Nucleation of Superconductivity in Decreasing Fields I

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Abstract

The bifurcation from a normally conducting to a superconducting state as an external magnetic field is lowered is examined using the Ginzburg-Landau theory. The results for three specific examples are reviewed, extended and unified in the framework of a systematic perturbation theory introduced in [13].

1 Introduction

If a superconducting body is placed in a sufficiently strong magnetic field, superconductivity will be destroyed and the body will revert to the normally conducting (normal) state. If the field is now reduced there will be a point at which the body begins to change back into the superconducting state. The way that the phase change occurs will depend upon the geometry of the sample and the type of superconductor, that is, whether it is of type I or type II.

A type I superconductor is expected to change phase by means of inwardly propagating phase boundaries separating normal regions from advancing superconducting regions, as described in [4]. The thickness of such a phase boundary is known as the penetration depth, and is typically of the order of 1 μm. The magnetic field is expelled from the superconducting regions, and the body will eventually be in a wholly superconducting state with the magnetic field in its interior being zero apart from in edge boundary layers (fig. 1a) (this is known as the Meissner Effect).

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Such a change of phase is expected to occur as the magnetic field passes through a value H_c, known as the thermodynamic critical field, which is the field at which the free energy of the superconducting state becomes lower than the free energy of the normal state. In fact it is found that a certain amount of 'supercooling' can occur, with the normal state persisting until the applied field reaches some lower value.

A type II superconductor, on the other hand, does not change phase in the above manner, but rather expels the magnetic field gradually as the strength of the applied field is reduced, as indicated by fig. 1b. It is found that for fields $h_{c_1} < h < h_{c_2}$ the superconductor is in a "mixed" state, consisting of normal and superconducting regions which are of a size comparable to the penetration depth. A typical observation for magnetic fields close to h_{c_1} is that the mixed state consists of cores of normal material, each surrounded by a vortex of superconducting current, embedded in a superconducting matrix [8]. Thus type II superconductors can be thought of as having three phases - normal, superconducting and mixed, whereas type I superconductors have only normal and superconducting phases.

The aim of this paper is to consider this behaviour in the light of the Ginzburg-Landau model [6, 10, 11]. We consider a superconducting body placed in an applied magnetic field and examine the bifurcation from the normal state solution to a partially superconducting solution as the external magnetic field is varied. This bifurcation has been previously studied in [1], [12], [13], [15], and [16] and the present paper will mainly form a review of this work, which centres on three specific examples, namely one-dimensional solutions on an infinite domain, one-dimensional solutions on a halfspace, and periodic two-dimensional solutions. We fill in some of the remaining gaps, and cast each example in the framework of the systematic perturbation theory introduced by Millman & Keller [13], who considered a body of arbitrary shape. In a following paper [5], hereafter referred to as II, we use the time-dependent Ginzburg-Landau theory developed by Gor'kov & Éliashberg [11] to perform linear and weakly-nonlinear stability analyses of the solution near the bifurcation points, and we consider the implications for each of our examples.

The analysis of the bifurcation will indicate the difference between type I and type II superconductors. It will explain the formation of the mixed state in a type II superconductor, and the "supercooling" observed in the change of phase of a type I superconductor.

We note that since none of our examples is on a bounded superconducting domain, they do not strictly follow from the general analysis of an arbitrary body in three dimensions. However, the methodology of the general analysis can be employed exactly, and the results are the same.

2 Nucleation of Superconductivity in Decreasing Fields

2.1 Superconductivity in a Body of Arbitrary Shape in an External Magnetic Field

This problem was considered by Millman & Keller [13]. We work through their analysis as it provides the basic framework for the examples which follow, and for the linear and weakly nonlinear stability analyses of II.

Consider a superconducting body occupying a region Ω bounded by a surface $\partial\Omega$, placed in an originally uniform magnetic field of strength h. Under isothermal conditions the steady state Ginzburg-Landau equations, together with boundary and other conditions are, in appropriate units and on the lengthscale of the penetration depth [10, 6]

$$
\begin{aligned}
((i/\kappa)\nabla + \boldsymbol{A})^2\Psi &= \Psi(1- \mid\Psi\mid^2) \quad \text{in } \Omega, & (1)\\
-(\text{curl})^2\boldsymbol{A} &= (i/2\kappa)(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + \mid\Psi\mid^2 \boldsymbol{A} \quad \text{in } \Omega, & (2)\\
(\text{curl})^2\boldsymbol{A} &= \boldsymbol{0}, \qquad \text{outside } \Omega, & (3)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi &= 0 \quad \text{on } \partial\Omega & (4)\\
[\boldsymbol{n}\wedge\boldsymbol{A}] &= \boldsymbol{0}, & (5)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{A}] &= \boldsymbol{0}, & (6)\\
\text{curl } \boldsymbol{A} &\to h\hat{\boldsymbol{z}} \quad \text{as } r\to\infty. & (7)
\end{aligned}
$$

Here Ψ is the (complex) superconducting order parameter, such that $\mid\Psi\mid^2$ represents the number density of superconducting electrons, varying from $\mid\Psi\mid= 0$ (normal) to $\mid\Psi\mid^2= 1$ (superconducting); $\boldsymbol{A}$ is the (real) magnetic vector potential (so that the magnetic field is given by $\boldsymbol{H} = \text{curl } \boldsymbol{A}$), $\hat{\boldsymbol{z}}$ is a unit vector in the z-direction, r is the distance from the origin, $\boldsymbol{n}$ is the outward normal on $\partial\Omega$, $[\,]$ stands for the jump in the enclosed quantity across $\partial\Omega$, and $*$ denotes complex conjugation. The constant κ determines the type of superconducting material, $\kappa < 1/\sqrt{2}$ describing a type I superconductor, and $\kappa > 1/\sqrt{2}$ a type II superconductor. Equation (3) states that there is no current in the region external to the superconductor. Equations (5) and (6) are the usual boundary conditions on the vector potential at an interface between two magnetic media. Equation (7) states that the field reduces to the applied field far from the body.

The boundary condition (4) is the natural boundary condition associated with the minimisation of free energy which leads to the Ginzburg-Landau equations, and corresponds to the superconductor being adjacent to a vacuum or insulator. In [9] it was proposed that this condition should be modified to

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi = -(i/d)\Psi \quad \text{on } \partial\Omega \tag{8}$$

for a boundary between a superconductor and another material, where d is a constant dependent on the external material, ranging from $d = \infty$ when the superconductor is adjacent to an insulator or vacuum, to $d = 0$ when the superconductor is adjacent to a magnetic material. In [13] only the case $d = \infty$ is considered, but the extra complication introduced by finite d is not great.

Equations (1)-(8) are gauge invariant in the sense that if $(\Psi, \boldsymbol{A})$ is a solution, then so is $(e^{i\kappa\eta}\Psi, \boldsymbol{A} + \nabla\eta)$, for an arbitrary function η. This freedom allows us to impose the additional gauge constraint

$$\operatorname{div} \boldsymbol{A} = 0, \tag{9}$$

which proves convenient in later calculations. We note that even (9) does not fix the solution uniquely (since $(e^{i\kappa\eta}\Psi, \boldsymbol{A} + \nabla\eta)$ will also be a solution, where η is any harmonic function), but this does not affect the physical quantities since curl $\boldsymbol{A}$ and $\mid \Psi \mid$ are invariant under such a transformation.

The solution of (1)-(9) that corresponds to the normal state is

$$\Psi \equiv 0, \qquad \boldsymbol{A} = h\boldsymbol{A}_N, \tag{10}$$

where, by inserting (10) into (1)-(9), $\boldsymbol{A}_N$ satisfies

$$\begin{aligned}
(\mathrm{curl})^2 \boldsymbol{A}_N &= \boldsymbol{0} \quad \text{except on } \partial\Omega, && (11)\\
[\boldsymbol{n} \wedge \boldsymbol{A}_N] &= \boldsymbol{0}, && (12)\\
[\boldsymbol{n} \wedge (1/\mu)\mathrm{curl}\ \boldsymbol{A}_N] &= \boldsymbol{0}, && (13)\\
\mathrm{curl}\ \boldsymbol{A}_N &\to \hat{\boldsymbol{z}} \quad \text{as } r \to \infty, && (14)\\
\mathrm{div}\ \boldsymbol{A}_N &= 0. && (15)
\end{aligned}$$

Equations (11)-(15) correspond to the problem of determining the vector potential for a permeable body carrying no current placed in an originally uniform unit external magnetic field, and have well-known methods of solution.

