# Asymptotic analysis of a secondary bifurcation of the one-dimensional Ginzburg-Landau equations of superconductivity

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**Abstract**

The bifurcation of asymmetric superconducting solutions from the normal solution is considered for the one-dimensional Ginzburg-Landau equations by the methods of formal asymptotics. The behaviour of the bifurcating branch depends of the parameters $d$, the size of the superconducting slab, and $\kappa$, the Ginzburg-Landau parameter. The secondary bifurcation in which the asymmetric solution branches reconnect with the symmetric solution branch is studied for values of $(\kappa, d)$ for which it is close to the primary bifurcation from the normal state. These values of $(\kappa, d)$ form a curve in the $\kappa d$-plane, which is determined. At one point on this curve, called the quintuple point, the primary bifurcations switch from being subcritical to supercritical, requiring a separate analysis. The results answer some of the conjectures of [3].

## 1 Introduction

When a superconducting slab of constant thickness, between the planes $x = -d$ and $x = d$ is submitted to an exterior magnetic field $(0, 0, H)$, the state of the slab can be described by the real functions $(\Psi(x), A(x))$ satisfying the Ginzburg-Landau system:

$$\begin{cases} \frac{1}{\kappa^2}\Psi'' = \Psi(\Psi^2 + A^2 - 1) \quad \text{in} \quad (-d, d), \\ \Psi'(\pm d) = 0, \\ A'' = \Psi^2 A \quad \text{in} \quad (-d, d), \\ A'(\pm d) = H. \end{cases} \tag{$GL_d$}$$

Here, $\Psi$ is the superconducting order parameter, which can be thought of as an averaged wave function of the superconducting electrons, and $(0, A, 0)$ is the magnetic vector potential, so that $(0, 0, A')$ is the magnetic field. This model was first introduced by Ginzburg and Landau [16]. For a more detailed description of the model, one may refer to [3], [12] or [14] and the references therein.

Notice that $\Psi \equiv 0$ and $A = H(x + e)$ is always a solution for any real $e$. From now on, we will call this a *normal solution*. It is well known that when $H$ is too large, superconductivity is destroyed and the only solution of $(GL_d)$ is the normal solution.

Let us first recall some basic properties of solutions of $(GL_d)$.

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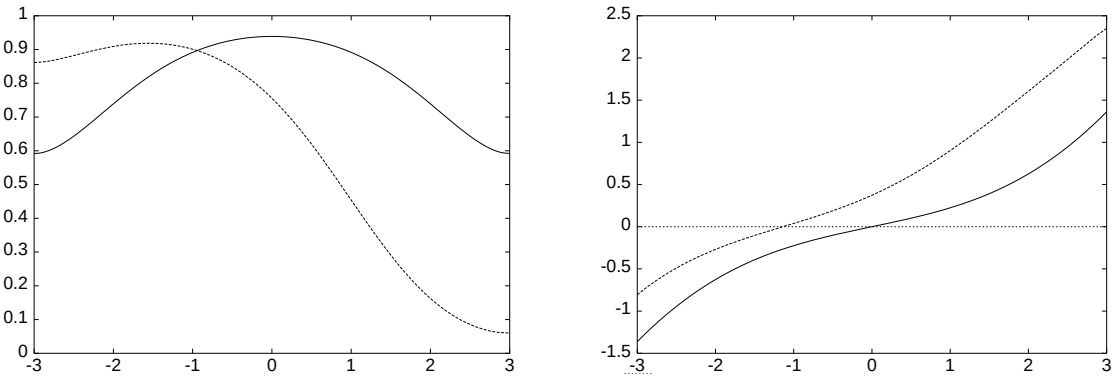


(a) (b)

Figure 1: Symmetric (solid) and asymmetric (dashed) solutions of $(GL_d)$ showing (a) $\Psi$ and (b) $A$, for $d = 3$, $\kappa = 0.9$.

**Proposition 1.1** *If $(\Psi, A)$ is a solution of $(GL_d)$ and if $\Psi$ is positive then*

*(i)* $|\Psi| \leq 1$ *in* $(-d, d)$,

*(ii)* *$A$ has a unique zero $d_0$ in $(-d, d)$, $A$ is increasing on $(-d, d)$, $A'$ is decreasing on $(-d, d_0)$ and increasing on $(d_0, d)$.*

*(iii)* *There exist $x_1$ and $x_2$ with $-d \leq x_1 \leq d_0 \leq x_2 \leq d$ and $x_0 \in [x_1, x_2]$ such that $\Psi'$ is increasing on $[-d, x_1] \cup [x_2, d]$ and decreasing on $[x_1, x_2]$, $\Psi$ is increasing on $[-d, x_0]$ and decreasing on $[x_0, d]$.*

The proof of $(i)$ and $(ii)$ can be found for instance in [7] and of $(iii)$ in [1].

There are two types of physically important solutions of $(GL_d)$: symmetric solutions and asymmetric solutions. We define a *symmetric solution* to be a solution of $(GL_d)$ such that $\Psi > 0$, $\Psi$ is even and $A$ is odd on $[-d, d]$. We define an *asymmetric solution* to be a solution of $(GL_d)$ which satisfies $\Psi > 0$ on $[-d, d]$, yet which is not symmetric, that is, $\Psi'(0) \neq 0$ or $A(0) \neq 0$. Typical symmetric and asymmetric solutions for $\Psi$ and $A$ are shown in Figures 1(a) and 1(b) respectively.

In sufficiently large slabs it has been discovered experimentally (see [21] or [23]) that as $H$ is decreased superconductivity nucleates first in surface layers of thickness $1/\kappa$. These surface layers can be explained using the asymmetric solution $(GL_d)$; in a decreasing field, the first solution to bifurcate from the normal state is the asymmetric solution, which is concentrated near the boundary [20].

It is interesting to study the bifurcation diagrams of solutions of $(GL_d)$, which show the norm of $\Psi$ as a function of $H$ (see for instance [15], [18] or [22]). In particular, the number of intersections of the bifurcation curve with the line $H = H_0$ provides the number of solutions of $(GL_d)$ for each $H_0$ (note in this respect that the asymmetric solutions come in pairs: if $\Psi(x)$ is a solution of $(GL_d)$ then so is $\Psi(-x)$). These bifurcation diagrams for both symmetric and asymmetric solutions have been computed in [3] using AUTO, for various values of the parameters $d$ and $\kappa$. For symmetric solutions, we refer the reader to [3] and the asymptotic analysis in [2]. For the branch of asymmetric solutions,

it is found in [3] that there are two possible behaviours according to the type of bifurcation, as illustrated by Figures 2 and 3. In both figures, the branch of symmetric solutions is that bifurcating from the normal solution at the smaller value of $H$, and the bifurcation of the asymmetric solutions from the symmetric solution is illustrated by a square (we call this the secondary bifurcation, the bifurcation from the normal solution being the primary bifurcation). In the case of Figure 2 the bifurcation of asymmetric solutions is subcritical[1] and there is at most one pair of asymmetric solutions for each given $H$, while in Figure 3 the bifurcation is supercritical and there are at most two pairs of asymmetric solutions.

In [3], the regimes of the parameters $d$ and $\kappa$ where each behaviour of the bifurcation diagram for asymmetric solutions holds have been classified. The results of these numerical investigations are shown graphically in Figure 4. They indicate that the $(d, \kappa)$ plane is the union of three connected sets $A_0$, $A_1$ and $A_2$. In $A_0$, there are no asymmetric solutions, in $A_1$, the behaviour of the asymmetric branch of Figure 2 holds and in $A_2$, the behaviour of the asymmetric branch of Figure 3.

In [3], the authors make the following conjecture.

**Conjecture 1.2** *([3]) There exist two continuous functions $\kappa_4(d)$ and $\kappa_5(d)$ separating the $(d, \kappa)$ plane into three connected regions $A_0$, $A_1$ and $A_2$. There exists exactly one point $d^*$ called the quintuple point (with approximate value 1.23) such that $\kappa_4(d^*) = \kappa_5(d^*) = \kappa^*$. Moreover, $\kappa_4(d) = C/d$ with $C$ approximately equal to 0.90, and $\kappa_5(d)$ is defined and monotone decreasing on $[d^*, \infty)$ with $\lim_{d\to\infty} \kappa_5(d) = \kappa_{as} \simeq 0.4$.*

The curve $\kappa_4(d)$ is the curve across which the asymmetric bifurcation disappears, while $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical. The point $(d^*, \kappa^*)$ is known as the quintuple point since, as shown numerically in [3], a third curve $\kappa_1(d)$ also passes through this point, dividing the plane into five regions locally (see figure 5). This curve is the curve across which the bifurcation of the symmetric solution switches from being subcritical to supercritical, and it has been analysed asymptotically in [2]. Indeed, once we know that the curves $\kappa_4$ and $\kappa_5$ meet, it is clear that $\kappa_1$ must also pass through this point.

