A semi-elliptic system arising in the theory of type-II superconductivity

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Abstract

A reduced form of the Ginzburg-Landau equations of superconductivity is considered, corresponding to the formal limit as the Ginzburg-Landau parameter κ tends to infinity. Existence and uniqueness of the solution is established, up to the point at which the magnitude of the potential first becomes equal to $1/\sqrt{3}$, when the solution becomes linearly unstable. The instability is shown to occur first on the boundary of the sample. Finally a more complete study of one-dimensional and radially-symmetric cases is presented.

1 Introduction

When a sufficiently strong magnetic field is applied to a superconducting material, superconductivity will be destroyed and the sample will revert to the normally conducting (normal) state. The magnetic field strength at which this transition occurs for a type-I superconductor is known as the thermodynamic critical field and is denoted by H_c; for $H < H_c$ the sample will be in the superconducting state, while for $H > H_c$ it will be in the normal state.

Type-II superconductors are characterized by the existence of a third state, known as the mixed state, which exists for magnetic fields $H_{c_1} < H_0 < H_{c_2}$ (see figure 1). The mixed state, as its name suggests, is neither wholly superconducting nor wholly normal, but consists of many "filaments" of normal material embedded in a superconducting matrix, each carrying with it a quantised amount of magnetic flux and circled by a vortex of superconducting current (thus these filaments are often known as vortices) (see figure 2).

The transition from the normal state to the mixed state takes place via a bifurcation as the magnetic field is lowered through the critical value H_{c_2} (known as the upper critical field), and is described in [1, 4, 6, 10, 11]. This bifurcation is subcritical for type-I superconductors, but supercritical for type-II superconductors; hence the observation of the mixed state only for type-II superconductors.

The transition between superconducting and mixed states is less well studied mathematically and some aspects of this question are the subject of the present manuscript. The critical field H_{c_1} plotted in figure 1 (known as the lower critical field) is calculated on the basis of an energy argument; it is the field at which the free energy of the wholly superconducting solution becomes equal to the free energy of the single vortex solution for an infinite superconductor. In fact, as H_0 is raised, there is a barrier to the generation of vortices, and there exists a superheating field H_{sh}, such that for fields $H_{c_1} < H_0 < H_{sh}$ the wholly superconducting solution is still linearly stable, even though it is not the global minimum energy solution. The superheating field for a halfspace was calculated recently in [5]. The aim of the present paper is to consider the existence and stability of the wholly superconducting solution for an arbitrary sample in two dimensions. The position of the maximum of the current density $|\nabla H|$ is also studied. Finally, a more complete study of the one-dimensional and radially-symmetrical cases is presented.

2 Formulation

Here we merely state the dimensionless Ginzburg-Landau equations. For a more complete introduction to the Ginzburg-Landau theory of superconductivity the reader is referred to [7, 8, 9] and the references therein. We recall that the Ginzburg-Landau free energy is given by

$$\int_\Omega \left| \left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right) \Psi \right|^2 - |\, \Psi\,|^2 + \frac{|\, \Psi\,|^4}{2} + (\mathrm{curl}\ \boldsymbol{A})^2 \tag{1}$$

The stationary points of this energy satisfy the following Ginzburg-Landau equations:

$$\left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right)^2 \Psi = (|\, \Psi\,|^2 - 1)\Psi \qquad \text{in } \Omega, \tag{2}$$

$$-(\mathrm{curl})^2 \boldsymbol{A} = \frac{i}{2\kappa} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) + |\, \Psi\,|^2 \boldsymbol{A} \qquad \text{in } \Omega, \tag{3}$$

where Ω is the region occupied by the superconducting sample, Ψ is the (complex) superconducting order parameter (when $|\, \Psi\,| = 0$ the sample is normal, when $|\, \Psi\,| > 0$ it is superconducting), and $\boldsymbol{A}$ is the (real) magnetic vector potential, which is such that

$$\boldsymbol{H} = \text{curl } \boldsymbol{A}, \tag{4}$$

where $\boldsymbol{H}$ is the magnetic field; $\boldsymbol{A}$ is defined only up to the addition of a gradient. Here κ is the Ginzburg-Landau parameter of the introduction.

We will be concerned only with the two-dimensional case in which the magnetic field is perpendicular to the plane of interest. In this case $\boldsymbol{A} = (A_1(x,y), A_2(x,y), 0)$, $\boldsymbol{H} = (0, 0, H(x,y))$, and the boundary conditions appropriate to (2)-(4) are

$$\boldsymbol{n} \cdot \left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right) \Psi = 0 \qquad \text{on } \partial\Omega, \tag{5}$$

$$H = H_0 \qquad \text{on } \partial\Omega, \tag{6}$$

where $\boldsymbol{n}$ is the unit outward normal to the boundary $\partial\Omega$.

In the units we are using the thermodynamic critical field is given by $H_c = 1/\sqrt{2}$, the upper critical field is given by $H_{c_2} = \kappa$ (for an infinite sample), and the lower critical field is given by $H_{c_1} = (1/2\kappa)\log\kappa$, for large κ.

