groups
- $(G\times H) \circlearrowright X, Y$ smooth manifolds
- $C \subset X \times Y$ $(G\times H)$-invariant correspondence

- $H_*(G), H_*(H)$ Hopf algebras
- $(H_*(G), H_*(H)) \circlearrowright H_*(X), H_*(Y)$ Hopf bimodules
- $H_*C : H_*(X) \to H_*(Y)$ morphism of Hopf bimodules.

$$\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
& \mathcal{L}ie_{\mathbb{R}} \arrow[u, hook, "T^*"] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}$$

Ma'u-Wehrheim-Woodward

A_∞-functor

Fukaya categories

$$\begin{tikzcd}
G \arrow[r, bend left=50, "M"{name=U}] \arrow[r, bend right=50, "N"'{name=D}] & H \arrow[Rightarrow, from=U, to=D, "L"]
\end{tikzcd}
\longmapsto
\begin{tikzcd}
CM_*(G) \arrow[r, bend left=50, "\mathrm{Fuk}(M)"{name=U}] \arrow[r, bend right=50, "\mathrm{Fuk}(N)"'{name=D}] & CM_*(H) \arrow[Rightarrow, from=U, to=D, "\Phi_L"]
\end{tikzcd}$$

- G, H : Lie groups
- $(G\times H) \circlearrowright M, N$: Hamiltonian manifolds
- $L \subset M^- \times N$ $(G\times H)$-Lagrangian correspondence

- $CM_*(G), CM_*(H)$: "Hopf_∞ algebras"
- $(CM_*(G), CM_*(H)) \circlearrowright \mathrm{Fuk}(M), \mathrm{Fuk}(N)$ "Hopf_∞ bimodules"
- $\Phi_L : \mathrm{Fuk}(M) \to \mathrm{Fuk}(N)$ "morphism of Hopf_∞ bimodules".

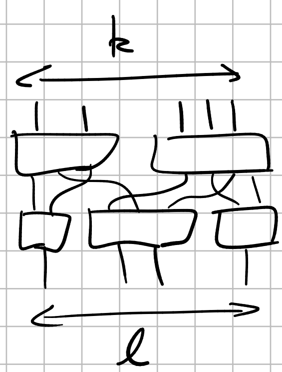
$\underline{\text{Hopf}_\infty\text{-algebras}}$ (Saneblidze – Umble)

$\approx$ A_∞-bialgebras + incorporate units, counits, antipodes

Hopf_∞-algebras (Saneblidze – Umble)

$\approx$ A_∞-bialgebras + incorporate units, counits, antipodes

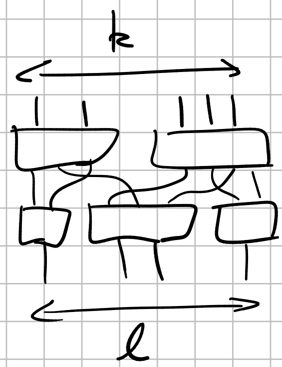
Def [Saneblidze-Umble] An A_∞-bialgebra $\left(H, \{P^k_\ell\}_{k,\ell\geq 1}\right)$ is a family of maps $P^k_\ell : H^{\otimes k} \to H^{\otimes \ell}$ satisfying a family of relations (R^k_ℓ) determined by the combinatorics of the "biassociahedra" $\{KK_{k,\ell}\}_{k,\ell \geq 1}$.

(R^k_ℓ) : " $\sum$  $= 0$ "

Hopf_∞ - algebras (Saneblidze - Umble)