The secondary bifurcation (shown as a square in Figures 2 and 3) in which the two asymmetric solutions and the symmetric solution reconnect to become a single symmetric solution will be our primary interest in this paper. We will be concerned with the situation in which the secondary bifurcation is close to the primary bifurcations, so that the amplitude of the solutions is still small when it takes place. Thus we will be concerned with the bifurcation diagram in the vicinity of the curve $\kappa_4(d)$.

The bifurcation of the asymmetric solution from the normal solution has been widely studied. Saint-James and De Gennes [20] examined the bifurcation in a half-space, and discovered that the bifurcation of asymmetric solutions occurs at a higher field than that of symmetric solutions in an infinite slab. They also established that the bifurcation switches from being supercritical to subcritical for $\kappa \simeq 0.4$. Since then, there has been a lot of rigorous work on the topic, including that of Bolley-Helffer [5, 6, 7, 8, 9] (computation of the field where the bifurcation happens and nature of the bifurcation in various limiting cases, namely $\kappa d$ small and large, $d$ small and large) and Hastings-Troy [17] (existence of only asymmetric solutions for a certain range of magnetic field). While this work was in progress, Dancer-Hastings [13] have studied the global bifurcation

[1]Since we are not investigating stability here, we need to assign a "direction" to the bifurcation to use the labels sub- and supercritical; we choose the direction of decreasing $H$. This definition coincides with the usual definition based on the stability properties of the asymmetric bifurcation investigated in [10, 11].

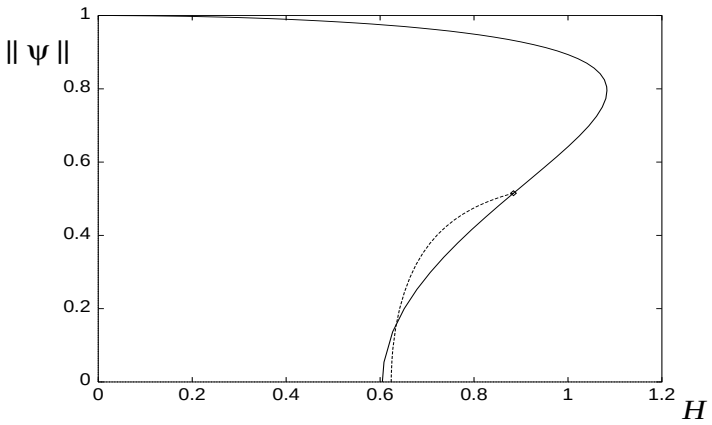


Figure 2: Bifurcation diagrams for $a = 3$ and $\kappa = 0.35$ showing symmetric (solid) and asymmetric (dashed) solution branches.

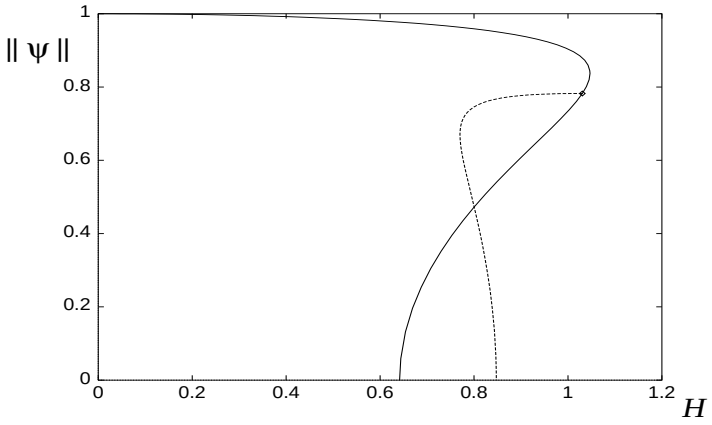


Figure 3: Bifurcation diagrams for $a = 3$ and $\kappa = 0.5$ showing symmetric (solid) and asymmetric (dashed) solution branches.

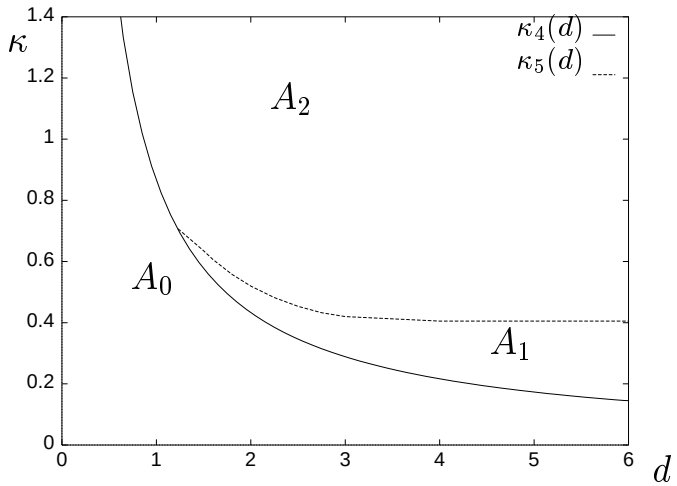


Figure 4: Curves $\kappa_4(d)$ and $\kappa_5(d)$, which divide the $\kappa d$-plane into the regions $A_0$, $A_1$ and $A_2$ where there zero, at most one, and at most two asymmetric solutions respectively. The curve $\kappa_4(d)$ is the curve across which the primary asymmetric bifurcation from the normal solution disappears, while $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical.

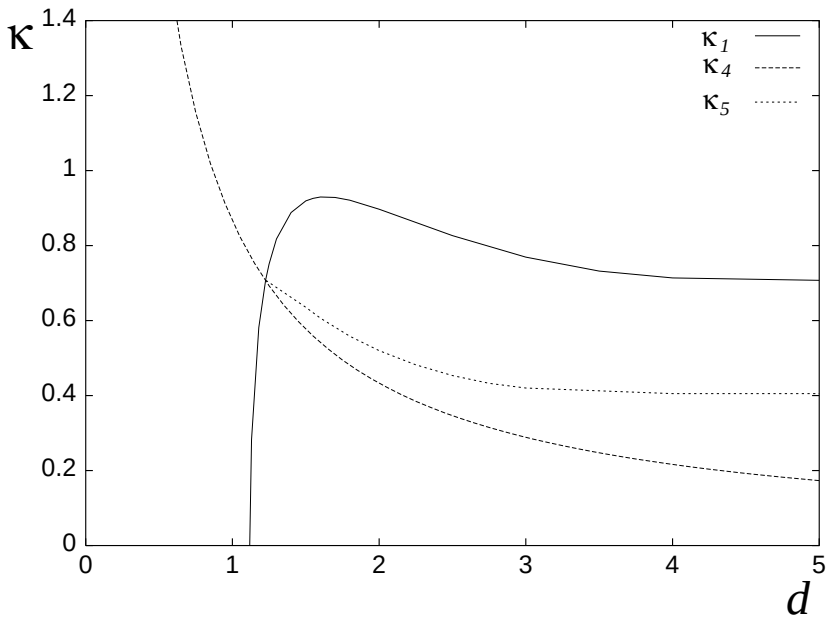


Figure 5: The curve $\kappa_1(d)$, which is the curve across which the bifurcation of the symmetric solution switches from being subcritical to supercritical, shown in conjunction with the curves $\kappa_4(d)$ and $\kappa_5(d)$.

diagrams and proved in particular that when $\kappa d$ is large, there exists a branch of asymmetric solutions connecting the symmetric curve to the normal solution.

In Section 2, we use asymptotic analysis to study the behaviour of the symmetric and asymmetric branches when their amplitudes are small and identify the curve $\kappa_4(d)$ and the quintuple point. Then in Section 3, we study the response diagram in the vicinity of the quintuple point. Finally, in Section 4, we present our conclusions.