Equations (2)-(4) are gauge invariant in the sense that they are invariant under transformations of the form

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla w, \qquad \Psi \to \Psi e^{i\kappa w}.$$

We may write the equations in terms of real variables by introducing the new, gauge invariant potential and order parameter:

$$\boldsymbol{Q} = \boldsymbol{A} - \frac{1}{\kappa}\nabla\chi, \;\; \Psi = f e^{i\chi} \tag{7}$$

where f and χ are real. We then obtain the following equations for f and $\boldsymbol{Q}$:

$$\frac{1}{\kappa^2}\nabla^2 f = f^3 - f + f\,|\boldsymbol{Q}|^2 \qquad \text{in } \Omega, \tag{8}$$

$$-(\text{curl})^2 \boldsymbol{Q} = f^2 \boldsymbol{Q} \qquad \text{in } \Omega, \tag{9}$$

with boundary conditions

$$\frac{\partial f}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{10}$$

$$\boldsymbol{Q} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{11}$$

$$H = H_0 \qquad \text{on } \partial\Omega. \tag{12}$$

Remark: It should be noted that the transformation from $\boldsymbol{A}$, Ψ to $\boldsymbol{Q}$, f in (7) which allows us to formulate the problem as (8)-(12) is only formal. Indeed, when Ψ is exactly 0, χ is not well defined. These points are precisely the vortices in the mixed state of a type-II superconductor. However, as we will explain below we can still observe — at a formal level — the phenomenom of vortex nucleation with these equations (for a more detailed discussion see [5]).

We consider the behaviour of (8)-(12) for large κ. Formally letting $\kappa \to \infty$ we find

$$\begin{aligned} f^2 &= 1- |\boldsymbol{Q}|^2, & (13)\\ -(\mathrm{curl})^2\boldsymbol{Q} &= \left(1- |\boldsymbol{Q}|^2\right)\boldsymbol{Q}, & (14) \end{aligned}$$

in Ω. Since the coefficient of the highest derivative in equation (8) vanishes in this limit the perturbation is a singular perturbation, and there will be a boundary layer near $\partial\Omega$, which needs to be examined before boundary conditions can be applied to (14). A full formal asymptotic analysis of this system has been performed in [5], and it is found that in this layer f, $\boldsymbol{Q}$ and H are constant to leading order, with the boundary layer simply serving to match the derivative of the outer solution for f (i.e. that given by (13)) to the Neuman boundary condition (10). Hence the boundary conditions appropriate to (14) are

$$\begin{aligned} \boldsymbol{Q}\cdot\boldsymbol{n} &= 0, & (15)\\ H &= H_0, \quad \text{on } \partial\Omega & (16) \end{aligned}$$

on $\partial\Omega$.

It was shown formally in [5] that in one dimension the solution to (14)-(16) ceases to exist for $H_0 > 1/\sqrt{2}$ (at which point $|\boldsymbol{Q}|$ passes through unity). A local analysis about this point revealed a turning round of the solution branch.

Furthermore, a linear stability analysis using the time-dependent Ginzburg-Landau equations revealed that while the solution was stable to one-dimensional perturbations until the nose was reached (i.e. as long as $|\boldsymbol{Q}|$ is less than unity and the solution exists), it was unstable to short-wavelength two-dimensional perturbations whenever $|\boldsymbol{Q}| > 1/\sqrt{3}$. It is conjectured that it is this instability which leads to the generation of vortices.

The linear stability analysis gives a simple criterion for the stability of the solution of (14)-(16): it will be stable if and only if $|\boldsymbol{Q}| < 1/\sqrt{3}$ everywhere.

Here we consider the problem (14)-(16) on a general domain Ω. We show that the solution exists and is unique so long as $|\boldsymbol{Q}| \leq 1/\sqrt{3}$, and that the maximum value of $|\boldsymbol{Q}|$ is always achieved on the boundary $\partial\Omega$. This means that the solution will always become unstable first on the boundary, i.e. vortices always enter a sample from the boundary, and are not spontaneously nucleated in the interior.

The proofs involve in particular refined forms of the maximum principle.

Before we proceed with our analysis we recast the problem in terms of the variables of primary interest, namely the magnetic field H, and the square modulus of the potential $\boldsymbol{Q}$, which we denote by u. If $\boldsymbol{Q} = (Q_1, Q_2, 0)$ then $H = \partial Q_2/\partial x - \partial Q_1/\partial y$, $u = Q_1^2 + Q_2^2$, and (14)-(16) become

$$-\frac{\partial H}{\partial y} = (1-u)Q_1 \qquad \text{in } \Omega, \tag{17}$$

$$\frac{\partial H}{\partial x} = (1-u)Q_2 \qquad \text{in } \Omega, \tag{18}$$

$$H = H_0 \qquad \text{on } \partial\Omega, \tag{19}$$

(note that (15) follows from (16) and (17)-(18) since H_0 is constant). Hence

$$\nabla \cdot \left(\frac{\nabla H}{1-u}\right) = H \qquad \text{in } \Omega, \tag{20}$$

$$u(1-u)^2 = |\nabla H|^2 \qquad \text{in } \Omega, \tag{21}$$

$$H = H_0 \qquad \text{on } \partial\Omega. \tag{22}$$

3 Qualitative properties of the solutions of (20)-(22).

We first want to know whether the solutions of (20)-(22) are bounded. This is given by