Millman & Keller go on to seek a superconducting solution (i.e. one in which $\Psi \not\equiv 0$) in which $\mid \Psi \mid \ll 1$, which depends continuously on a parameter ϵ (which measures the magnitude of $\mid \Psi \mid^2$), and which reduces to (10) for $\epsilon = 0$. They introduce ϵ, ψ and $\boldsymbol{a}$ through the equations

$$\begin{aligned}
\Psi &= \epsilon^{1/2}\psi, && (16)\\
\boldsymbol{A} &= h\boldsymbol{A}_N + \epsilon\boldsymbol{a}, \quad 0 < \epsilon \ll 1. && (17)
\end{aligned}$$

The relative scalings of ψ and $\boldsymbol{a}$ here are motivated by requiring that $\Psi \mid \Psi \mid^2$ balances with $\boldsymbol{A} - h\boldsymbol{A}_N$ in equation (1). We expand h, $\boldsymbol{a}$ and ψ in powers of ϵ:

$$
\begin{aligned}
h &= h^{(0)} + \epsilon h^{(1)} + \cdots, && (18)\\
\boldsymbol{a} &= \boldsymbol{a}^{(0)} + \epsilon \boldsymbol{a}^{(1)} + \cdots, && (19)\\
\psi &= \psi^{(0)} + \epsilon \psi^{(1)} + \cdots. && (20)
\end{aligned}
$$

Insertion of (16)-(20) into (1)-(9) yields the leading-order problem

$$
\begin{aligned}
((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)^2\psi^{(0)} - \psi^{(0)} &= 0 \quad \text{in } \Omega, && (21)\\
-(\mathrm{curl})^2\boldsymbol{a}^{(0)} &= (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(0)} - \psi^{(0)}\nabla\psi^{(0)*}) \\
&\quad + h^{(0)} \mid \psi^{(0)} \mid^2 \boldsymbol{A}_N \quad \text{in } \Omega, && (22)\\
(\mathrm{curl})^2\boldsymbol{a}^{(0)} &= \boldsymbol{0} \quad \text{outside } \Omega, && (23)\\
\boldsymbol{n} \cdot ((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi^{(0)} &= -(i/d)\psi^{(0)} \quad \text{on } \partial\Omega, && (24)\\
[\boldsymbol{n} \wedge \boldsymbol{a}^{(0)}] &= \boldsymbol{0}, && (25)\\
[\boldsymbol{n} \wedge (1/\mu)\mathrm{curl}\ \boldsymbol{a}^{(0)}] &= \boldsymbol{0}, && (26)\\
\mathrm{curl}\ \boldsymbol{a}^{(0)} &\to \boldsymbol{0} \text{ as } r \to \infty, && (27)\\
\mathrm{div}\ \boldsymbol{a}^{(0)} &= 0. && (28)
\end{aligned}
$$

Equations (21) and (24) constitute an eigenvalue problem for $h^{(0)}$. Because of the gauge invariance of the equations mentioned previously the eigenvalues are independent of the gauge of $\boldsymbol{A}_N$. In the examples we consider they will also be found to be real and discrete, as was postulated by [13]. The upper critical field h_{c_2} is defined to be the largest positive eigenvalue. Let the normalised eigenfunction corresponding to $h^{(0)}$ be θ, i.e. θ is such that

$$\int_\Omega \mid \theta \mid^2 \, dV = 1.$$

Then $\psi^{(0)} = C\theta$ where C is constant, and $\boldsymbol{a}^{(0)} = \mid C \mid^2 \hat{\boldsymbol{a}}^{(0)}$, where

$$-(\mathrm{curl})^2\hat{\boldsymbol{a}}^{(0)} = (i/2\kappa)(\theta^*\nabla\theta - \theta\nabla\theta^*) + h^{(0)} \mid \theta \mid^2 \boldsymbol{A}_N \quad \text{in } \Omega, \tag{29}$$

and $\hat{\boldsymbol{a}}$ satisfies (23) and (25)-(28), which is the problem of determining the vector potential $\hat{\boldsymbol{a}}^{(0)}$ due to a permeable body carrying a specified real current distribution, since the right-hand side of (29) is known having determined θ. Again, well-known methods of solution are available.

We have now determined the upper critical field h_{c_2} and leading order approximations to ψ and $\boldsymbol{a}$, once we have determined the constant C.

Equating coefficients of ϵ in (1)-(9) yields

$$((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)^2\psi^{(1)} - \psi^{(1)} = -\mid\psi^{(0)}\mid^2\psi^{(0)} - 2h^{(1)}h^{(0)}\mid\boldsymbol{A}_N\mid^2\psi^{(0)}$$
$$- 2h^{(1)}(i/\kappa)(\boldsymbol{A}_N\cdot\nabla\psi^{(0)}) - 2h^{(0)}\psi^{(0)}(\boldsymbol{A}_N\cdot\boldsymbol{a}^{(0)}) - 2(i/\kappa)(\boldsymbol{a}^{(0)}\cdot\nabla\psi^{(0)}) \text{ in } \Omega, \quad (30)$$

$$-(\text{curl})^2\boldsymbol{a}^{(1)} = (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(1)} + \psi^{(1)*}\nabla\psi^{(0)})$$
$$-(i/2\kappa)(\psi^{(0)}\nabla\psi^{(1)*} + \psi^{(1)}\nabla\psi^{(0)*}) + h^{(0)}\boldsymbol{A}_N(\psi^{(0)}\psi^{(1)*} + \psi^{(0)*}\psi^{(1)})$$
$$+ h^{(1)}\mid\psi^{(0)}\mid^2\boldsymbol{A}_N + \mid\psi^{(0)}\mid^2\boldsymbol{a}^{(0)} \text{ in } \Omega, \quad (31)$$

$$\begin{aligned}
(\text{curl})^2\boldsymbol{a}^{(1)} &= \boldsymbol{0} \text{ outside } \Omega, && (32)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi^{(1)} + (i/d)\psi^{(1)} &= -\boldsymbol{n}\cdot(\boldsymbol{a}^{(0)} + h^{(1)}\boldsymbol{A}_N)\psi^{(0)} \text{ on } \partial\Omega, && (33)\\
[\boldsymbol{n}\wedge\boldsymbol{a}^{(1)}] &= \boldsymbol{0}, && (34)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}^{(1)}] &= \boldsymbol{0}, && (35)\\
\text{curl } \boldsymbol{a}^{(1)} &\to \boldsymbol{0} \text{ as } r\to\infty, && (36)\\
\text{div } \boldsymbol{a}^{(1)} &= 0. && (37)
\end{aligned}$$

Assuming $\psi^{(1)}$ and $h^{(1)}$ were known these equations would again correspond to the problem of determining the vector potential due to a permeable body carrying a known current distribution. Thus $\boldsymbol{a}^{(1)}$ is fixed once $\psi^{(1)}$ and $h^{(1)}$ are given.

Now (30) and (33) are inhomogeneous versions of (21) and (24) and therefore have a solution if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying both sides of (30) by $\psi^{(0)*}$ and integrating over Ω. After considerable manipulation, integration by parts, and use of (15), (21), (22), (24), (28) and (33) the result is

$$\begin{aligned}
h^{(1)} &= \frac{\int_\Omega\mid\psi^{(0)}\mid^4\,dV - 2\int_\Omega\boldsymbol{a}^{(0)}\cdot(\text{curl})^2\boldsymbol{a}^{(0)}\,dV}{2\int_\Omega\boldsymbol{A}_N\cdot(\text{curl})^2\boldsymbol{a}^{(0)}\,dV}, && (38)\\
&= \mid C\mid^2\frac{\int_\Omega\mid\theta\mid^4\,dV - 2\int_\Omega\hat{\boldsymbol{a}}^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}^{(0)}\,dV}{2\int_\Omega\boldsymbol{A}_N\cdot(\text{curl})^2\hat{\boldsymbol{a}}^{(0)}\,dV}. && (39)
\end{aligned}$$

Thus $\mid C\mid$ is given in terms of $h^{(1)}$. Since $\mid C\mid^2 > 0$ this determines the sign of $h^{(1)}$, which in turn determines whether the solution exists for values of the magnetic field slightly greater than or slightly less than h_{c_2}. When $\mid C\mid$ is given by (39), (30) and (33) have a solution $\psi^{(1)}$, which will still contain an undetermined constant. This constant is determined by a solvability condition for the second order equations.

We have now determined a solution in the form

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots \tag{40}$$

$$\Psi = \epsilon^{1/2}[\psi^{(0)} + \epsilon\psi^{(1)} + \cdots] \tag{41}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \epsilon[\boldsymbol{a}^{(0)} + \epsilon\boldsymbol{a}^{(1)} + \cdots] \tag{42}$$

Equation (42) leads to a magnetic field

$$\text{curl } \boldsymbol{A} = \boldsymbol{H} = h \,\text{curl } \boldsymbol{A}_N + \epsilon \,\text{curl } \boldsymbol{a}^{(0)} + \cdots \tag{43}$$

If $\hat{\boldsymbol{z}} \cdot \text{curl } \boldsymbol{a}^{(0)} < 0$ in Ω, as is the case in the examples which follow, then the magnetic field within the body would be less than its value in the normal state, in agreement with fig. 1b.