## 2 Asymptotic analysis

Since the size of the domain $d$ is a parameter which we will want to vary, we begin by rescaling distance with $d$ so that the domain will remain fixed as $-1 < x < 1$. The Ginzburg-Landau system is then

$$\frac{1}{\kappa^2 d^2}\Psi'' = \Psi^3 - \Psi + \Psi A^2, \tag{2.1}$$

$$A'' = d^2\Psi^2 A, \tag{2.2}$$

$$\Psi'(\pm 1) = 0, \tag{2.3}$$

$$A'(\pm 1) = Hd. \tag{2.4}$$

We wish to examine the bifurcation from the normal state to a superconducting state. Close to the bifurcation point $\Psi$ will be small. We quantify this smallness by introducing a small parameter $\varepsilon$ and setting

$$\Psi = \varepsilon^{1/2} f, \tag{2.5}$$

$$A = Hd(x+e) + \varepsilon q, \tag{2.6}$$

as in [4, 10, 11, 19]. The relative scaling of $f$ and $q$ here is motivated by the fact that we want the nonlinear term $\Psi^3$ to balance with the correction to $\Psi A^2$ in (2.1). For ease of notation we set

$$p = \frac{1}{\kappa^2 d^2}, \qquad h = Hd.$$

Then $f$ and $q$ satisfy

$$\begin{aligned} pf'' &= \varepsilon f^3 - f + f(h^2(x+e)^2 + 2\varepsilon h(x+e)q + \varepsilon^2 q^2), && (2.7)\\ q'' &= d^2 f^2 (h(x+e) + \varepsilon q), && (2.8)\\ f'(\pm 1) &= 0, && (2.9)\\ q'(\pm 1) &= 0. && (2.10) \end{aligned}$$

We formally expand all quantities in powers of $\varepsilon$:

$$\begin{aligned} f &= f_0 + \varepsilon f_1 + \cdots, && (2.11)\\ q &= q_0 + \varepsilon q_1 + \cdots, && (2.12)\\ e &= e_0 + \varepsilon e_1 + \cdots, && (2.13)\\ h &= h_0 + \varepsilon h_1. && (2.14) \end{aligned}$$

**Leading order** Substituting the expansions (2.11)-(2.14) into (2.7)-(2.8) and equating coefficients of powers of $\varepsilon$ we find that at leading order

$$\begin{aligned} pf_0'' + f_0 - f_0 h_0^2 (x+e_0)^2 &= 0, && (2.15)\\ q_0'' &= d^2 f_0^2 h_0 (x+e_0), && (2.16) \end{aligned}$$

with homogeneous Neumann boundary data. Now, for each $e_0$ (2.15) is an eigenvalue problem for $h_0$. Let $\theta_0$ be the corresponding normalised eigenfunction (with the $L^\infty$ norm, say). Then $f_0 = C\theta_0$, where $C$ is undetermined at present. Now, the homogeneous version of (2.16) with homogeneous Neumann boundary conditions is satisfied by a constant, so that by the Fredholm alternative there is a solvability condition that the integral of the right-hand side is zero. Thus

$$\int_{-1}^{1} \theta_0^2 (x+e_0)\, dx = 0. \qquad (2.17)$$

This is the equation which determines $e_0$ (remember that $\theta_0$ is a function of $e_0$). It is easy to see that $e_0 = 0$ is always a solution, since in this case $\theta_0$ will be even. This is the symmetric superconducting solution. However, for some values of $p$ and $d$ (which we will determine later) there is also a nonzero solution for $e_0$. Note that if $(\theta_0(x), e_0)$ is a solution of (2.15), (2.17) then $(\theta_0(-x), -e_0)$ is also a solution. These two solutions are the asymmetric superconducting solutions.

We are interested in the secondary bifurcation in which the two asymmetric solutions and the symmetric solution reconnect to become a single symmetric solution. In particular, we will be concerned with the situation in which the secondary bifurcation is close to the primary bifurcations, so that the amplitude of the solutions is still small when it takes place (the square in Figures 2 and 3 is close to $\Psi = 0$). In this case we expect that $e$ will be zero for all three solutions to leading order, but will be nonzero for the asymmetric solutions at a higher order. Therefore, we expand in powers of $\varepsilon$ as:

$$f = f_0 + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \cdots, \qquad (2.18)$$

$$q = q_0 + \varepsilon^{1/2} q_1 + \varepsilon q_2 + \cdots, \tag{2.19}$$
$$e = \varepsilon^{1/2} e_1 + \varepsilon e_2 + \cdots, \tag{2.20}$$
$$h = h_0 + \varepsilon h_2 + \cdots, \tag{2.21}$$
$$p = p_0 + \varepsilon p_2 + \cdots, \tag{2.22}$$
$$d = d_0 + \varepsilon d_2 + \cdots. \tag{2.23}$$

The motivation for the $\varepsilon^{1/2}$ scaling on $e$ here is difficult to give. It occurs because the solvability conditions on $q$ at first and second order are satisfied automatically, leaving $e_1$ to be determined by the third order solvability condition as we will see. Once $e$ has assumed to be of order $\varepsilon^{1/2}$ the expansion for $f$ and $q$ must be in powers of $\varepsilon^{1/2}$. However, we still expand $h$ in powers of $\varepsilon$, since we know that this is the correct scaling for the symmetric solution. Finally note that we also expand $p$ and $d$ to allow us to consider what happens for $(d, p)$ close to a point $(d_0, p_0)$ on the curve $\kappa_4$. Again, the $O(\varepsilon^{1/2})$ term in these expansions is zero, so that the corrections to $p_0$ and $d_0$ come into the solvability condition at the right order. The final justification for the form of the expansions (2.18)-(2.23) is the ability to develop them self-consistently.

**Leading order** Substituting the expansions (2.18)-(2.23) into (2.7)-(2.8) and equating coefficients of powers of $\varepsilon$ we find that at leading order

$$Lf_0 \equiv p_0 f_0'' + f_0 - f_0 h_0^2 x^2 = 0, \tag{2.24}$$
$$q_0'' = d_0^2 f_0^2 h_0 x, \tag{2.25}$$

with homogeneous Neumann boundary data. Let $h_0$ be the leading eigenvalue of (2.24)[2], and $\theta_0$ the corresponding normalised eigenfunction (with the $L^\infty$ norm, say). Then $f_0 = C\theta_0$, where $C$ is undetermined at present. We see that $q_0 = C^2 a_0$, where

$$a_0'' = d_0^2 \theta_0^2 h_0 x, \tag{2.26}$$

with homogeneous Neumann boundary data. Thus $q_0$ is determined once $C$ is known. Note that $\theta_0$ is even, while $a_0$ is odd, so that the solvability condition on (2.26) is automatically satisfied.

**First order** At first order in (2.7), (2.8) we find

$$Lf_1 = 2 f_0 h_0^2 x e_1, \tag{2.27}$$
$$q_1'' = 2 d_0^2 f_0 f_1 h_0 x + d_0^2 f_0^2 h_0 e_1, \tag{2.28}$$

with homogeneous Neumann boundary data. Since the operator $L$ is self-adjoint and $f_0$ satisfies the homogeneous version of (2.27), there is a solvability condition for $f_1$ by the Fredholm alternative, which is that the right-hand side of (2.27) must be orthogonal to $f_0$. However, this condition is automatically satisfied since $f_0$ is even and the right-hand side of (2.27) is odd. Hence

$$f_1 = C e_1 \theta_1 + D\theta_0, \tag{2.29}$$

where

$$L\theta_1 = 2 h_0^2 x \theta_0. \tag{2.30}$$

[2]We choose to examine the leading eigenfunction, since it is shown in [10, 11] that this is the one which has the possibility of being stable. Of course, a similar analysis can be performed using any eigenfunction.

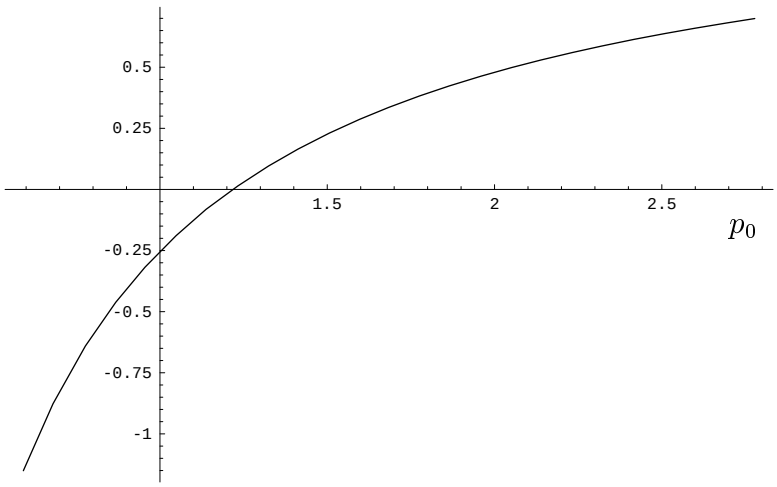


Figure 6: The left-hand side of (2.33) as a function of $p_0$.