Proposition 3.1. *Let $H_0 > 0$ and (H, u) be a solution of:*

$$\operatorname{div}\left(\frac{1}{1-u}\nabla H\right) = H \quad \text{in } \Omega, \tag{23}$$

$$u(1-u)^2 = |\nabla H|^2 \quad \text{in } \Omega, \tag{24}$$

$$H = H_0 \quad \text{on } \partial\Omega, \tag{25}$$

such that $0 \leq u < 1$. Then $H < H_0$ in Ω.

Proof: Indeed, equation (24) implies $|\nabla H|^2 \leq 4/27$. Therefore $H \leq C\ \mathrm{diam}(\Omega) + H_0$. We prove that the maximum of H cannot be reached at an interior point.

By contradiction, suppose H reaches a maximum at a point P inside Ω. Then $\nabla H = 0$ and $u = 0$ at P. If $u(P) = 0$, then $u < 1/3$ in a neighborhood $\mathcal{U}$ of P, whence (23) can be written as an elliptic equation (see Section 4) and the maximum principle used in $\mathcal{U}$ for (20)-(22) gives that H is constant in $\mathcal{U}$. Then $u = 0$ on $\mathcal{U}$ and a connexity argument shows that H is constant in Ω, with $u \equiv 0$. Then, (20)-(22) $\Rightarrow H \equiv 0$ and $H_0 = 0$.

Remark: Notice that when $u = 1$ equation (23) is not properly defined. We may consider a situation where $u = 1$ at an isolated point P and prove that such a situation cannot occur if H is not singular at P. Consider a level curve l of H around P. Then

$$\int_l \boldsymbol{Q} \cdot \boldsymbol{ds} = \int_S \text{curl } \boldsymbol{Q} \cdot \boldsymbol{dS} = \int_S H \, dS, \qquad \text{where } l = \partial S. \tag{26}$$

We may take l very close to P, such that $\text{diam}(S) = \epsilon \to 0$. Since u is close to unity (i.e. $|\ \boldsymbol{Q}\ |$ is close to unity), and since $\boldsymbol{Q} \cdot \boldsymbol{ds} = |Q| \, ds$, we deduce that $\int_l \boldsymbol{Q} \cdot \boldsymbol{ds} \geq \mathcal{O}(\epsilon)$ and $\int_S H dS \leq \mathcal{O}(\epsilon^2)$, contradicting (26).

We now prove that for a regular domain Ω, there is no solution of (20)-(22) for sufficiently large H_0.

Theorem 3.2. *There is a critical magnetic field H_M such that for $H_0 > H_M$ the system (20)-(22) does not have a solution.*

Remark. When $H_0 > H_M$ we expect there to be no superconducting (i.e. $f > 0$ everywhere) solution of the Ginzburg-Landau equations (2)-(3). There may however be vortex solutions (in which f varies on a lengthscale $1/\kappa$ and is zero at isolated points) of that system. Note also that there is the normal solution $f = 0$ ($u = 1$). Neither the vortex solution nor the normal solution are solutions of the reduced system (20)-(22).

Proof. Let us integrate (20) over Ω:

$$\int_\Omega H = \int_\Omega \text{div } (\frac{1}{1-u} \nabla H) = \int_{\partial\Omega} \frac{\nabla H \cdot \nu}{1-u} < \int_{\partial\Omega} \frac{|\nabla H|}{1-u}. \tag{27}$$

Since (21) gives $\frac{|\nabla H|}{1-u} = \sqrt{u}$, the right-hand side of (27) is less than the length $|\partial\Omega|$ of $\partial\Omega$. Now, since $|\nabla H| \leq \sqrt{4/27}$ the left-hand side of (27) is more than $|\Omega|(H_0 - \sqrt{4/27}\text{diam}(\Omega)/2)$. Consequently, if there is a solution of (20)-(22) then $H_0 < |\partial\Omega|/|\Omega| + \sqrt{4/27}\text{diam}(\Omega)/2)$. Which completes the proof of the theorem.