Since θ is normalised we have that $\|\Psi\| = \left(\int_\Omega |\Psi|^2 \, dV\right)^{1/2} = \epsilon^{1/2} \mid C \mid$ and so $\|\Psi\|$ increases as $|h^{(0)} - h|^{1/2}$ for h close to $h^{(0)}$, giving the pitchfork bifurcation shown in Fig. 2.

2.2 A one-dimensional example

In one dimension we may choose Ψ to be real and the vector potential $\boldsymbol{A}$ to be directed along the y-axis, so that $\boldsymbol{A} = (0, A, 0)$ [10]. We consider an infinite sample. The Ginzburg-Landau equations (1), (2), together with the appropriate boundary conditions are then

$$\kappa^{-2}\Psi'' = \Psi^3 - \Psi + \Psi A^2, \tag{44}$$

$$A'' = \Psi^2 A, \tag{45}$$

$$\Psi' \to 0 \quad \text{as } |x| \to \infty, \tag{46}$$

$$A' \to h \quad \text{as } |x| \to \infty, \tag{47}$$

where $' = d/dx$. The solution corresponding to the normal state is $\Psi \equiv 0, A = hx$. (We note that the equations and boundary conditions are translationally invariant. Thus, although we could add an arbitrary constant to A such a solution is simply a translation of this solution.) The leading-order equations (21)-(28) become

$$\kappa^{-2}\psi^{(0)''} = -\psi^{(0)} + (h^{(0)})^2 x^2 \psi^{(0)}, \tag{48}$$

$$a^{(0)''} = h^{(0)} x (\psi^{(0)})^2, \tag{49}$$

$$\psi^{(0)'} \to 0 \quad \text{as } |x| \to \infty, \tag{50}$$

$$a^{(0)'} \to 0 \quad \text{as } |x| \to \infty. \tag{51}$$

Equation (48) with the boundary condition (50) is Schrödinger's equation for a harmonic oscillator, and has solutions

$$\exp\left(\frac{-\kappa^2 x^2}{2(2n+1)}\right) H_n\left(\frac{\kappa\sqrt{2}x}{\sqrt{2n+1}}\right),$$

when $h^{(0)} = \kappa/(2n+1)$, where H_n is the nth Hermite polynomial. The upper critical field is given by the largest of these eigenvalues, namely $h^{(0)} = h_{c_2} = \kappa$. This is where we begin to notice the difference between type I and type II superconductors. In these units the thermodynamic critical field, that is, the field at which the free energy of the superconducting state becomes lower than that of the normal state, is given by $H_c = 1/\sqrt{2}$ [10]. Hence we have

$$h_{c_2} \lesseqgtr H_c \text{ as } \kappa \lesseqgtr 1/\sqrt{2}.$$

For a type II superconductor $\kappa > 1/\sqrt{2}$, $h_{c_2} > H_c$, and, as the external magnetic field is lowered, the bifurcation point will be reached before the thermodynamic critical field. For a type I superconductor however, $\kappa < 1/\sqrt{2}$, $h_{c_2} < H_c$, and before the external field reaches the bifurcation point there is the possibility that the superconductor will undergo a phase change to the superconducting state by means of an inwardly propagating phase boundary as described in a previous paper [4].

The eigensolution corresponding to $h^{(0)} = \kappa$ is

$$\psi^{(0)} = C\frac{\kappa^{1/2}}{\pi^{1/4}}e^{-\frac{\kappa^2 x^2}{2}}. \tag{52}$$

The solution of (49) is

$$a^{(0)} = - \mid C \mid^2 \frac{1}{2\pi^{1/2}} \int_{-\infty}^{x} e^{-\kappa^2\xi^2}\, d\xi, \tag{53}$$

where again the arbitrary constant is irrelevant. Notice that $a^{(0)\prime} < 0$ so that the magnetic field in the sample is less than the applied field (the Meissner effect.) From (52), (53) and (39) we find

$$h^{(1)} = \frac{\mid C \mid^2}{\sqrt{2\pi}} \left[\frac{1}{2} - \kappa^2\right]. \tag{54}$$

Here again we notice a difference between type I and type II superconductors. We have

$$h^{(1)} < 0 \text{ if and only if } \kappa > 1/\sqrt{2}.$$

Thus for type II superconductors we have a solution for all values of the applied magnetic field slightly below the critical value. For type I superconductors however we have a solution for all values of the applied magnetic field slightly above the critical value. The response diagrams for κ less than, equal to and greater than $1/\sqrt{2}$ are shown in Fig. 3.

When $\kappa = 1/\sqrt{2}$ we have $h^{(1)} = 0$, and the question arises as to what the behaviour of Ψ is in the neighbourhood of $h = \kappa$ in this case. In fact we find that the response diagram is vertical, i.e. when $h = \kappa$ there exists a family of solutions of arbitrary amplitude (similar vertical bifurcations have been studied recently in [14], for example). We demonstrate this fact by noting that in one dimension, when $\kappa = 1/\sqrt{2}$, the Ginzburg-Landau equations exhibit what has been termed self-duality [2], so that solutions of (44)-(47) are given by solutions of the following pair of first order equations:

$$\begin{aligned} \sqrt{2}A' &= 1 - \Psi^2, && (55) \\ \sqrt{2}\Psi' &= -\Psi A, && (56) \end{aligned}$$

which in turn have the solution

$$x = \int_{\Psi(0)^2}^{\Psi^2} \frac{-d(\Psi^2)}{\sqrt{2}\,\Psi^2(\Psi^2 - c - \log\Psi^2)^{1/2}}, \tag{57}$$

where, in order for the solution to be bounded, c must be greater than 1 but is otherwise arbitrary. A number of these solutions, for different values of c, are shown in Fig. 4. The solutions take the form of a superconducting "blip" that grows into a completely superconducting region separated from the remaining normal region by two phase boundaries. However, we should remember that these are all steady state solutions, and the diagrams in Fig. 4 do not represent the evolution of a small blip into two travelling waves. It is not clear whether the solutions along the superconducting branch in Figs. 3a,c take the same form as those in Fig. 4.

2.3 Surface superconductivity

We have seen that as an external magnetic field is lowered a superconducting solution first appears in an infinite superconductor when $h = \kappa$. Any real superconducting body is of course finite, and it is of interest to consider the effects of the surface on the nucleation of superconductivity. If the superconducting body is large (compared to the penetration depth) and smooth we may consider surface boundary layers by measuring length from the boundary on the lengthscale of the penetration depth. Then the body appears as a half-space.

We consider here the problem of a superconducting half-space $x > 0$, in an external magnetic field which is parallel to the boundary. The problem was first considered in [16] for the case $d = \infty$, although they did not proceed further than finding the bifurcation point $h^{(0)}$. When the field is perpendicular to the boundary, or at any other angle, the problem is much more difficult, since there is no longer a one-dimensional solution. It is claimed in [16] that the nucleation field for a perpendicular magnetic field is exactly that of bulk nucleation (i.e. that for an infinite superconductor), and that the parallel magnetic field is the one of greatest interest.

We take the field to be in the z-direction, so that we may still take the vector potential to be in the y-direction, $\boldsymbol{A} = (0, A, 0)$. We then look for a one-dimensional solution $A = A(x)$,

$\Psi = F(x)$, where F is real. Here we are fixing the phase of Ψ by requiring that F is real, but allowing the gauge of A to be arbitrary. Because the equations are invariant under transformations of the form

$$\Psi \to e^{i\kappa c y}\Psi, \qquad A \to A + c,$$

this is equivalent to fixing the gauge of A by requiring that $A(0) = 0$, and seeking a solution $\Psi = e^{-i\kappa c y}F(x)$. In fact we prefer the latter viewpoint, since it is an extension of this idea that forms the basis of the following example.

The Ginzburg-Landau equations and boundary conditions become:

$$\kappa^{-2}F'' = F^3 - F + (A+c)^2 F, \tag{58}$$

$$A'' = F^2(A+c), \tag{59}$$

$$F'(0) = (\kappa/d)F(0), \qquad F' \to 0 \text{ as } x \to \infty, \tag{60}$$

$$A'(0) = h, \qquad A' \to h \text{ as } x \to \infty, \tag{61}$$

where $' = d/dx$. The normal state solution is given by $F \equiv 0, A = hx$.