Since $L\theta_0 = 0$ we may add any multiple of $\theta_0$ to $\theta_1$. However, it is convenient to fix $\theta_1(0) = 0$, so that $\theta_1$ is odd. We see that

$$q_1 = C^2 e_1 a_1 + 2CDa_0, \tag{2.31}$$

where

$$a_1'' = 2d_0^2\theta_0\theta_1 h_0 x + d_0^2 h_0 \theta_0^2. \tag{2.32}$$

Now the homogeneous version of (2.32) is satisfied by a constant. Thus there is a solvability condition on $a_1$, which is

$$\int_{-1}^{1} \left(2\theta_0\theta_1 x + \theta_0^2\right)\, dx = 0. \tag{2.33}$$

This condition defines the curve $\kappa_4$. Note that since $\theta_0$ and $\theta_1$ are functions of $p_0$ only, that is, they are independent of $d_0$, condition (2.33) gives the curve $\kappa_4$ as $p =$ constant $\approx 1.218$. The left-hand side of (2.33) is shown as a function of $p_0$ in Figure 6, from which it appears that there is a unique $p_0$ satisfying (2.33). We believe this to be true, but note that it is not necessary for the following analysis, for which $p_0$ may be any solution of (2.33).

With this solvability condition satisfied we may add any constant to $a_1$. For definiteness we fix $a_1(0) = 0$. Note that $a_1$ is even.

**Second order** Proceeding to the next order in the expansion in (2.7) we find

$$Lf_2 = -p_2 f_0'' + f_0^3 + 2h_0 h_2 x^2 f_0 + 2h_0 x q_0 f_0 + f_0 h_0^2(e_1^2 + 2xe_2) + 2xe_1 f_1 h_0^2, \tag{2.34}$$

$$\begin{aligned} q_2'' = {} & 2d_0 d_2 f_0^2 h_0 x + 2d_0^2 f_0 f_2 h_0 x + d_0^2 f_1^2 h_0 x + d_0^2 f_0^2 h_2 x \\ & + d_0^2 f_0^2 q_0 + d_0^2 f_0^2 h_0 e_2 + 2d_0^2 f_0 f_1 h_0 e_1. \end{aligned} \tag{2.35}$$

As before, since $f_0$ satisfies the homogeneous version of equation (2.34) there is a solvability condition which is derived by multiplying by $f_0$ and integrating. We find, after some manipulation that

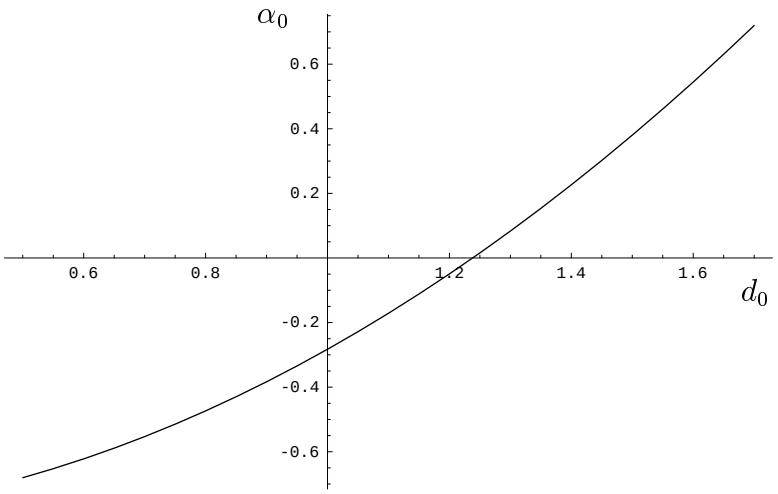


Figure 7: $\alpha_0$ as a function of $d_0$ on the curve $\kappa_4$.

the terms involving $D$ and $e_1$ cancel, leaving

$$h_2 = \alpha_0 C^2 + H_2 p_2, \tag{2.36}$$

where

$$\alpha_0 = -\frac{\int_{-1}^{1} \left(\theta_0^4 - \frac{2(a_0')^2}{d_0^2}\right) dx}{2h_0 \int_{-1}^{1} x^2 \theta_0^2 \, dx}, \tag{2.37}$$

$$H_2 = -\frac{\int_{-1}^{1} (\theta_0')^2 \, dx}{2h_0 \int_{-1}^{1} x^2 \theta_0^2 \, dx}. \tag{2.38}$$

Thus, when $\alpha_0 \neq 0$, $C$ is determined in terms of $h_2$. Figure 7 shows $\alpha_0$ as a function of $d_0$ on the curve $\kappa_4$. The leading order solution is now determined. However, we need to proceed further down the expansions to determine the coefficients $e_1$ and $D$ of the first order solution, in order to distinguish between the symmetric and asymmetric branches. We have

$$f_2 = C^3 \theta_2 + C p_2 \phi_2 + C e_1^2 \Omega_2 + (C e_2 + D e_1)\theta_1 + E\theta_0, \tag{2.39}$$

where

$$L\theta_2 = \theta_0^3 + 2h_0 \alpha_0 x^2 \theta_0 + 2h_0 x a_0 \theta_0, \tag{2.40}$$

$$L\phi_2 = -\theta_0'' + 2h_0 H_2 x^2 \theta_0, \tag{2.41}$$

$$L\Omega_2 = h_0^2 \theta_0 + 2x h_0^2 \theta_1. \tag{2.42}$$

Again, we can add any multiple of $\theta_0$ to $\theta_2$, $\phi_2$ and $\Omega_2$. For definiteness we fix these functions by requiring that they are equal to zero at the origin. The solvability condition on $q_2$ from (2.35) turns

out to be automatically satisfied due to (2.33). Then we find

$$q_2 = C^4 a_2 + \frac{2C^2 d_2 a_0}{d_0} + C^2 p_2 b_2 + C^2 e_1^2 g_2 + (C^2 e_2 + 2CDe_1)a_1 + (D^2 + 2CE)a_0, \tag{2.43}$$

where

$$\begin{aligned} a_2'' &= 2d_0^2\theta_0 h_0 x\theta_2 + d_0^2\theta_0^2\alpha_0 x + d_0^2\theta_0^2 a_0, & (2.44)\\ b_2'' &= 2d_0^2\theta_0 h_0 x\phi_2 + d_0^2\theta_0^2 H_2 x, & (2.45)\\ g_2'' &= 2d_0^2\theta_0 h_0 x\Omega_2 + d_0^2 h_0 x\theta_1^2 + 2d_0^2\theta_0\theta_1 h_0. & (2.46) \end{aligned}$$

Again, we may add a constant to $a_2$, $b_2$ and $c_2$. We fix them by requiring that they are zero at the origin. Then $\theta_2$, $\phi_2$ and $\Omega_2$ are even, while $a_2$, $b_2$ and $g_2$ are odd.

**Third order** Proceeding to third order in (2.7) we find

$$\begin{aligned} Lf_3 &= -p_2 f_1'' + 3f_0^2 f_1 + 2xe_3 h_0^2 f_0 + 2e_1 e_2 h_0^2 f_0 + 2h_0 e_1 q_0 f_0 + 4h_0 h_2 x e_1 f_0 \\ &\quad + 2f_0 h_0 x q_1 + 2h_0 h_2 x^2 f_1 + (2xe_2 + e_1^2)h_0^2 f_1 + 2h_0 x q_0 f_1 + 2xe_1 h_0^2 f_2, & (2.47)\\ q_3'' &= 2d_0 d_2(2f_0 f_1 h_0 x + f_0^2 h_0 e_1) + d_0^2 f_0^2(h_0 e_3 + h_2 e_1 + q_1) + 2f_0 f_1 d_0^2(h_0 e_2 + q_0 + h_2 x) \\ &\quad + d_0^2(2f_0 f_2 + f_1^2)h_0 e_1 + d_0^2(2f_0 f_3 + 2f_1 f_2)h_0 x, & (2.48) \end{aligned}$$

As usual there are solvability conditions. Multiplying (2.47) by $f_0$ and integrating, using the expressions for $f_1$ etc., we find after some manipulation that the only term remaining is that multiplied by $C^3 D$. Hence we must have $D = 0$, so that $f_1$ is odd. Then

$$f_3 = C^3 e_1\theta_3 + Ce_1 p_2\phi_3 + Ce_1^3\Omega_3 + Ce_1 e_2\Omega_2 + (Ee_1 + Ce_3)\theta_1 + F\theta_0, \tag{2.49}$$

where

$$\begin{aligned} L\theta_3 &= 3\theta_0^2\theta_1 + 2\theta_0 h_0 a_0 + 4\theta_0 h_0\alpha_0 x + 2\theta_1 h_0\alpha_0 x^2 + 2h_0 x a_0\theta_1 + 2\theta_2 x h_0^2 + 2\theta_0 h_0 x a_1, & (2.50)\\ L\phi_3 &= -\theta_1'' + 4h_0 H_2\theta_0 x + 2\theta_1 h_0 x^2 H_2 + 2xh_0^2\phi_2, & (2.51)\\ L\Omega_3 &= \theta_1 h_0^2 + 2xh_0^2\Omega_2. & (2.52) \end{aligned}$$

Again, we choose $\theta_3$, $\phi_3$ and $\Omega_3$ to be zero at the origin, so that all are odd.