4 A multi-dimensional elliptic model

We are interested now in those solutions (H, u) for which $u < 1/3$ on Ω. Under this hypothesis, the function $|\nabla H|^2 = u(1-u)^2$, giving $|\nabla H|^2$ in terms of u, may be inverted. For $|\nabla H|^2 < 4/27$ we denote by $k(|\nabla H|^2)$ the unique $u < 1/3$ such that $u(1-u)^2 = |\nabla H|^2$. We introduce $F(|\nabla H|^2) = \frac{1}{1-k(|\nabla H|^2)}$ to get a new formulation of the problem:

$$\begin{aligned} \mathrm{div}\,(F(|\nabla H|^2)\nabla H) &= H \quad \text{in } \Omega, & (28)\\ H &= H_0 \quad \text{on } \partial\Omega. & (29)\end{aligned}$$

We will be addressing the question of uniqueness, maximum of u (or equivalently $|\nabla H|$), and existence for small H_0. We note that all the results presented in this section are valid in $\mathbb{R}^n$.

Theorem 4.1. *There is at most one solution of*

$$\begin{aligned} \mathrm{div}\,(F(|\nabla H|^2)\nabla H) &= H \quad \text{in } \Omega, & (30)\\ H &= H_0 \quad \text{on } \partial\Omega, & (31)\end{aligned}$$

with $|\nabla H|^2 < 4/27$ *in* Ω.

Proof: We recall that $k(|\nabla H|^2) = 1 - \frac{1}{F(|\nabla H|)}$ is the inverse of $u :\mapsto u(1-u)^2$ for $u \in [0, 1/3)$. The functions k and F are defined on $(0, 4/27)$ and k', $F' > 0$ on this interval. As F is $\mathcal{C}^\infty$, a bootstrap argument shows that any solution of (30)-(31) is $\mathcal{C}^\infty$. Equation (30) can be written as:

$$F(|\nabla H|^2)\Delta H + 2F'(|\nabla H|^2)^t\nabla H(D^2H)\nabla H = H\ . \tag{32}$$

Let H_1 and H_2 be two solutions of (30)-(31), let $W = H_2 - H_1 + \mu$. For μ large enough, $W > 0$. There is a minimum $\mu = \mu_0$ for which $W \geq 0$. If $\mu_0 > 0$, then $\exists P \in \Omega$ such that $W(P) = 0$. Then, at P we have $\nabla H_1 = \nabla H_2 = B$ and we may compute at P:

$$F(B^2)\Delta W + 2F'(B^2)^t B(D^2W)B = -\mu_0,$$

that is $LW = -\mu_0$, where L is an elliptic operator since $F' > 0$. Since P is a point of minimum of W, $LW \geq 0$, a contradiction as $\mu_0 > 0$.

Remark: Notice that in the language of fully nonlinear operators, the equation (32) reads $\Phi(x, u, Du, D^2u) = 0$. In this framework, the function $\Phi(x, u, t, p)$ defines an elliptic operator if the matrix (Φ_{p_i,p_j}) is positive definite and $\Phi_u \leq 0$. Here, Φ_{p_i,p_j} takes the value $\alpha\delta_{ij} + \beta b_i b_j$ where $\alpha = F(B^2) > 0$, $\beta = F'(B^2) \geq 0$ and $B = (b_i)$, and also $\Phi_H = -1$. Therefore the operator defined by (32) is elliptic and uniqueness to the Dirichlet problem (30)-(31) is a consequence of this property. For the sake of completness we have given here a direct proof of the uniqueness of the solution of the Dirichlet problem.

We now come to the determination of the maximum of $|\nabla H|$.

Theorem 4.2. *When $0 < u < 1/3$, $|\nabla H|$ reaches its maximum on the boundary.*

Proof: The proof relies essentially on a version of the maximum principle for the fully nonlinear equation (32). For any direction $\boldsymbol{v}$ in $I\!R^2$, $(|\boldsymbol{v}|^2 = 1)$, let $V = \partial H/\partial v = \nabla H \cdot \boldsymbol{v}$. We may differentiate (32) with respect to $\boldsymbol{v}$ to get the linear partial differential equation

$$F(|\nabla H|^2)\Delta V + 2F'(|\nabla H|^2)^t\nabla H(D^2V)\nabla H + K(\nabla H, D^2H)\cdot\nabla V = V.$$

where K is a vector valued function. This is an elliptic operator satisfying the maximum principle (F' is positive). Then V reaches its maximum at a boundary point. This is true for any direction $\boldsymbol{v}$, which completes the proof of the theorem.

Theorem 4.3. *There exists $H_0^* > 0$ such that for any $0 \leq H_0 \leq H_0^*$ there is a solution of:*

$$\begin{aligned} \operatorname{div}(F(|\nabla H|^2)\nabla H) &= H \quad \text{in } \Omega, &(33)\\ H &= H_0 \quad \text{on } \partial\Omega, &(34) \end{aligned}$$

such that $|\nabla H|^2 \leq 4/27$. The critical value $H_0^* > 0$ of the magnetic field is reached when $|\nabla H|^2$ first reaches the value $4/27$.