The leading-order equations (21)-(28) become

$$\kappa^{-2}f^{(0)\prime\prime} = -f^{(0)} + (h^{(0)}x + c^{(0)})^2 f^{(0)}, \tag{62}$$

$$a^{(0)\prime\prime} = (h^{(0)}x + c^{(0)})(f^{(0)})^2, \tag{63}$$

$$f^{(0)\prime}(0) = (\kappa/d)f^{(0)}(0), \qquad f^{(0)\prime} \to 0 \text{ as } x \to \infty, \tag{64}$$

$$a^{(0)\prime}(0) = 0, \qquad a^{(0)\prime} \to 0 \text{ as } x \to \infty, \tag{65}$$

where $F = \epsilon^{1/2}f$. We now have a double eigenvalue problem for $h^{(0)}$ and $c^{(0)}$. For each fixed $c^{(0)}$, equations (62) and (64) determine a set of eigenvalues for $h^{(0)}$. Equations (63) and (65) then determine $c^{(0)}$. Integrating (63) and applying the boundary condition (65) we find $c^{(0)}$ must satisfy

$$\int_0^\infty (h^{(0)}\xi + c^{(0)})(f^{(0)})^2\, d\xi = 0. \tag{66}$$

We note that (62) and $f^{(0)\prime} \to 0$ as $x \to \infty \Rightarrow f^{(0)\prime\prime} \to 0$ as $x \to \infty$ and $x^2 f^{(0)} \to 0$ as $x \to \infty$. We multiply (62) by $f^{(0)\prime}$ and integrate over $[0,\infty)$ to give

$$0 = \left[\frac{(f^{(0)\prime})^2}{2\kappa^2}\right]_0^\infty + \left[\left(1 - (h^{(0)}x + c^{(0)})^2\right)\frac{(f^{(0)})^2}{2}\right]_0^\infty + \int_0^\infty (h^{(0)}x + c^{(0)})(f^{(0)})^2\, dx,$$

on integration by parts. Hence

$$0 = \left[d^{-2} + \left(1 - (c^{(0)})^2\right)\right](f^{(0)}(0))^2,$$

by (66) and (64). Therefore, for $d \neq 0$,

$$(c^{(0)})^2 = 1 + d^{-2}. \tag{67}$$

The substitution $w = (2\kappa/h^{(0)})^{1/2}(h^{(0)}x + c^{(0)})$ converts (62), (64) into

$$\frac{d^2 f^{(0)}}{dw^2} + \left(\frac{\mu^2}{2} - \frac{w^2}{4}\right) f^{(0)} \quad = \quad 0, \tag{68}$$

$$\frac{df^{(0)}}{dw}(\sqrt{2}\,\mu c^{(0)}) \quad = \quad \frac{\mu}{\sqrt{2}\,d} f^{(0)}(\sqrt{2}\,\mu c^{(0)}), \tag{69}$$

$$\frac{df^{(0)}}{dw} \quad \to \quad 0, \qquad \text{as } w \to \infty, \tag{70}$$

where $\mu^2 = \kappa/h^{(0)}$. Equation (68) is Weber's equation of index ν, where $\nu = \mu^2/2 - 1/2$. The solution that decays at infinity is

$$f^{(0)}(w) = C D_\nu(w), \tag{71}$$

where D_ν is the parabolic cylinder function. μ is given by the relation

$$D_\nu'(\sqrt{2}\,\mu c^{(0)}) = \frac{\mu}{\sqrt{2}\,d} D_\nu(\sqrt{2}\,\mu c^{(0)}). \tag{72}$$

Using an integral representation of D_ν and writing $\sinh\gamma = 1/d$ we find that μ satisfies

$$\int_0^\infty e^{-(\tau \pm \mu\cosh\gamma)^2} \tau^{-\mu^2/2-1/2}(2\tau \pm \mu e^{\pm\gamma})\, d\tau = 0. \tag{73}$$

Figure 5 is a plot of the lowest eigenvalue against γ. The eigenvalue $h^{(0)}$ corresponding to this lowest eigenvalue of μ is usually denoted by h_{c_3}. Not suprisingly the lowest value of μ^2 (and so the highest value of h_{c_3}) occurs when $\gamma = 0$, $d = \infty$, and the superconductor is adjacent to a vacuum. In this case $\mu^2 \approx 0.59$ so that $h_{c_3} \approx 1.7\,\kappa$. Decreasing the value of d decreases the value of the nucleation field h_{c_3}. As $d \to 0$, the smallest eigenvalue tends to 1. Thus we see that unless the superconductor is coated with a magnetic material, which strongly interferes with the superconducting electrons near the surface (in which case the value of d is small), then the value of the surface nucleation field is higher than the value of the bulk nucleation field (that is, the nucleation field for an infinite superconductor). This means that, when the solution above is stable, as the external field is lowered a superconducting sheath will first form on the surface of the sample.

Integrating (63) we now find

$$a^{(0)} \quad = \quad \int_0^x (x - \xi)(h^{(0)}\xi + c^{(0)})(f^{(0)}(\xi))^2\, d\xi. \tag{74}$$

Figure 6 shows the form of $f^{(0)}$ when $\gamma = 0$.

Continuing as in the general case, we find that equation (39) becomes

$$h^{(1)} = -\frac{\int_0^\infty (f^{(0)})^4 + 2(h^{(0)}x + c^{(0)})a^{(0)}(f^{(0)})^2\,dx}{\int_0^\infty 2x(h^{(0)}x + c^{(0)})(f^{(0)})^2\,dx}. \tag{75}$$

When we substitute in the solution (71) this equation gives C in terms of $h^{(1)}$, and also determines the sign of $h^{(1)}$. For small κ we find that $h^{(1)}$ is positive, while for large κ it is negative. The value of κ at which $h^{(1)}$ switches from being positive to negative is shown in figure 7. Also shown here is the value of κ at which $h_{c_3} = H_c$ ($h_{c_3} > H_c$ is often taken as the criterion for the observation of surface superconductivity). Notice that the two values differ.

2.4 The mixed state

Our examples thus far have been essentially one-dimensional. In order to examine the structure of the mixed state of a type II superconductor we need to consider higher dimensions. Here we consider the simplest two-dimensional situation, namely an infinite superconductor with the applied magnetic field perpendicular to the plane of interest. This problem was first studied by Abrikosov [1]. Odeh [15] has also studied the bifurcation to and existence of the mixed state, and more recently numerical work has been performed by Gunzburger et al. [7]. Since the superconductor is infinite we must decide how to implement the natural boundary conditions (4). One possibility which is both interesting and tractable is to consider periodic solutions. The natural boundary conditions are then that $\boldsymbol{H}$ and $((i/\kappa)\nabla + \boldsymbol{A})\Psi$ are periodic in x and y, with period L_x and L_y respectively. We choose the field $\boldsymbol{H}$ to be directed along the z-axis and choose the gauge so that $\boldsymbol{A}$ is directed along the y-axis; then $\boldsymbol{H} = (0, 0, H(x, y))$, $\boldsymbol{A} = (0, A(x, y), 0)$ and $H = \partial A/\partial x$. (Note that with this gauge div $\boldsymbol{A} \neq 0$. It is more convenient in the present situation to have a single scalar variable A than to have div $\boldsymbol{A} = 0$.)

The solution corresponding to the normal state is again $\Psi \equiv 0$, $A = hx$. The leading order equations (21)-(28) become

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi^{(0)}}{\partial x^2} + \frac{\partial^2\psi^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial\psi^{(0)}}{\partial y} = \psi^{(0)} - (h^{(0)})^2x^2\psi^{(0)}, \tag{76}$$

$$-\frac{\partial^2 a^{(0)}}{\partial x\partial y} = \frac{i}{2\kappa}\left(\psi^{(0)*}\frac{\partial\psi^{(0)}}{\partial x} - \psi^{(0)}\frac{\partial\psi^{(0)*}}{\partial x}\right), \tag{77}$$

$$\frac{\partial^2 a^{(0)}}{\partial x^2} = h^{(0)}x\,|\psi^{(0)}|^2 + \frac{i}{2\kappa}\left(\psi^{(0)*}\frac{\partial\psi^{(0)}}{\partial y} - \psi^{(0)}\frac{\partial\psi^{(0)*}}{\partial y}\right), \tag{78}$$

with $\partial a^{(0)}/\partial x$, $\partial\psi^{(0)}/\partial x$ and $(i/\kappa)\partial\psi^{(0)}/\partial y + h^{(0)}x\psi^{(0)}$ periodic. We found previously that equation (76) had the one-dimensional solutions

$$\psi^{(0)} = \exp\left(\frac{-\kappa^2 x^2}{2(2n+1)}\right) \mathrm{H}_n\left(\frac{\sqrt{2}\,\kappa x}{\sqrt{2n+1}}\right), \tag{79}$$

when $h^{(0)} = \kappa/(2n+1)$. In addition to (79), (76) is also satisfied by the function

$$\psi^{(0)} = \exp\left(iky - \frac{\kappa^2(x-(2n+1)k/\kappa^2)^2}{2(2n+1)}\right) \mathrm{H}_n\left(\frac{\sqrt{2}\,\kappa(x-(2n+1)k/\kappa^2)}{\sqrt{2n+1}}\right), \tag{80}$$

for any k. The largest eigenvalue is $h^{(0)} = \kappa$, with the set of corresponding eigenfunctions

$$\psi^{(0)} = \exp\left\{iky - \frac{\kappa^2}{2}\left(x - \frac{k}{\kappa^2}\right)^2\right\}. \tag{81}$$