The solvability condition on $q_3$ is obtained by integrating (2.48) to give, after some manipulation

$$e_1(\alpha_1 C^2 + p_2\beta_1 + \gamma_1 e_1^2) = 0, \tag{2.53}$$

where

$$\begin{aligned} \alpha_1 &= \int_{-1}^{1}\left[\theta_0^2 a_1 + 2\theta_0\theta_1 a_0 + 2h_0\theta_0\theta_2 + 2\theta_0 h_0 x\theta_3 + 2h_0 x\theta_1\theta_2\right]\,dx, & (2.54)\\ \beta_1 &= \int_{-1}^{1}\left[\theta_0^2 H_2 + 2\theta_0\theta_1 H_2 x + 2h_0\theta_0\phi_2 + 2h_0 x\theta_0\phi_3 + 2h_0 x\theta_1\phi_2\right]\,dx, & (2.55)\\ \gamma_1 &= \int_{-1}^{1}\left[2h_0\theta_0\Omega_2 + h_0\theta_1^2 + 2\theta_0 h_0 x\Omega_3 + 2h_0 x\theta_1\Omega_2\right]\,dx & (2.56) \end{aligned}$$

Hence, for $\alpha_1 C^2/\gamma_1 < -p_2\beta_1/\gamma_1$ there are three solutions for $e_1$, namely

$$e_1 = 0, \qquad e_1 = \pm\left(\frac{-p_2\beta_1 - \alpha_1 C^2}{\gamma_1}\right)^{1/2}, \tag{2.57}$$

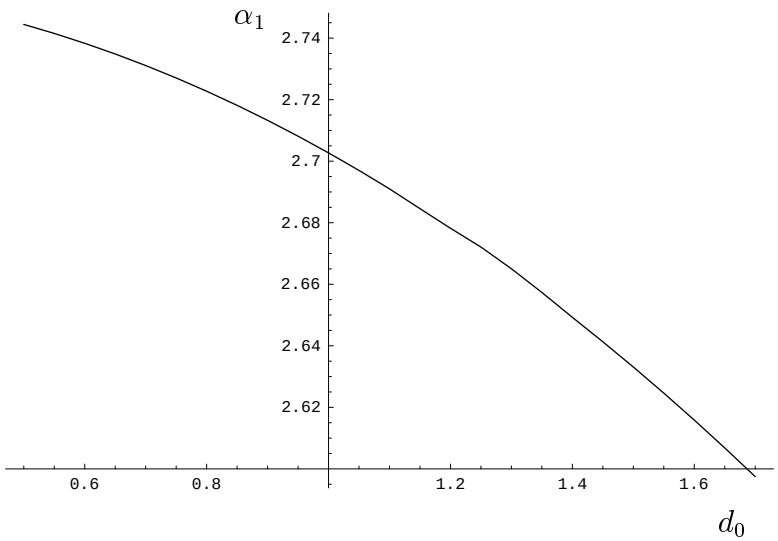


Figure 8: $\alpha_1$ as a function of $d_0$ on the curve $\kappa_4(d)$.

while for $\alpha_1 C^2/\gamma_1 > -p_2\beta_1/\gamma_1$ there is only the zero solution. Thus for the secondary bifurcation point to exist we require that $p_2\beta_1/\gamma_1 < 0$ (so that the primary asymmetric bifurcation exists) and $\alpha_1/\gamma_1 > 0$ (so that the asymmetric branch then reconnects with the symmetric branch), in which case it occurs when $C^2 = -p_2\beta_1/\alpha_1$.

Figure 8 shows $\alpha_1$ as a function of $d_0$ on the curve $\kappa_4$. We find that $\beta_1$ and $\gamma_1$ are independent of $d_0$, and are approximately given by $\beta_1 = 1.734$, $\gamma_1 = 3.889$. Thus we see that the asymmetric solution exists for $p_2 < 0$.

We have now determined the solution at leading and first order. We see that the norms of the three solutions are the same to leading order. Note that $e_1$ is monotone decreasing to zero as $C$ increases from zero to its value at the secondary bifurcation, so that the asymmetric solution becomes symmetric as the secondary bifurcation is approached.

# 3 The quintuple point

Suppose now that $p_0$ and $d_0$ are such that $\alpha_0 = 0$, which corresponds to the quintuple point since the sign of $\alpha_0$ determines whether the primary bifurcations are sub- or supercritical. Then $C$ is not determined by (2.36). In this case (2.53) gives $e_1$ in terms of $C$ for each solution, but we need to proceed to fourth order to determine $C$.

**Fourth order** At fourth order in equation (2.7) we find

$$\begin{aligned} Lf_4 \;=\; & -p_4 f_0'' - p_2 f_2'' + 3f_0^2 f_2 + 3f_0 f_1^2 + 2h_0^2 x e_4 f_0 + (2e_1e_3 + e_2^2)h_0^2 f_0 + 2h_0h_2(2xe_1 + e_1^2)f_0 \\ & + (2h_0h_4 + h_2^2)x^2 f_0 + 2h_0 x q_2 f_0 + 2h_0 e_1 q_1 f_0 + 2h_0 e_2 q_0 f_0 + 2h_2 x q_0 f_0 + q_0^2 f_0 \\ & + 2xh_0^2 e_3 f_1 + 2h_0^2 e_1 e_2 f_1 + 2h_0 q_0 e_1 f_1 + 4h_0 h_2 x e_1 f_1 + 2h_0 x q_1 f_1 + 2h_0 h_2 x^2 f_2 \\ & + (2xe_2 + e_1^2)h_0^2 f_2 + 2h_0 x q_0 f_2 + 2xh_0^2 e_1 f_3. \end{aligned} \tag{3.58}$$

As usual, the solvability condition is derived by multiplying by $f_0$ and integrating. After some manipulation, we find that the terms involving $E$ cancel, to leave

$$h_4 = \alpha_2 C^4 + \beta_2 C^2 + \gamma_2 + p_2\delta_2 e_1^2 + \eta_2 C^2 e_1^2 + \nu_2 e_1^4, \tag{3.59}$$

where

$$\alpha_2 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \left[3\theta_0^3\theta_2 + 2h_0 x a_2\theta_0^2 + \theta_0^2 a_0^2 + 2h_0 x a_0\theta_0\theta_2\right]\,dx, \tag{3.60}$$

$$\beta_2 = \lambda_2 d_2 + \mu_2 p_2, \tag{3.61}$$

$$\lambda_2 = -\frac{2}{d_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 x\theta_0^2 a_0\,dx, \tag{3.62}$$

$$\mu_2 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \Big[\theta_0'\theta_2' + 3\theta_0^3\phi_2 + 2h_0 x\theta_0^2 b_2 + 2x a_0\theta_0^2 H_2 + 2h_0 H_2 x^2\theta_0\theta_2 + 2h_0 x a_0\theta_0\phi_2\Big]\,dx, \tag{3.63}$$

$$\gamma_2 = p_4 H_2 + p_2^2 H_4, \tag{3.64}$$

$$H_4 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \left[\theta_0^2 x^2 H_2^2 + \theta_0'\phi_2' + 2h_0 H_2 x^2\theta_0\phi_2\right]\,dx, \tag{3.65}$$

$$\delta_2 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \Big[\theta_0'\Omega_2' + 2h_0 H_2\theta_0^2 + 4h_0 H_2 x\theta_0\theta_1 + 2h_0 H_2 x^2\theta_0\Omega_2 + h_0^2\theta_0\phi_2 + 2h_0^2 x\theta_0\phi_3\Big]\,dx, \tag{3.66}$$

$$\eta_2 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \Big[3\theta_0^3\Omega_2 + 3\theta_0^2\theta_1^2 + 2h_0 x\theta_0^2 g_2 + 2h_0\theta_0^2 a_1 + 2h_0\theta_1\theta_0 a_0 + 2h_0 x a_1\theta_0\theta_1 + h_0^2\theta_0\theta_2 + 2h_0 x a_0\theta_0\Omega_2 + 2h_0^2 x\theta_0\theta_3\Big]\,dx, \tag{3.67}$$

$$\nu_2 = -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\int_{-1}^1 \left[h_0^2\theta_0\Omega_2 + 2h_0^2 x\theta_0\Omega_3\right]\,dx. \tag{3.68}$$