Corollary 4.4. *For $0 \leq H_0 \leq H_0^*$, there is one and only one solution satisfying $|\nabla H|^2 \leq 4/27$ in Ω.*

A natural framework to prove the existence of solutions of (33)-(34) is the Schauder theory for Hölder spaces $C^{k,\delta}$ (here δ is a fixed number such that $0 < \delta < 1$). In order to prove Theorem 4.3, we need to find $H \in \mathcal{C}_0^{2,\delta} \equiv \{H \in \mathcal{C}^{2,\delta}(\Omega),\ H = 0 \text{ on } \partial\Omega\}$ such that $|\nabla H|^2 \leq 4/27$ and div $(F(|\nabla H|^2)\nabla H) = H + H_0$, or equivalently,

$$F(|\nabla H|^2)\Delta H + 2F'(|\nabla H|^2)^t\nabla H(D^2H)\nabla H = H + H_0\ .$$

As pointed out before, this is an equation of the type $\Phi(H, \nabla H, D^2 H) = H_0$ where $\Phi(u, t, p)$ is a $\mathcal{C}^\infty$ real function for $|t|^2 < 4/27$. The operator $g(H) = \Phi(H, \nabla H, D^2 H)$ acts from $\mathcal{C}_0^{2,\delta}$ into $\mathcal{C}^{0,\delta}$. Let us linearize g. For $V \in \mathcal{C}_0^{2,\delta}$ we compute

$$g'(H)\,V = F(|\nabla H|^2)\Delta V + 2F'(|\nabla H|^2)^t \nabla H D^2 V \nabla H + K(\nabla H, D^2 H) \cdot \nabla V - V$$

(Where $K = \Phi_t$ is the partial derivative of Φ with respect to the second variable t.)

We remark that since $|\nabla H|^2 \leq 4/27$ we have $F' \geq 0$ and g' is a linear elliptic operator. Moreover it satisfies the maximum principle as $g'_V = -1 < 0$. We intend to apply the local inversion theorem to find an inverse $A : \mathcal{C}^{0,\delta} \to \mathcal{C}_0^{2,\delta}$ such that

$$g'(H)\,.\,A = id_{\mathcal{C}^{0,\delta}}.$$

¿From the maximum principle it follows that

$$\text{for } V \in \mathcal{C}_0^{2,\delta}, \quad g'(H)\,V = 0 \Rightarrow V = 0.$$

Hence, by the usual Schauder theory and Fredholm alternative, we know that for any given $\phi \in \mathcal{C}^{0,\delta}$ there exists a unique $V \in \mathcal{C}_0^{2,\delta}$, which we denote by $V = A\phi$ such that

$$g'(H)\,V = \phi\,.$$

Hence,

$$g'(H)\,.\,A = id_{\mathcal{C}^{0,\delta}}.$$

In this situation, we may apply the implicit function theorem (see e.g., [2]) using H_0 as a parameter. Therefore since for $H_0 = 0$, zero is a solution of the problem, we obtain a branch of solution for all values of H_0 such that $0 \leq H_0 \leq H_0^*$. The field H_0^* is the value of the magnetic field applied at the boundary of the domain for which the square of the modulus of the gradient of the solution first reaches the value $4/27$, at which points $F'(|\nabla H|^2) \to +\infty$.

We remark that for $H_0 = H_0^*$ there are some points at the boundary, where $|\nabla H|^2 = 4/27$. However, inside Ω, $|\nabla H|^2 < 4/27$ and therefore the mappings g and $g'(H)$ are still defined in Ω, and although $g'(H)$ is no longer uniformly elliptic, it is uniformly elliptic on any strict-subdomain $\Omega' \subset \Omega$. This is a limiting case of the local inversion theorem, which is more delicate.

5 Complete study of the one dimensional case

We are now interested in the one dimensional formulation of problem (20)-(22), corresponding to an infinite plate of constant thickness. In the one-dimensional setting (20)-(22) read:

$$\left(\frac{1}{1-u}H'\right)' = H, \tag{35}$$

$$u(1-u)^2 = (H')^2, \tag{36}$$

under the condition $0 \leq u < 1$.

In this case, however, it is more convenient to work with the model (14)-(16) directly. We have $\boldsymbol{Q} = (0, Q(x), 0)$, with

$$Q'' = Q(1-Q^2), \tag{37}$$

$$Q'(\pm a) = H_0, \tag{38}$$

and $-1 < Q < 1$.

Lemma 5.1. *Any solution on $I = [-a, a]$ of:*

$$Q'' = Q(1-Q^2), \tag{39}$$

$$Q'(\pm a) = H_0 > 0, \tag{40}$$

with $-1 < Q < 1$ is odd ($Q(-x) = -Q(x)$).

Proof: For any solution of (40), there is a point x_0 where Q vanishes. Indeed, if not, either $1 > Q > 0$ on I and $Q'' > 0$ on I, which gives $Q'(+a) > Q'(-a)$, or $-1 < Q < 0$ and $Q'' < 0$ on I, which gives $Q'(+a) < Q'(-a)$. This contradicts the condition $Q'(+a) = Q'(-a)$.