Since we are looking for a solution which is periodic in y the general solution to (76) with $h^{(0)} = \kappa$ is

$$\psi^{(0)} = \sum_{-\infty}^{\infty} C_n e^{inky}\psi_n(x), \tag{82}$$

where

$$\psi_n(x) = \exp\left\{-\frac{\kappa^2}{2}\left(x - \frac{nk}{\kappa^2}\right)^2\right\}, \tag{83}$$

$L_y = 2\pi/k$, and C_n and k are as yet arbitrary. Periodicity in x now implies $L_x = kN/\kappa^2$, where N is an integer, and gives the following simple recursion relation for the C_n:

$$C_{n+N} = C_n. \tag{84}$$

The resulting solution satisfies

$$\begin{aligned} \psi^{(0)}(x, y+2\pi/k) &= \psi^{(0)}(x,y), & (85)\\ \psi^{(0)}(x+kN/\kappa^2) &= e^{ikNy}\psi^{(0)}(x,y). & (86) \end{aligned}$$

Note that this implies that the phase of $\psi^{(0)}$ varies by $2\pi\kappa N$ around the boundary of the unit cell, which corresponds to the cell containing N zeros of $\psi^{(0)}$ (or vortices).

The solution of (77), (78) is then

$$\begin{aligned} a^{(0)} &= -\frac{1}{2\kappa}\sum_{m,n=-\infty}^{\infty} C_n^* C_m e^{ik(m-n)y}\int^x \psi_m(x)\psi_n(x)\,dx, & (87)\\ &= -\frac{1}{2\kappa}\int^x |\psi^{(0)}|^2\ dx. & (88) \end{aligned}$$

Notice again that the field in the sample is reduced.

The solvability condition (39) now becomes

$$\frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2}-1\right)\sum_{r,m} C_{n-r+m}C_mC_r \exp\left\{-\frac{k^2}{2\kappa^2}\left[(r-n)^2+(r-m)^2\right]\right\} - \frac{h^{(1)}C_n}{\kappa} = 0. \quad (89)$$

This equation determines both $h^{(1)}$ and the allowed coefficients C_n. To determine $h^{(1)}$ we multiply by C_n^* and sum over n to give

$$\left(\frac{1}{2\kappa^2}-1\right)\overline{|\psi^{(0)}|^4} - \frac{h^{(1)}}{\kappa}\overline{|\psi^{(0)}|^2} = 0. \quad (90)$$

Hence

$$h^{(1)} = \kappa\left(\frac{1}{2\kappa^2}-1\right)\frac{\overline{|\psi^{(0)}|^4}}{\overline{|\psi^{(0)}|^2}}. \quad (91)$$

As in the one-dimensional case we see that $h^{(1)} < 0$ if and only if $\kappa > 1/\sqrt{2}$, i.e. for type II superconductors. Note also that $h^{(1)} = 0$ when $\kappa = 1/\sqrt{2}$. We expect that there will be a vertical bifurcation when $\kappa = 1/\sqrt{2}$, since the Ginzburg-Landau equations are again self-dual [17]. The average magnetic field in the specimen is given by

$$\begin{aligned}\overline{H} &= h^{(0)} + \epsilon h^{(1)} - \epsilon\overline{|\psi^{(0)}|^2}/2\kappa + \cdots, \\ &= h^{(0)} + \epsilon h^{(1)} - \epsilon h^{(1)}/(2\kappa^2-1)\beta + \cdots,\end{aligned}$$

where

$$\beta = \frac{\overline{|\psi^{(0)}|^4}}{(\overline{|\psi^{(0)}|^2})^2},$$

is independent of the amplitude of $\psi^{(0)}$. Thus $\overline{H}$ is linear in h near $h = \kappa$ with a gradient that approaches infinity as $\kappa \to 1/\sqrt{2}$. This is in agreement with fig. 1b.

Abrikosov calculates the free energy per unit volume and finds it to be proportional to

$$\frac{1}{2} + \overline{H}^2 - \frac{(\kappa-\overline{H})^2}{1+(2\kappa^2-1)\beta}.$$

Thus, for $\kappa > 1/\sqrt{2}$ and fixed $\overline{H}$ the free energy is minimised by minimising β. This is the traditional way of deciding which of the possible mixed state solutions will be stable for a type II superconductor. In II we will compare this to the classical stability of the solutions. We note that the solvability condition (89) can be written as

$$\frac{\partial\beta}{\partial C_n^*} = 0.$$

We now approach the problem of choosing the C_n.

The simplest case $N = 1$ was analysed by Abrikosov [1]. In this case $C_n = C$, for all n, and (89) simply gives C in terms of $h^{(1)}$. The resulting solution $\psi^{(0)}$ has one zero in the unit cell, at its centre $x = k/2\kappa^2, y = \pi/k$, and $|\psi^{(0)}|^2$ has the symmetry of a rectangular lattice. We find that

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa} S_{1,0}^2,$$

where

$$S_{p,q} = \sum_{n=-\infty}^{\infty} e^{-\frac{k^2}{2\kappa^2}[pn+q]^2}.$$

($S_{p,q}$ is related to Jacobi theta functions.) The minimum value of β occurs when $k = \kappa\sqrt{2\pi}$, giving

$$\beta = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 1.18.$$

In this case the $|\psi^{(0)}|^2$ has the symmetry of a square lattice, as shown in Fig. 8.

The next simplest case $N = 2$ was considered in [12]. In this case (89) becomes

$$\frac{2\sqrt{2}\,\kappa h^{(1)} C_0}{(1-2\kappa^2)} = |C_0|^2\, C_0 S_{2,0}^2 + C_0^* C_1^2 S_{2,1}^2 + 2\,|C_1|^2\, C_0 S_{2,0} S_{2,1}, \tag{92}$$

$$\frac{2\sqrt{2}\,\kappa h^{(1)} C_1}{(1-2\kappa^2)} = |C_1|^2\, C_1 S_{2,0}^2 + C_1^* C_0^2 S_{2,1}^2 + 2\,|C_0|^2\, C_1 S_{2,0} S_{2,1}, \tag{93}$$

with solutions

$$C_1 = \pm C_0,\ \pm i C_0.$$

The case $C_1 = C_0$ corresponds to the case $N = 1$. The case $C_1 = -C_0$ also corresponds to the $N = 1$ solution, but translated by π/k in y. Similarly the case $C_1 = -iC_0$ corresponds to a translation of the case $C_1 = iC_0$. Thus there is only one new case to consider, namely $C_1 = iC_0$, $C_{n+2} = C_n$. In this case the unit cell has dimensions $L_x = 2k/\kappa^2$, $L_y = 2\pi/k$, and $\psi^{(0)}$ vanishes at the points $x = k/2\kappa^2$, $y = \pi/2k$, and $x = 3k/2\kappa^2, y = 3\pi/2k$.

When $k = \kappa\sqrt{\pi}$, $\psi^{(0)}$ has the symmetry of a square lattice, and [12] show that the solution is identical to the square lattice of the $N = 1$ case rotated by 45° and translated.

We find that

$$\begin{aligned}\beta &= \frac{2k}{\sqrt{2\pi}\,\kappa}\left\{\frac{(|C_0|^4 + |C_1|^4)S_{2,0}^2 + 4\,|C_0|^2|C_1|^2\, S_{2,0}S_{2,1} + 2\Re((C_0^*)^2 C_1^2) S_{2,1}^2}{(|C_0|^2 + |C_1|^2)^2}\right\}, \\ &= \frac{k}{\sqrt{2\pi}\,\kappa}\left\{S_{2,0}^2 + S_{2,1}(2S_{2,0} - S_{2,1})\right\}.\end{aligned}$$

The minimum of β is obtained when $k = \kappa\sqrt{\pi\sqrt{3}}$, giving $\beta \approx 1.16$. In this case $|\psi^{(0)}|^2$ has the symmetry of a hexagonal lattice, as shown in Fig. 9.

Let us now consider the cases $N = 3$ and $N = 4$. When $N = 3$, (89) has solutions

$$C_1, C_2 = C_0, \mu C_0, \mu^2 C_0,$$

where μ is a complex cube root of unity. There are nine cases to consider, as tabulated below.