Thus we have a coupled system (2.53), (3.59) for $C$ and $e_1$. Note that $\alpha_1$, $\beta_1$, $\gamma_1$, $\alpha_2$, $\lambda_2$, $\mu_2$, $\eta_2$, $H_2$, $H_4$, $\delta_2$, and $\nu_2$ depend only on $p_0$ and $d_0$, evaluated at the quintuple point, and are therefore simply fixed numbers. Numerically we find that at the quintuple point

$$d_0 = 1.24, \qquad \kappa_0 = 0.73, \qquad \alpha_1 = 2.67, \qquad \beta_1 = 1.734, \qquad \gamma_1 = 3.889.$$

$$\alpha_2 = -0.39, \qquad \lambda_2 = 1.32, \qquad \mu_2 = 0.16, \qquad \eta_2 = -5.57,$$
$$H_2 = -0.053, \qquad H_4 = 0.048, \qquad \delta_2 = -1.81, \qquad \nu_2 = -2.03.$$

Let us consider the behaviour of the solution of (2.53), (3.59) as $p_2$ and $d_2$ vary. The first important curve is that given by $p_2 = 0$, which determines whether the asymmetric solution exists or not, and therefore corresponds to the curve $\kappa_4$. We find that the asymmetric solution exists only for $p_2 < 0$. The second important curve is $\beta_2 = 0$, since this decides whether the symmetric bifurcation is sub or supercritical, and corresponds to the curve $\kappa_1$. Hence $\kappa_1$ corresponds to

$$\lambda_2 d_2 + \mu_2 p_2 = 0. \tag{3.69}$$

The final curve $\kappa_5$ determines whether the asymmetric bifurcation is sub or supercritical. If the asymmetric solution exists we may replace $e_1^2$ in (3.59) with

$$-\frac{\alpha_1 C^2 + p_2\beta_1}{\gamma_1}, \tag{3.70}$$

giving

$$\begin{aligned} h_4 &= \left(\alpha_2 - \frac{\alpha_1\eta_2}{\gamma_1} + \frac{\alpha_1^2\nu_2}{\gamma_1^2}\right) C^4 \\ &\quad + \left(\lambda_2 d_2 + p_2\left(\mu_2 - \frac{\alpha_1\delta_2}{\gamma_1} - \frac{\beta_1\eta_2}{\gamma_1} + \frac{2\alpha_1\beta_1\nu_2}{\gamma_1^2}\right)\right) C^2 + \gamma_2 + p_2^2\left(\frac{\beta_1\nu_2}{\gamma_1^2} - \frac{\beta_1\delta_2}{\gamma_1}\right). \end{aligned} \tag{3.71}$$

Whether the bifurcation is sub or supercritical depends on the coefficient of $C^2$. Thus we see that $\kappa_5$ is given by

$$\lambda_2 d_2 + p_2\left(\mu_2 - \frac{\alpha_1\delta_2}{\gamma_1} - \frac{\beta_1\eta_2}{\gamma_1} + \frac{2\alpha_1\beta_1\nu_2}{\gamma_1^2}\right) = 0, \tag{3.72}$$

with the constraint that $p_2 < 0$. Thus we have been able to determine the lines in the $p_2 d_2$-plane (two infinite, one semi-infinite) across which the behaviour of the bifurcation diagram changes.

These may be translated to the $\kappa d$-plane by noting that

$$p_2 = -\frac{2(d_2\kappa_0 + d_0\kappa_2)}{d_0^3\kappa_0^3}.$$

In Figures 9 and 10 we show the curves $\kappa_1$, $\kappa_4$ and $\kappa_5$ in the $pd$- and $\kappa d$-planes respectively. In Figures 11-16 we show a selection of bifurcation diagrams. Figures 11 and 12 correspond to regions where no asymmetric solutions exist and the curve is a curve of symmetric solutions. In Figures 13 to 16, the asymmetric branch is the one that bifurcates from the normal solution at the higher field.

Note that the bifurcation point for the symmetric solution is $h_4 = \gamma_2$, while that for the asymmetric solution is

$$h_4 = \gamma_2 + p_2^2\left(\frac{\beta_1\nu_2}{\gamma_1^2} - \frac{\beta_1\delta_2}{\gamma_1}\right) \approx \gamma_2 + 0.577 p_2^2.$$

The coefficient of $p_2^2$ here is positive, so that the asymmetric solution always bifurcates at a higher value of $h$ in the vicinity of the quintuple point (in fact numerical simulations indicate that this is always true).

Note also that we may calculate the angle between the two solutions at the secondary bifurcation point. We find that

$$\frac{dh_4}{dC^2} = \beta_2 - \frac{2\alpha_2\beta_1 p_2}{\alpha_1}, \tag{3.73}$$

for the symmetric branch, and

$$\frac{dh_4}{dC^2} = \beta_2 - \frac{2\alpha_2\beta_1 p_2}{\alpha_1} - \frac{\alpha_1\delta_2 p_2}{\gamma_1} + \frac{\beta_1\eta_2 p_2}{\gamma_1}, \tag{3.74}$$

for the asymmetric branch. Since $p_2 < 0$ for the asymmetric branch to exist we find that $dh_4/dC^2$ is larger for the asymmetric branch than for the symmetric branch at the secondary bifurcation point. This means that the asymmetric branch lies to the left of the symmetric branch close to the

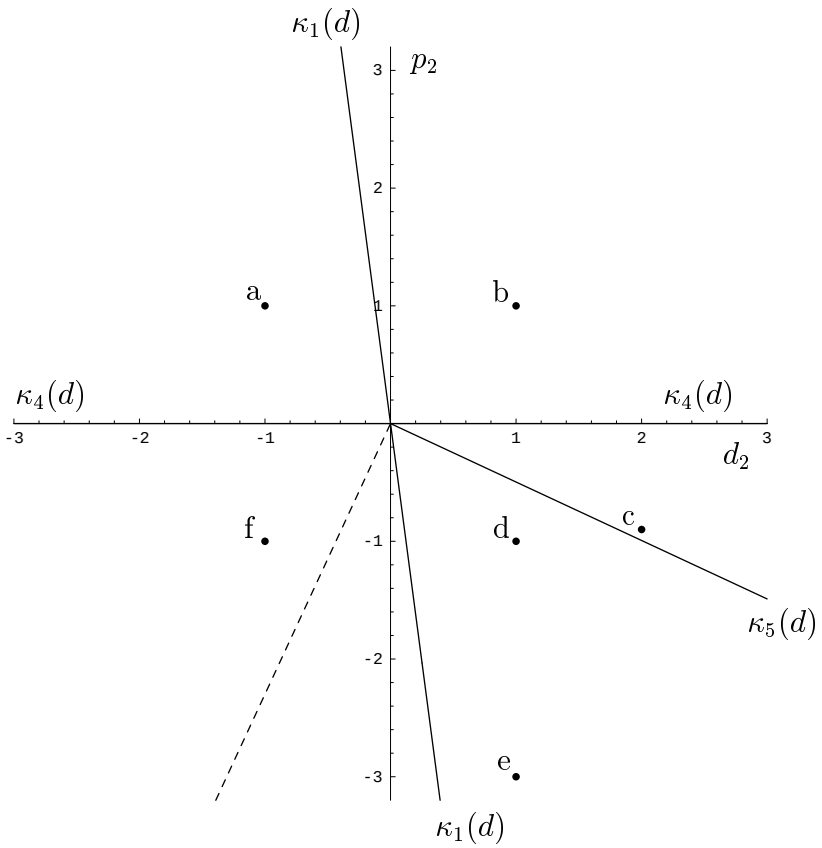


Figure 9: The curves $\kappa_1$, $\kappa_4$ and $\kappa_5$ in the $pd$-plane. The bifurcation diagrams for the points a-f are shown in Figures 11-16. The curve $\kappa_4$ is the curve across which the primary bifurcation to an asymmetric solution from the normal solution disappears, $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical, and $\kappa_1$ is the curve across which the primary bifurcation to a symmetric solution switches from being subcritical to supercritical. Also shown (dashed) is the curve across which the nose in the asymmetric solution branch disappears by meeting the secondary bifurcation. The nose is present between this curve and the curve $\kappa_5$.