Now, since $Q'' = \phi(Q)$ where ϕ is odd, we deduce that Q is symmetrical with respect to x_0:

$$Q(2x_0 - x) = -Q(x). \tag{41}$$

Since Q is monotonic on $(-a, x_0)$ and $(x_0, +a)$, and $Q'(+a) = Q'(-a) = H_0 > 0$, we have that Q and Q'' are positive on $(x_0, +a)$ (respectively Q and Q'' are negative on $(-a, x_0)$).

Finally, if we assume that $x_0 < 0$, we have from (41) that $Q'(-a) = Q'(2x_0 + a)$, and from (40) that $Q'(-a) = Q'(+a)$, which implies $2x_0 + a = a$, since $Q'' > 0$ on $(x_0, +a)$.

The case $x_0 > 0$ is treated by a similar argument.

The existence and uniqueness theorem in the one-dimensional setting is more general than in the multi-dimensional one which was restricted to the case $u < 1/3$ (or equivalently $|Q| < 1/\sqrt{3}$):

Theorem 5.2. *There exists $H_0^{**} > 0$ such that $\forall\, 0 \le H_0 \le H_0^{**}$, there is an unique solution of:*

$$Q'' = Q(1 - Q^2), \tag{42}$$

$$Q'(\pm a) = H_0, \tag{43}$$

such that $|Q| \le 1$.

Proof: We know that a solution of:

$$Q'' = Q(1 - Q^2) \tag{44}$$

on $[-a, +a]$ is odd $(Q(-x) = -Q(x))$. Let us define $Q'(0) = \alpha$ and make a shooting argument from 0 with the parameter α. An integration of (44) multiplied by Q' on $[0, +a]$ gives at $+a$:

$$\frac{(Q')^2(+a)}{2} = \frac{\alpha^2}{2} + \frac{Q^2(+a)}{2} - \frac{Q^4(+a)}{4}. \tag{45}$$

We differentiate this identity with respect to α:

$$Q'(+a)\partial Q'(+a)/\partial\alpha = \alpha + Q(+a)(1 - Q^2(+a))\partial Q(+a)/\partial\alpha. \tag{46}$$

¿From the boundary conditions we compute $\partial Q'(0)/\partial\alpha = 1$. Then, when $x > 0$ goes to 0, $\partial Q(x)/\partial\alpha \sim x$ and is positive.

We want to prove that $Q'(+a)$ is monotonic increasing with respect to α while $|Q| < 1$. By contradiction, if not, there is a $\alpha_0 > 0$ such that

$$\frac{\partial Q'(+a)}{\partial\alpha} \le 0 \tag{47}$$

For these values of a and α, Q' is positive. Indeed, while $|Q| < 1$ on $(0, +a)$ one has $Q'' > 0$ and then, Q' is monotonic increasing with respect to $x \in (0, +a)$ and $Q' > \alpha$ on $(0, +a)$.

Now, equation (46) gives that at $\alpha = \alpha_0$, $\partial Q(+a)/\partial\alpha < 0$ and $0 < Q(+a) < 1$. Therefore, $\partial Q/\partial\alpha$, which was positive for x small, changes sign at a point $a' < a$. We have then $\partial Q/\partial\alpha > 0$ on $[0, +a'))$ and $\partial Q/\partial\alpha(a') = 0$ which gives $(\partial Q/\partial\alpha)'(a') \le 0$. Finally, equation (46) gives $\partial(Q'(+a'))/\partial\alpha = \alpha/Q'(a')$ which is positive, and a contradiction, since $(\partial Q/\partial\alpha)'(a') = \partial(Q'(a'))/\partial\alpha$.

Let us now study where $|\nabla H|$ achieves its maximum in the one-dimensional case. We recall that:

$$|\nabla H| = H' = Q'' = Q(1-Q^2).$$

The maximum is therefore reached at points where $Q(1-Q^2)$ is maximum.

Theorem 5.3. *If $Q(+a) \leq 1/\sqrt{3}$ (that is $|Q| \leq 1/\sqrt{3}$ on $[-a,+a]$), then the maximum of $|\nabla H|$ is reached strictly at the boundary points $+a$ and $-a$.*
If $Q(+a) > 1/\sqrt{3}$, then the maximum of $|\nabla H|$ is reached inside $\Omega = [-a,+a]$ at points x_m and $-x_m$ where $Q(x_m) = 1/\sqrt{3}$, the maximum of $|\nabla H|$ is then $\sqrt{4/27}$.

6 More general existence and uniqueness in the case of a disc

In the case of a disc we can derive the following symmetry property:

Theorem 6.1 *If Ω is a disc B_R then any solution such that $|\nabla H|^2 \leq 4/27$ is spherically symetric: $H = H(r)$ with $H_r > 0$ for $0 < r < R$.*

Proof: This is a consequence of a result in [3], using the form (32) for the equation and noticing that it satisfies the assumptions in [3].