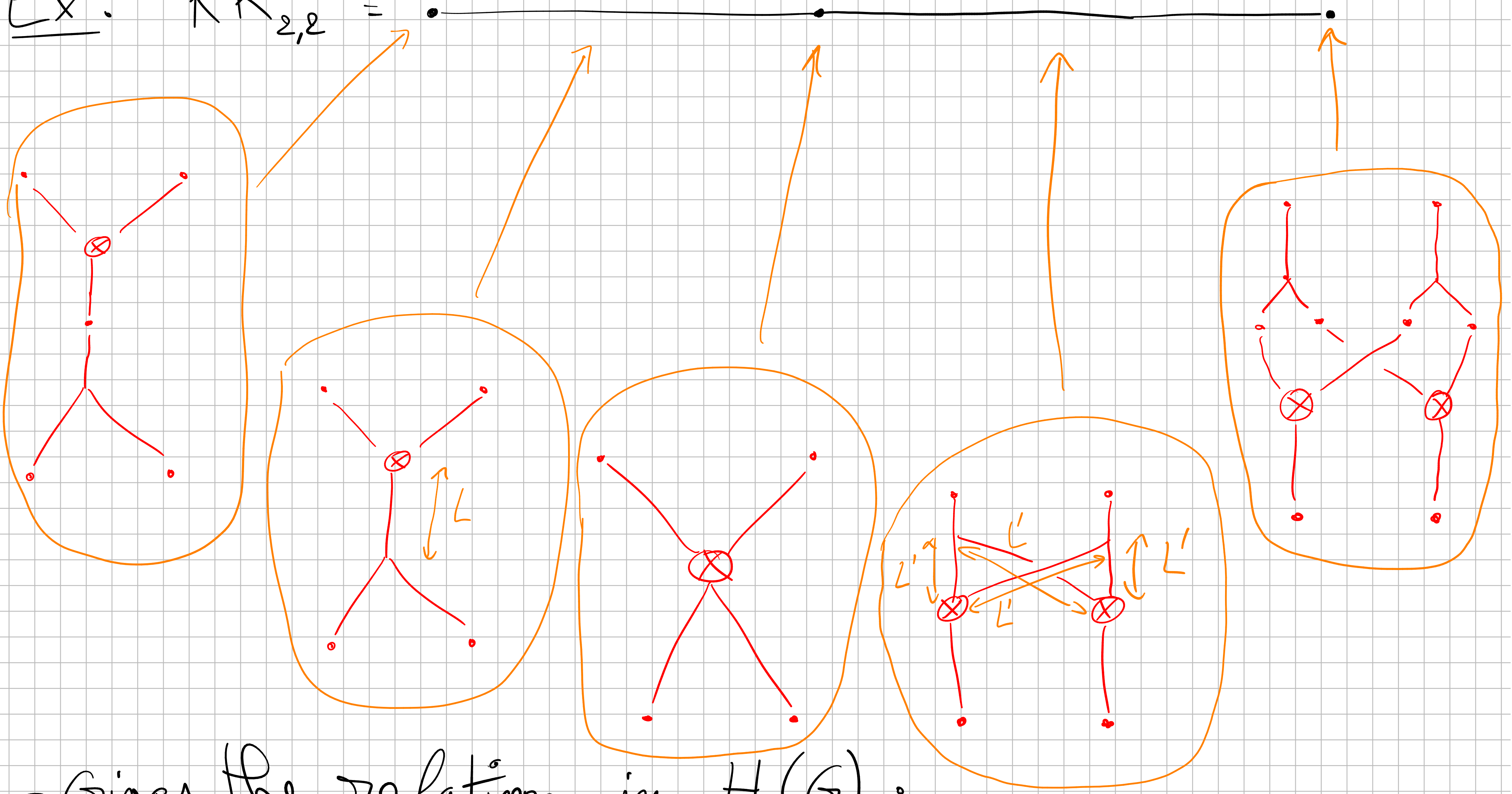
$\approx$ A_∞ - bialgebras + incorporate units, counits, antipodes

Def [Saneblidze-Umble] An A_∞-bialgebra $\left(H, \{P^k_\ell\}_{k,\ell \geq 1}\right)$ is a family of maps $P^k_\ell : H^{\otimes k} \to H^{\otimes \ell}$ satisfying a family of relations (R^k_ℓ) determined by the combinatorics of the "biassociahedra" $\{KK_{k,\ell}\}_{k,\ell \geq 1}$.

(R^k_ℓ) : " $\sum'$  $= 0$ "

Rk: Contains
- A_∞-algebra $(H, \{P^k_1\}_k)$
- A_∞-coalgebra $(H, \{P^1_\ell\}_\ell)$

Ex: $KK_{2,2} =$



→ Gives the relation in $H_*(G)$:

$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

Foam version:

$KK_{2,2} =$