		C_2/C_0		
		1	μ	μ^2
C_1/C_0	1	$N=1$ case	New soln. 1a	New soln. 1b
	μ	1a translated in y	1b translated in y	$N=1$ translated in y
	μ^2	1b translated in y	$N=1$ translated in y	1a translated in y

In fact, solution 1b can be obtained from solution 1a by reflection in the x-axis and conjugation. Thus there is only one new solution to consider. When $C_1 = C_0$, $C_2 = \mu C_0$, the unit cell has dimensions $L_x = 3k/\kappa^2$, $L_y = 2\pi/k$ and $\psi^{(0)}$ has zeros at

$$x = k/2\kappa^2,\ y = \pi/k,$$
$$x = 3k/2\kappa^2,\ y = \pi/3k,$$
$$x = 5k/2\kappa^2,\ y = 5\pi/3k.$$

When $C_1 = C_0$, $C_2 = \mu C_0$, β is given by

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa}\left\{3S_{3,0}^2 - 2(S_{3,0} - S_{3,1})^2\right\}.$$

The minimum of β is obtained when $k \approx 1.355\sqrt{\pi}\,\kappa$, and is $\beta = 1.166$. Figure 10 shows a plot of $|\,\Psi^{(0)}|^2$ in this case.

We see that the free energy of this solution lies between those of the square and hexagonal lattice solutions.

Finally we examine the case $N = 4$. In this case equations (89) have solutions [3]

$$C_2 = \pm C_0, C_1 = \pm C_3 = \pm C_0, \pm i C_0,$$

and

$$C_2 = \pm C_0, C_1 = \mp C_3, |\,C_1|=|\,C_0\,|\,.$$

Solutions in the first set are simply translations of the cases $N = 1$ and $N = 2$. When $C_2 = \pm C_0$, $C_3 = \mp C_1$, $|\,C_1|=|\,C_0\,|$, then

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa}\left\{(S_{4,0} + S_{4,2})^2 + 4S_{4,1}(S_{4,0} - S_{4,2})\right\} \tag{94}$$
$$= \frac{k}{\sqrt{2\pi}\,\kappa}\left\{S_{2,0}^2 + 4S_{4,1}(S_{4,0} - S_{4,2})\right\}. \tag{95}$$

We find the minimum of β to occur when $k \approx 1.380\sqrt{\pi}\kappa$, giving $\beta \approx 1.172$. Figure 11 shows a plot of $|\,\Psi^{(0)}|^2$ in this case, with $C_1 = C_0$.

Thus we find that there are many periodic solutions, all with relatively close values of β. The lowest value of β yet discovered corresponds to the hexagonal lattice, with $N = 2$. It is conjectured that this solution gives the absolute minimum of β, and hence will be the energetically favoured solution.

3 Conclusion

We have found that as the magnitude of the external magnetic field is varied there is a series of bifurcations from the normal state solution to superconducting solution branches. The largest positive eigenvalue is known as the upper critical magnetic field, and denoted by h_{c_2}.

When the sample dimensions are large compared with the penetration depth, the sample may be considered infinite when we are using the penetration depth as our lengthscale, and this was our first example. In this case the upper critical field is equal to κ, the Ginzburg-Landau parameter. For type I superconductors we find that $h_{c_2} < H_c$, the thermodynamic critical field (at which the free energy of the wholly superconducting state becomes lower than that of the normal state), and the one-dimensional superconducting solution branch bifurcating at $h = h_{c_2}$ exists for values of the field slightly greater than h_{c_2}. For type II superconductors we find that $h_{c_2} > H_c$, and the superconducting solution branch bifurcating at $h = h_{c_2}$ exists for values of the field slightly less than h_{c_2}.

However, even for a large superconducting body things may be different near the surface of the sample. Our second example considered the effects of a surface on the nucleation of superconductivity. If the body is again large compared to the penetration depth, and we again use the penetration depth as our lengthscale, with lengths measured from the boundary, then the body appears as a halfspace, and we are considering surface boundary layers. It was found that the nucleation field, here known as h_{c_3}, is increased above that for bulk nucleation. Hence for any real sample superconductivity will nucleate first at the surface in the form of a superconducting sheath (known as surface superconductivity). Furthermore the value of κ at which the switch from a subcritical to supercritical bifurcation occurs is no longer $1/\sqrt{2}$, but is given by figure 7.

Our final example considered two-dimensional periodic solutions. Here we extended the work of Abrikosov [1] and Kleiner et al. [12] to include solutions with three and four zeros of the order parameter in the unit cell. The free energy of these additional solutions was found to lie between those of the square [1] and hexagonal [12] lattice solutions.

In II we will examine the linear stability of each of the solution branches in figure 2, and consider the implications for each of our examples. The stability of the bifurcating solution will determine whether phenomena such as surface superconductivity or the mixed state will be observed in practice. We will also perform a weakly-nonlinear stability analysis near the bifurcation point to examine how instability of the normal state solution leads to the formation of the mixed state.

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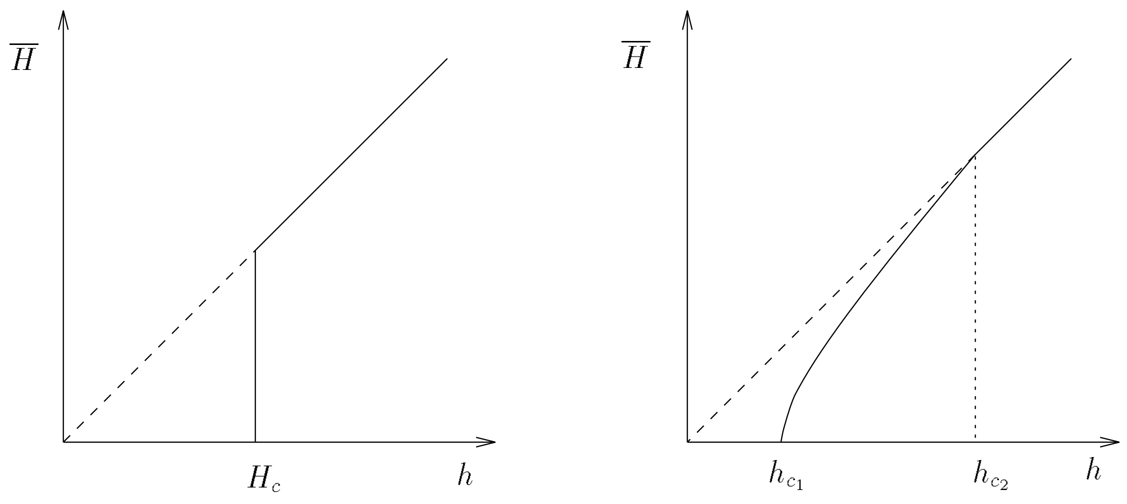

(a) Type I Superconductors

(b) Type II Superconductors

Figure 1: Average magnetic field $\overline{H}$ in bulk type I and type II superconductors as a function of the applied magnetic field h.

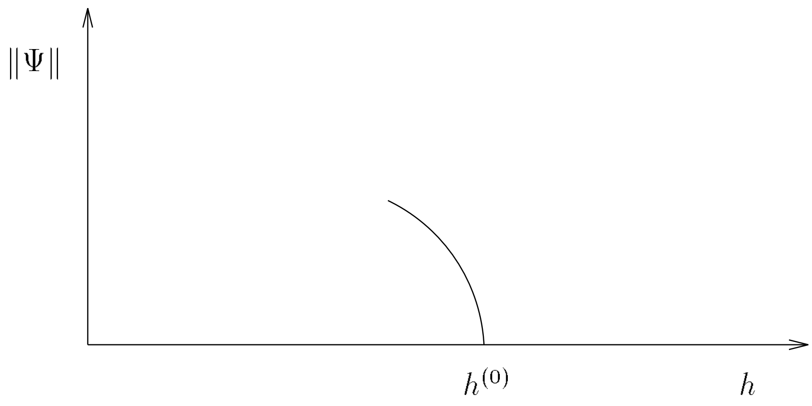

Figure 2: Pitchfork bifurcation from $\Psi \equiv 0$ ar $h = h^{(0)}$.

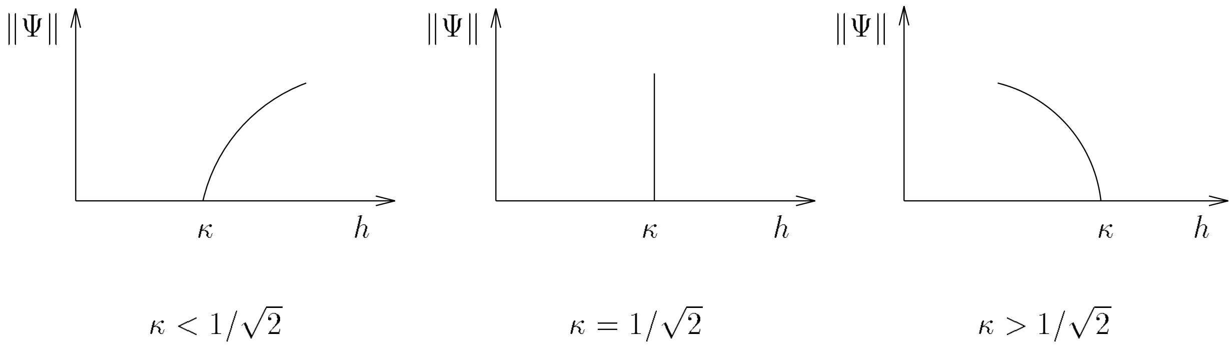

Figure 3: The response diagrams for κ less than, equal to and greater than $1/\sqrt{2}$.