As in the one-dimensional case, for radial solutions it is more convenient to work with the model (14)-(16) directly. Then $\boldsymbol{Q} = Q(r)\boldsymbol{e}_\theta$, where Q satisfies

$$Q'' + \frac{Q'}{r} - \frac{Q}{r^2} = Q - Q^3, \tag{48}$$

with boundary conditions

$$Q(0) = 0, \quad Q'(R) + \frac{Q(R)}{R} = H_0. \tag{49}$$

For these solutions we are able to obtain more precise results.

Theorem 6.2. *There exists H_0^{**} such that for any H_0, $0 < H_0 < H_0^{**}$ there is a solution of (48)-(49) satisfying $|Q| < 1$ on $(0,R)$.*

Proof: Since the operator in (48) is singular at the origin we cannot use a direct shooting method from the origin as in the one dimensional case (Theorem 5.2).

Let Q be the unique solution of (48) defined on a maximal subset $(\alpha, R]$ of $(0,R]$ and satisfying the boundary conditions:

$$Q'(R) + \frac{Q(R)}{R} = H_0 \ , \quad Q(R) = Q_0. \tag{50}$$

We introduce the sets

$$I^1 = \{(H_0, Q_0) \in I\!R^+ \times (0,1),\ \text{st } \exists \zeta \in (0,R),\ Q(\zeta) = 1 \text{ and } 0 < Q(r) < 1 \text{ on } (\zeta, 1)\}$$

$$I^0 = \{(H_0, Q_0) \in I\!R^+ \times (0,1),\ \text{st } \exists \zeta \in (0,R),\ Q(\zeta) = 0 \text{ and } 0 < Q(r) < 1 \text{ on } (\zeta, 1)\}$$

Lemma 6.3. *I^1 and I^0 are open sets.*

Proof: It is sufficient to show that $Q'(\zeta) \neq 0$ in the definition of I^1 and I^0. For I^1, $Q'(\zeta) = 0$ together with $Q(\zeta) = 1$ implies $Q''(\zeta) > 0$ which contradicts the condition $0 < Q(r) < 1$ on $(\zeta, 1)$. For I^0, $Q'(\zeta) = 0$ together with $Q(\zeta) = 0$ gives $Q \equiv 0$, a contradiction.

Lemma 6.4. *I^1 and I^0 are non-empty.*

Proof:

(i) $I^1 \neq \emptyset$.

If $H_0 < Q_0/R$ then $Q'(R) < 0$. Noting that $Q''(r) > -Q'(r)/r$ for $0 < Q(r) < 1$, we have that Q' is negative and increasing $\forall r$, $\alpha < r < R$. Consequently either there exists $\zeta \in (\alpha, R)$, $Q(\zeta) = 1$ or Q is decreasing on $(0,R)$ and stays in the interval $(0,1)$, contradicting Lemma 6.5.

(ii) $I^0 \neq \emptyset$.

We remark that the function

$$q(r) = \left(r - \frac{R^3}{1+R^2}\right)\frac{(1+R^2)}{R}Q_0$$

satisfies $q'' > -\frac{q'}{r} + \frac{q}{r^2} + q$ for r in $(\frac{R^3}{1+R^2}, R)$, and is such that $q(R) = Q_0$. Choosing $H_0 > (2/R + R)Q_0$ gives $Q'(R) > q'(R)$, and hence $Q < q$ on $(\frac{R^3}{1+R^2}, R)$. Indeed, if $q - Q$ is maximum at a point $r_0 \in (\frac{R^3}{1+R^2}, R)$ then $q(r_0) > Q(r_0)$ and $q'(r_0) = Q'(r_0)$. This leads eventually $q''(r_0) > Q''(r_0)$ which contradicts the maximality of $q - Q$. Consequently $Q(\frac{R^3}{1+R^2}) < 0$ and $(H_0, Q_0) \in I^0$.

Lemma 6.5. *If $0 < Q(r) < 1$ for all $r \in (0,R)$ then $Q(0) = 0$.*

Proof: Equation (48) reads

$$\left(\frac{1}{r}(rQ)'\right)' = Q - Q^3$$

Since Q is bounded, an integration gives that $(rQ(r))' = rg(r)$ where g is a continuous function in $[0,1]$. Since Q is bounded an other integration gives $rQ(r) = \int_0^r tg(t)dt$. This gives the estimation $Q(r) < Cr$ for some constant C and completes the proof of the lemma.

To conclude the proof of the Theorem 6.2 we remark that points of $\partial I^1 \cap (0,+\infty)\times(0,1)$ give solutions Q of (48), (50) staying between 0 and 1. Lemma 6.5 implies that Q is in fact a solution of (48)-(49). Note that I^1 may not be connected. Let J^1 be the connected component of I^1 containing $H_0 < Q_0/R$ and let H_0^{**} be the maximum of H_0 over points in J^1. Then, for all $H_0 < H_0^{**}$ there is a point (H_0, Q_0) of $\partial J_1 \cap (0,+\infty)\times(0,1)$ associated to a solution of (48)-(49). Note that for the solutions given by ∂J_1, the value of $Q(R)$ ranges from 0 to 1. We have the following result of uniqueness in the class of radial solutions.