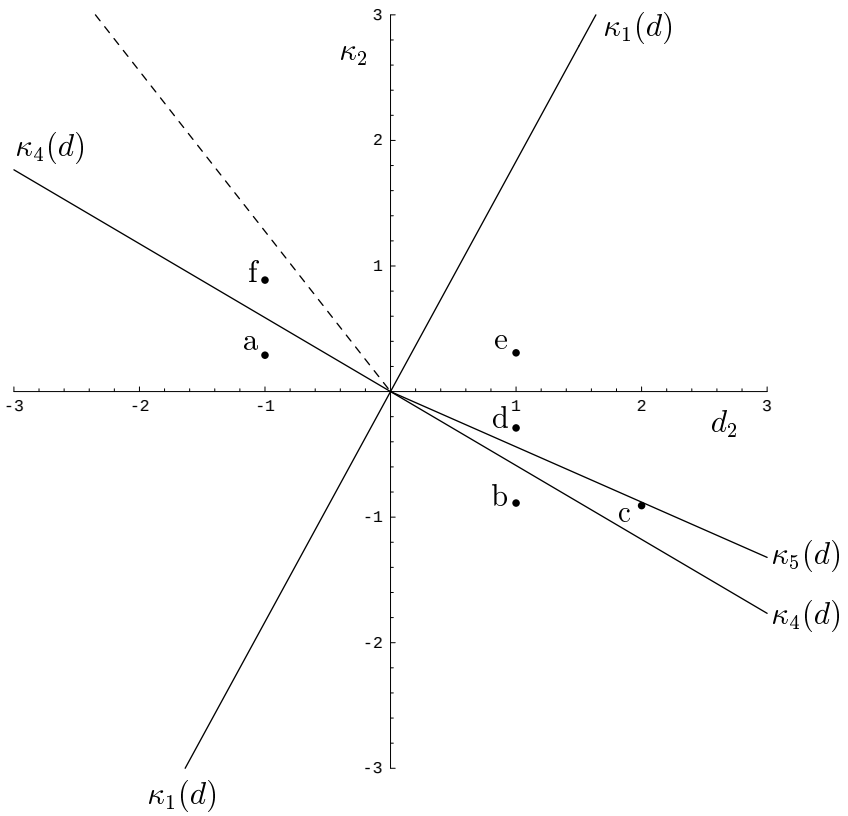


Figure 10: The curves $\kappa_1$, $\kappa_4$ and $\kappa_5$ in the $kd$-plane. The bifurcation diagrams for the points a-f are shown in Figures 11-16. The curve $\kappa_4$ is the curve across which the primary bifurcation to an asymmetric solution from the normal solution disappears, $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical, and $\kappa_1$ is the curve across which the primary bifurcation to a symmetric solution switches from being subcritical to supercritical. Also shown (dashed) is the curve across which the nose in the asymmetric solution branch disappears by meeting the secondary bifurcation. The nose is present between this curve and the curve $\kappa_5$.

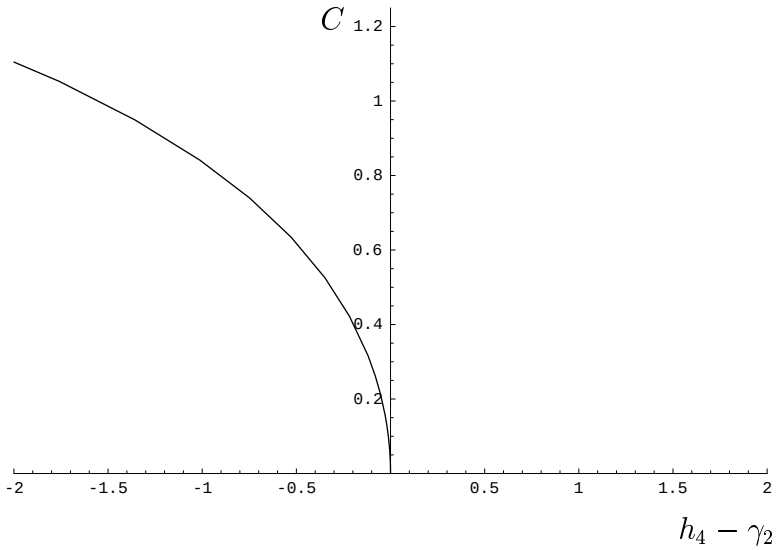


Figure 11: The bifurcation diagram for $d_2 = -1$, $p_2 = 1$, corresponding to point (a) in Figures 9 and 10. The symmetric solution is shown; there is no asymmetric solution.

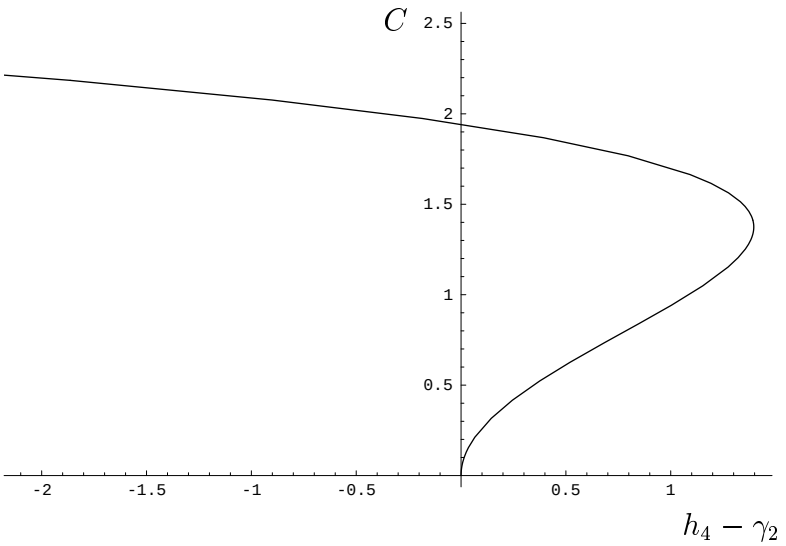


Figure 12: The bifurcation diagram for $d_2 = 1$, $p_2 = 1$, corresponding to point (b) in Figures 9 and 10. The symmetric solution is shown; there is no asymmetric solution.

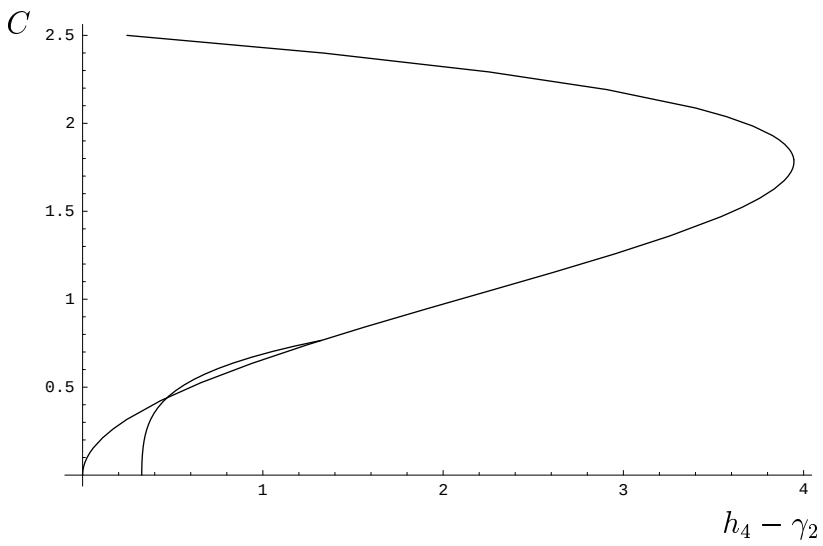


Figure 13: The bifurcation diagram for $d_2 = 2$, $p_2 = -0.9$, corresponding to point (c) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of $h_4$.