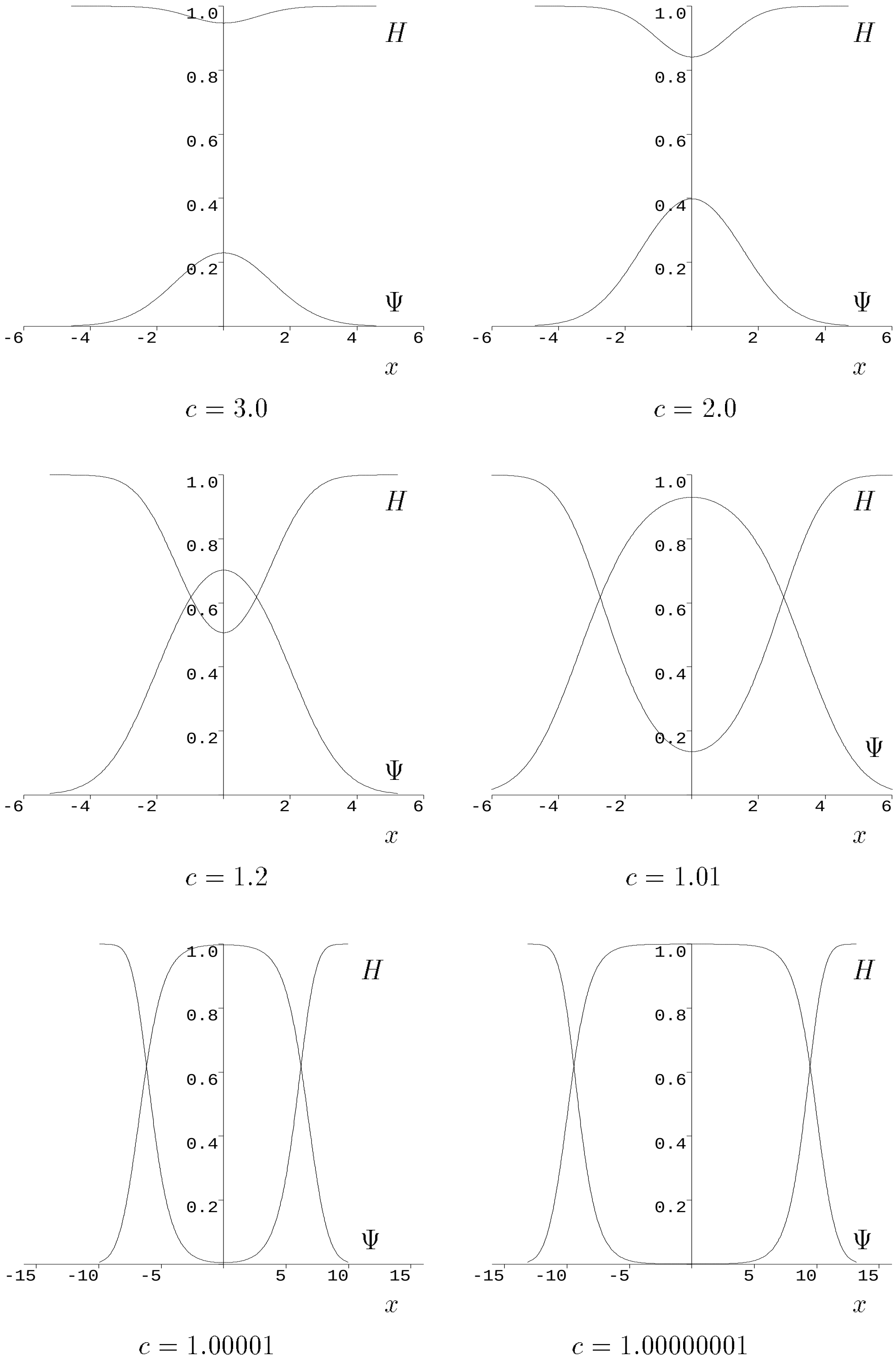

Figure 4: Solutions bifurcating from $\Psi \equiv 0$ at $h = 1/\sqrt{2}$ for $\kappa = 1/\sqrt{2}$.

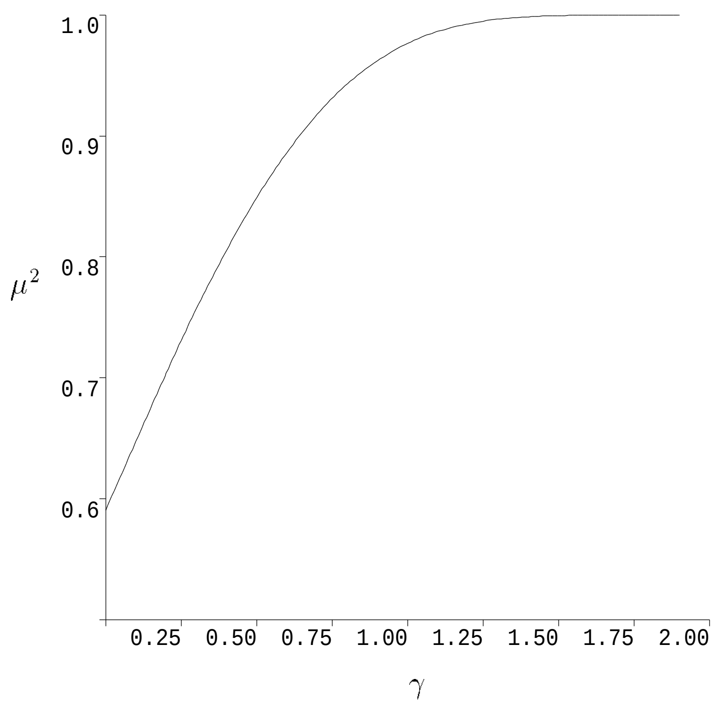

Figure 5: Lowest eigenvalue μ^2 as a function of γ.

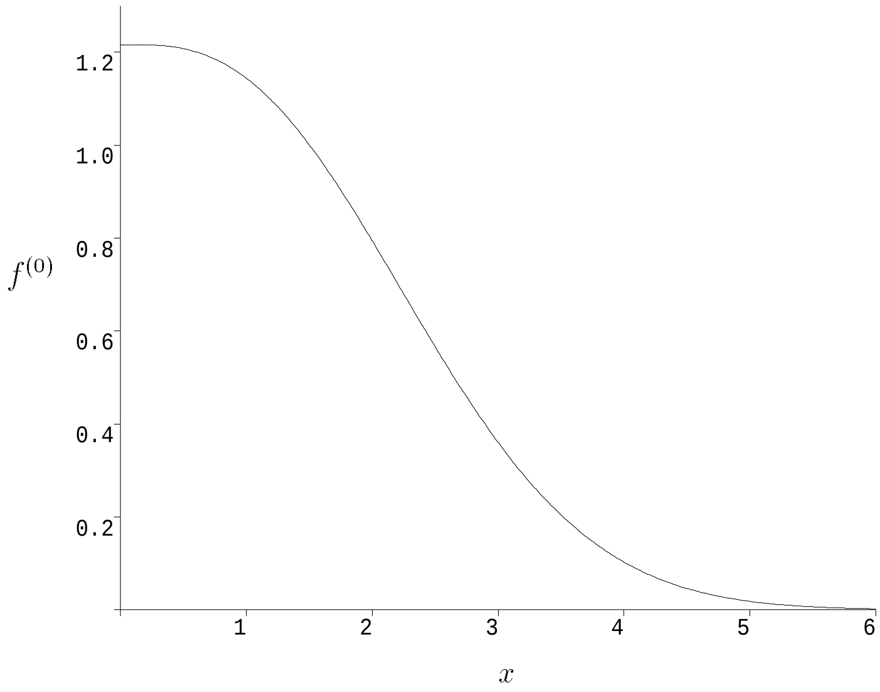

Figure 6: $f^{(0)}$ as a function of x for $\gamma = 0$. $f^{(0)}$ is concentrated in a region $x = O(1)$ from the boundary, hence the term 'surface superconductivity'.