Theorem 6.6 *If $R \le \frac{1}{\sqrt{2}}$ then there is a unique solution of (48)-(49) such that $|Q| \le 1$.*

Proof: Assume by contradiction that there are two solutions Q_1 and Q_2 of (48)-(49). The difference $w = Q_1 - Q_2$ is a solution of

$$w'' + \frac{w'}{r} = w\left(\frac{1}{r^2} + 1 - (Q_1^2 + Q_1Q_2 + Q_2^2)\right)$$

When $r < \frac{1}{\sqrt{2}}$ this reads $w'' + \frac{w'}{r} = Cw$ where C is positive. The maximum principle and the boundary conditions $w(0) = 0$ and $w'(R) + \frac{w(R)}{R} = 0$ give that $w = 0$.

Remark: For H_0 large enough, the solution of (48)-(49) is such that $u = Q^2 > 1/3$. In this case the maximum of u is still attained on the boundary, and the region $\{u > 1/3\}$ comprises an annulus adjacent to the boundary. However the maximum of $|\nabla H|$ is reached in the interior (the points where Q crosses the value $1/\sqrt{3}$).

7 Open Problems

The study of the solutions of system (20)-(22) raises a series of open questions which we now list. These problems are motivated by the physical model and lead to very interesting mathematical questions. The first concerns the derivation of the model; the others refer to the transition towards the mixed state of type-II superconductors.

* The system (20)-(22) is derived from the Ginzburg-Landau equations by a formal limiting process. Are there solutions of the Ginzburg-Landau equations which converge to a solution of (20)-(22) as κ tends to $+\infty$. Some preliminary results indicate that the global minimizer of the Ginzburg-Landau free energy would not converge to a solution of (20)-(22) as κ tends to $+\infty$. Hence this problem might be delicate.

The situation where u is larger than $1/3$ is not well understood in the multidimensional setting:

* Are there any solutions of (20)-(22) for H_0 small such that u is larger than $1/3$ in some region of Ω. (We know already that there is a unique solution such that $u < 1/3$ in Ω.)

* When $H > H_0^*$, does the system (20)-(22) have a solution for a general domain?

* In situations where $H > H_0^*$ vortices are expected to appear. These are points where the order parameter Ψ vanishes. At these points χ in (7) is not well defined, and further modelling is needed to take this fact into account.

8 Conclusion

We have shown the existence and uniqueness of the superconducting solution of the Ginzburg-Landau equations, in the formal limit as $\kappa \to \infty$, for values of the applied magnetic field H_0 up to some critical value H_0^*, at which point the modulus of the potential $\boldsymbol{Q}$ first becomes equal to $1/\sqrt{3}$. At this value of the magnetic field the superconducting solution becomes linearly unstable, and it is conjectured that this instability will lead to the formation of vortices.

We have shown that the point of first instability will occur on the boundary of the domain, so that vortices will always enter a sample from the boundary and will not be spontaneously created in the interior.

In the one-dimensional and radially-symmetric cases we were able to further establish the existence of the superconducting solution up to a second critical field $H_0^{**} > H_0^*$, at which point $|\boldsymbol{Q}|$ first becomes equal to 1. In one dimension the solution was shown to be unique, while for the radially-symmetric case uniqueness was shown providing the radius of the disc is sufficiently small.

For values of the applied magnetic field leading to values of $|\boldsymbol{Q}|$ greater than $1/\sqrt{3}$ we found that the maximum of $|\boldsymbol{Q}|$ was still obtained on the boundary of the domain. The region $|\boldsymbol{Q}| > 1/\sqrt{3}$ comprised two intervals adjacent to the boundary and an annulus adjacent to the boundary in the one-dimensional and radially symmetric cases respectively (see figure 3). These will be the regions of instability. For such values of the magnetic field the maximum of $|\nabla H|$ (which is also the maximum of the current density) no longer coincides with the maximum of $|\boldsymbol{Q}|$, but occurs at the interior points at which $|\boldsymbol{Q}| = 1/\sqrt{3}$. Thus the onset of instability coincides with the point of maximum of the current density leaving the boundary of the sample.

Finally, the model leads to many interesting open problems.

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Figure Captions

Figure 1. The preferred state of a superconductor as a function of the applied magnetic field H_0 and the Ginzburg-Landau parameter κ, a material constant which determines the type of superconducting material; $\kappa < 1/\sqrt{2}$ describes are known as type-I superconductors, while $\kappa > 1/\sqrt{2}$ describes what are known as type-II superconductors.

Figure 2. Schematic diagram of normal filaments in the mixed state of a type-II superconductor, showing the direction of the magnetic field $\boldsymbol{H}$ and the superconducting current $\boldsymbol{j}$.

Figure 3. The potential Q and the current $j = Q(Q^2 - 1)$ for the system (37)-(38), with $H_0 = 1$. The solution will be unstable in the region $|Q| > 1/\sqrt{3}$.