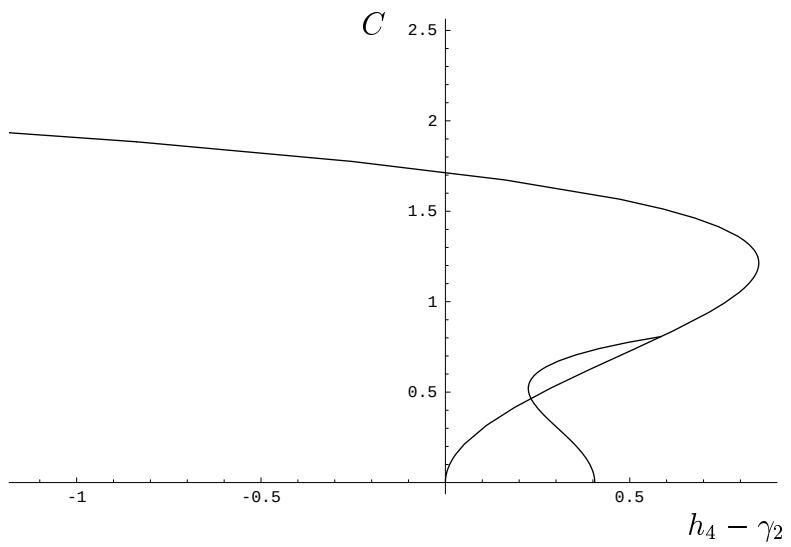


Figure 14: The bifurcation diagram for $d_2 = 1$, $p_2 = -1$, corresponding to point (d) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of $h_4$.

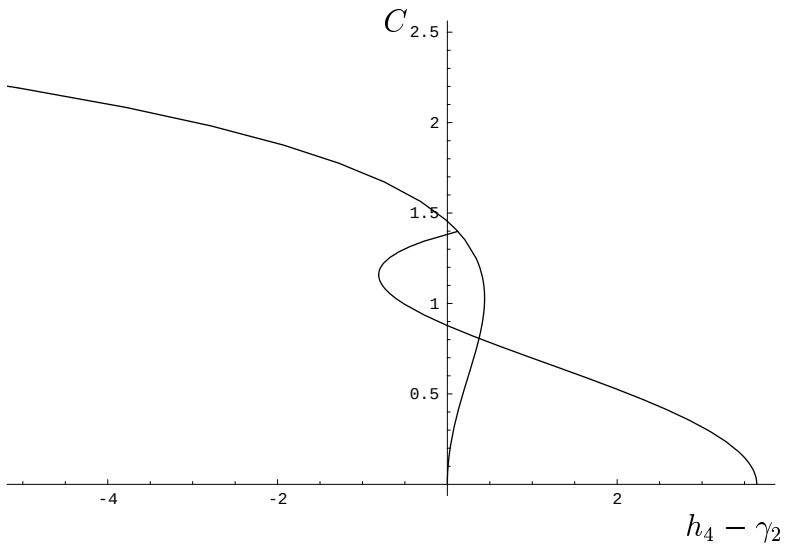


Figure 15: The bifurcation diagram for $d_2 = 1$, $p_2 = -3$, corresponding to point (e) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of $h_4$.

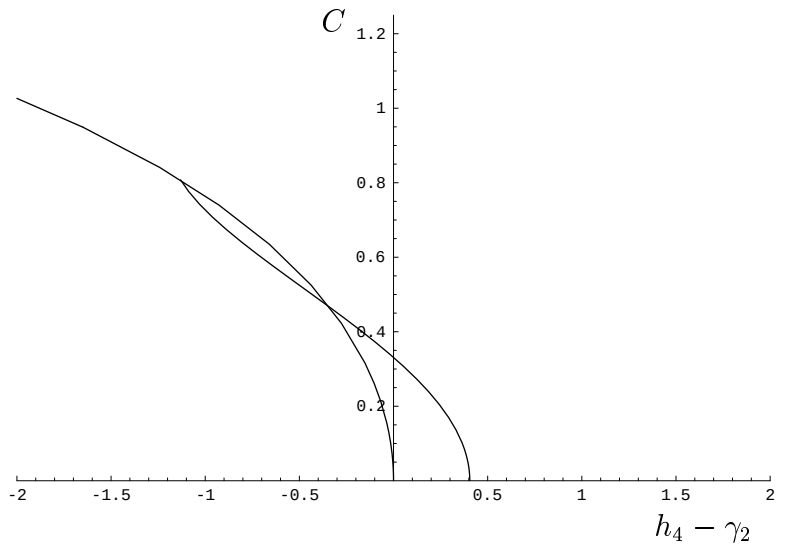


Figure 16: The bifurcation diagram for $d_2 = -1$, $p_2 = -1$, corresponding to point (f) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of $h_4$.

secondary bifurcation point, but to the right of it near the primary bifurcation points. Hence the two branches must always cross at some point.

Note that the secondary bifurcation point may be above or below the nose of the symmetric branch, and that there may or may not be a nose in the asymmetric branch. If there is a nose in the asymmetric branch, we find it occurs at

$$C^2 = \frac{(\alpha_1\delta_2\gamma_1 + \beta_1\eta_2\gamma_1 - 2\alpha_1\beta_1\nu_2 - \gamma_1^2\mu_2)p_2 - \gamma_1^2\lambda_2 d_2}{2(\alpha_2\gamma_1^2 + \alpha_1^2\nu_2 - \alpha_1\eta_2\gamma_1)} \approx -0.266d_2 - 0.536p_2. \tag{3.75}$$

Hence, the condition for there to be a nose in the asymmetric branch is that this lies below the value of $C$ at which the secondary bifurcation occurs, which is

$$C^2 = -p_2\beta_1/\alpha_1 \approx -0.652164p_2. \tag{3.76}$$

Hence there is a nose in the asymmetric branch if and only if

$$\alpha_1\gamma_1^2\lambda_2 d_2 + (\alpha_1\beta_1\eta_2\gamma_1 - \alpha_1^2\delta_2\gamma_1 - 2\alpha_2\beta_1\gamma_1^2 + \alpha_1\gamma_1^2\mu_2)p_2 > 0, \tag{3.77}$$

which approximates to

$$d_2 - 0.435p_2 > 0. \tag{3.78}$$

Since $p_2 < 0$ for the asymmetric branch to exist, we see that there will be a nose in this branch unless $d_2$ is sufficiently negative. Since $d_2 = p_2 = -1$ in Figure 16 there is no nose in the asymmetric branch there.

# 4 Conclusion

We have examined the bifurcation from the normal solution of superconducting solutions of the Ginzburg-Landau equations on a slab ($GL_d$), using the applied magnetic field as the bifurcation parameter. There are bifurcation branches corresponding to both symmetric and asymmetric solutions, and their existence and behaviour depends on the additional parameters $d$, the slab thickness, and $\kappa$, the Ginzburg-Landau parameter.

It was found numerically in [3] that there are three distinct regions of the $\kappa d$-plane, $A_0$, $A_1$ and $A_2$, in which there are respectively no asymmetric solutions, at most one and at most two asymmetric solutions.

Here we have been concerned with the curve they labelled $\kappa_4(d)$, which divides $A_0$ from $A_1 \cup A_2$. This is the curve across which the asymmetric solution ceases to exist, and is given by (2.33). We find it is of the form $\kappa d = C$, where $C$ is constant and approximately equal to 0.90.

Close to $\kappa_4$ the secondary bifurcation in which the asymmetric solutions connect with the symmetric solution to form a single symmetric solution lies close to the normal solution, i.e. close to the primary bifurcations. Thus, close to $\kappa_4$, we have been able to see the secondary bifurcation explicitly through our asymptotic analysis.

Since the leading-order problem corresponds to a linearization about the normal state, the amplitude of the leading-order solution is not determined at leading order but by a solvability condition at higher order. We found that along most of the curve $\kappa_4$ this solvability condition occurs at second order, and leads to the perturbation of the magnetic field from the bifurcation value being quadratic in the amplitude of the leading-order solution as given by (2.36). The symmetric and asymmetric solutions are the same to leading order, and we were able to distinguish between them at first order and identify the secondary bifurcation point.

However, at one point on $\kappa_4$, (called the quintuple point and given by $\alpha_0 = 0$, where $\alpha_0$ is defined by (2.37)) the primary bifurcations switch from being subcritical to supercritical, and at this point the amplitude of the solutions is determined by a solvability condition at fourth order, with the perturbation of the magnetic field from the bifurcation value being quartic in the amplitude of the leading-order solution, as given by (3.59). In this case the symmetric and asymmetric solutions have different amplitudes at leading order. This is the point at which $A_0$, $A_1$ and $A_2$ meet, and there are five different qualitative behaviours for the bifurcation diagram in the vicinity of this point. This rich structure is captured completely by our asymptotic analysis. In particular, we are able to establish that the asymmetric branch lies to the left of the symmetric branch close to the secondary bifurcation point, but to the right of it near the primary bifurcation points (near the normal solution), so that the two branches must always cross at some point. We have also determined the condition for there to be a nose in the asymmetric branch.

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