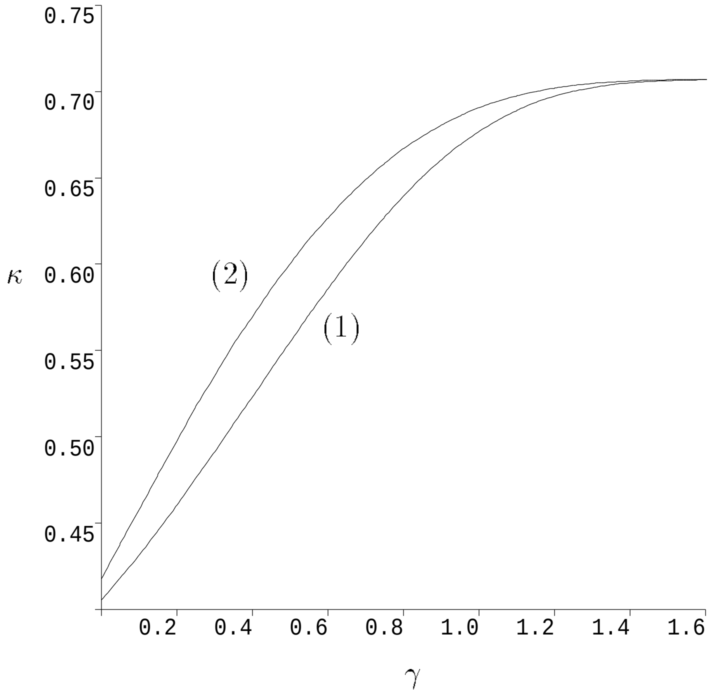

Figure 7: (1) The value of κ at which $h^{(1)}$ switches sign as a function of γ. (2) The value of κ at which the critical field h_{c_3} is equal to the thermodynamic critical field H_c.

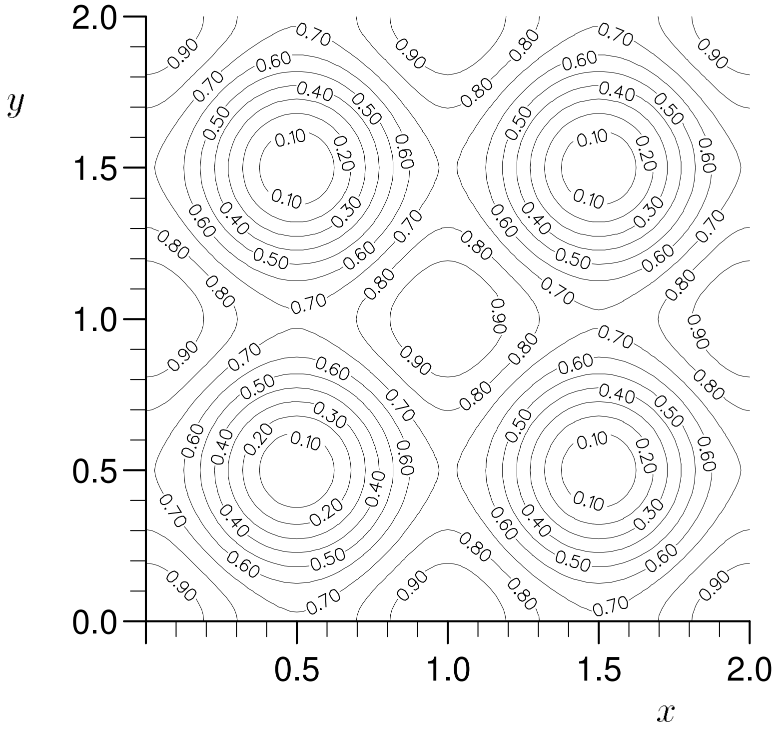

Figure 8: Mixed state solution $N = 1$, $C_n = C$, when $k = \sqrt{2\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

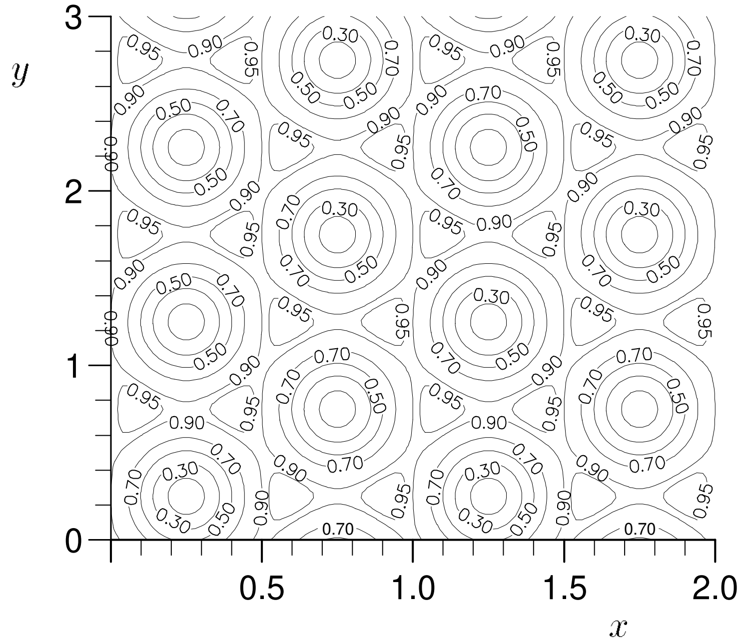

Figure 9: Mixed state solution $N = 2$, $C_1 = iC_0$, when $k = \sqrt{\pi\sqrt{3}}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

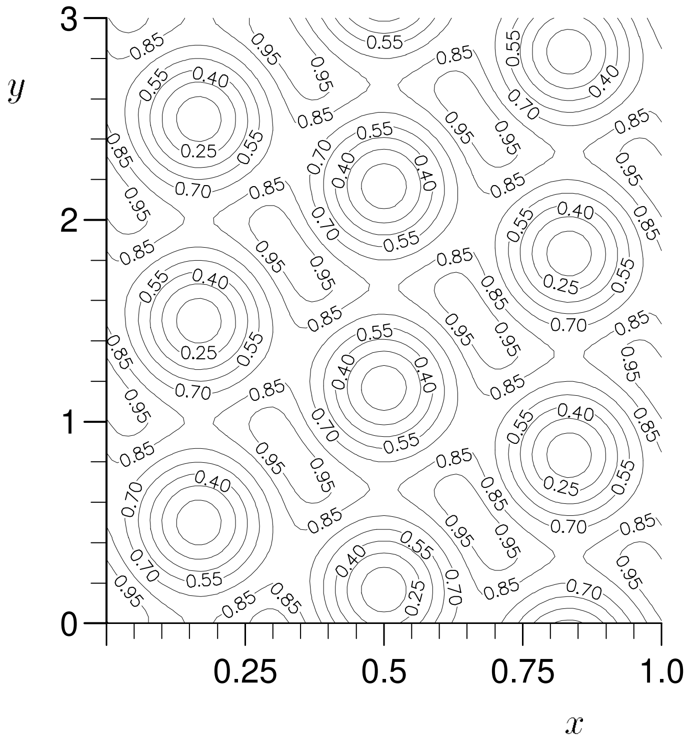

Figure 10: Mixed state solution $N = 3$, $C_1 = C_0$, $C_2 = \mu C_0$, when $k = 1.355\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

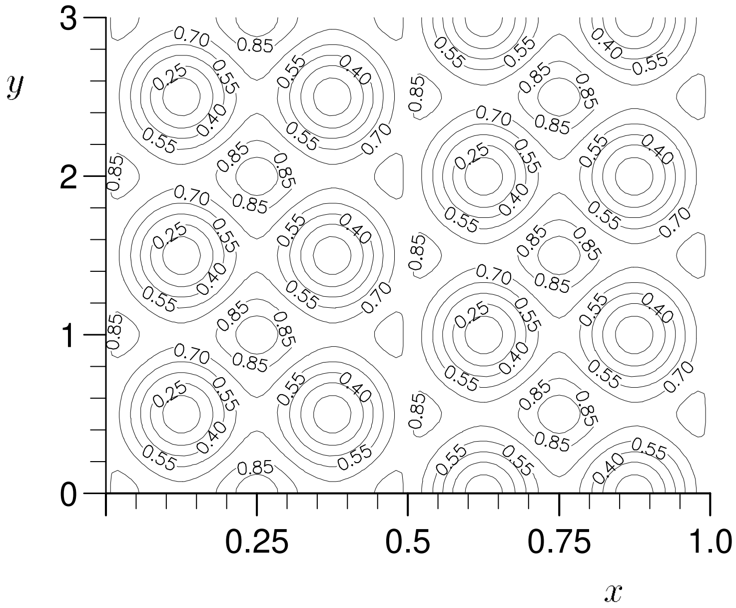

Figure 11: Mixed state solution $N=4$, $C_1=C_0$, $C_2=C_0$, $C_3=-C_0$, when $k=1.380\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.