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STOKES PHENOMENON AND MATCHED ASYMPTOTIC EXPANSIONS*

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Abstract. This paper describes the use of matched asymptotic expansions to illuminate the description of functions exhibiting Stokes phenomenon. In particular the approach highlights the way in which the local structure and the possibility of finding Stokes multipliers explicitly depends on the behaviour of the coefficients of the relevant asymptotic expansions.

Key words. Stokes' phenomenon, matched asymptotic expansions, Airy function, Error function

AMS(MOS) subject classifications. 41A60, 33C10, 33B20

1. Introduction. The role of Stokes phenomenon in describing the asymptotic behaviour of certain analytic functions as $|z| \to \infty$ has been discussed in recent years (Berry [1], Berry & Howls [2], McLeod [4], Olde Daalhuis & Olver [5], Olver [7], Paris [9]). The basic picture that has been emerged is that the asymptotic expansions of the solutions of certain classes of ordinary differential equations in the complex plane are only uniform in ph$z$ if the coefficients of the asymptotic expansions of the individual complementary functions are made to vary rapidly in the vicinity of the so-called Stokes lines. There are three important aspects of this phenomenon:

(i) the Stokes lines are characterised by the fact that one complementary function is maximally dominant over an other as $|z| \to \infty$.
(ii) the Stokes lines are separated by anti-Stokes lines, on which the complementary functions are of comparable size as $|z| \to \infty$.
(iii) the coefficients can be thought of as changing smoothly rather than discontinuously [1].

The purpose of this paper is to interpret this state of affairs in terms of the theory of matched asymptotic expansions (m.a.e.'s) (van Dyke [10]). In particular, we will use the m.a.e. format to set out a systematic procedure for finding the location of the Stokes lines and the local behaviour of the solutions in their vicinity, for as large a class of ordinary differential equations as we can. We will begin with a very simple example where the solutions have integral representations. Then we will use this to motivate the study of a class of differential equations that illustrates both the scope and limitations of our idea; in particular it will highlight the distinction between those problems for which the Stokes multipliers (the change in the coefficient referred to above) can be found explicitly, and those which demand numerical computation. It also sheds light on some special degenerate cases where these multipliers can be found by inspection.

Almost all the differential equations we shall study have already been analyzed by different methods ([4] and [5]), and our results are in agreement with those works.

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A crucial aspect of our discussion is our interpretation of the Stokes phenomenon in terms of the rapidity with which the difference between the solution of the differential equation and its optimally truncated asymptotic expansion varies as ph$z$ varies. Thus the phenomenon will not appear directly as a change in the coefficient of a complementary function, but rather as a change in the remainder of the asymptotic expansion of the dominant complementary function in the region where this remainder is comparable to the subdominant complementary function. This remainder or error will have relatively smooth variation away from a Stokes line, and we view the change usually ascribed to the Stokes multiplier as a *consequence* of the rapid variation in this error rather than being the basic mechanism underlying the phenomenon.

Throughout the paper we will only consider differential equations that have solutions with divergent asymptotic expansions; when all the solutions have convergent or terminating expansions, Stokes phenomenon does not occur.

2. Example: The complementary error function. The complementary error function is a well-known function defined by

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt, \qquad z \in \mathbb{C}. \tag{2.1}$$

It is an entire function with the following asymptotic behaviour

$$\text{erfc}(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{\infty} (-1)^s \frac{(2s)!}{s!(4z^2)^s}, \qquad |\text{ph}(z)| < \tfrac{3}{4}\pi, \tag{2.2.a}$$

$$\text{erfc}(z) \sim 2 + \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{\infty} (-1)^s \frac{(2s)!}{s!(4z^2)^s}, \qquad |\text{ph}(-z)| < \tfrac{3}{4}\pi, \tag{2.2.b}$$

as $|z| \to \infty$. Notice that both asymptotic expansions are valid in the sector $\frac{1}{4}\pi <$ ph$z < \frac{3}{4}\pi$. So, the asymptotic behaviour of erfc(z) is the infinite expansion of (2.2.a), plus a constant. In the sector $-\frac{1}{4}\pi <$ ph$z < \frac{1}{4}\pi$, where $\exp(-z^2)/z\sqrt{\pi}$ is subdominant, this constant is 0. In the sector $\frac{3}{4}\pi <$ ph$z < \frac{5}{4}\pi$, where, again, $\exp(-z^2)/z\sqrt{\pi}$ is subdominant, this constant is 2. And in the sector $\frac{1}{4}\pi <$ ph$z < \frac{3}{4}\pi$, where $\exp(-z^2)/z\sqrt{\pi}$ is dominant, this 'constant' changes from 0 to 2. See Figure (2.1).

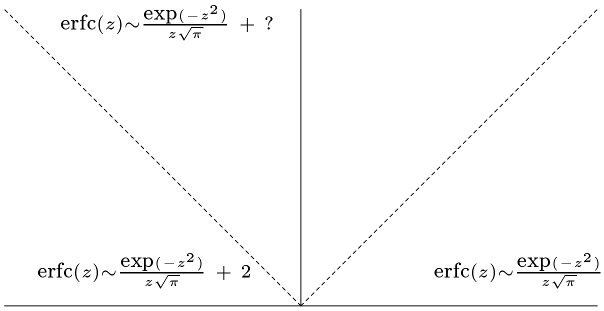

FIG. 2.1. *The asymptotic behaviour of* erfc(z) *for* $0 \le$ ph$z \le \pi$.

We want to obtain the change of the asymptotic behaviour directly from the differential equation

$$\frac{d^2}{dz^2} w(z) + 2z \frac{d}{dz} w(z) = 0, \tag{2.3}$$

without use of (2.1). Both $\operatorname{erfc}(z)$ and the constant function are solutions of this differential equation. To obtain more information on the change in the sector $\frac{1}{4}\pi < \operatorname{ph} z < \frac{3}{4}\pi$, we truncate (2.2.$a$) after N terms:

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{N-1} (-1)^s \frac{(2s)!}{s!(4z^2)^s} + R_N(z). \tag{2.4}$$

The remainder $R_N(z)$ is a solution of

$$\frac{d^2}{dz^2} R_N(z) + 2z \frac{d}{dz} R_N(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \frac{(-1)^{N-1}(2N)!}{(N-1)!4^{N-1}z^{2N}}. \tag{2.5}$$

To show where the significant changes take place, we introduce polar coordinates (2.6)

$$z = re^{i\theta}, \qquad 0 < \theta < \pi, \qquad \frac{d}{dz} = -\frac{ie^{-i\theta}}{r}\frac{d}{d\theta}, \qquad \frac{d^2}{dz^2} = \frac{ie^{-2i\theta}}{r^2}\frac{d}{d\theta} - \frac{e^{-2i\theta}}{r^2}\frac{d^2}{d\theta^2},$$

where we have deliberately written d/dz in terms of the "fast" variable θ rather than the "slow" variable r. Equation (2.5) in terms of polar coordinates is

$$\begin{aligned} -\frac{e^{-2i\theta}}{r^2}\frac{d^2}{d\theta^2} R_N(z) + i\Big(\frac{e^{-2i\theta}}{r^2} - 1\Big)\frac{d}{d\theta} R_N(z) \\ = \frac{\exp\big[-r^2e^{2i\theta} + (N-1)\pi i - (2N+1)i\theta\big](2N)!}{\sqrt{\pi}(N-1)!4^{N-1}r^{2N+1}}. \end{aligned} \tag{2.7}$$

The magnitude of the right-hand side of (2.7), as a function of N, is minimal for $N \sim r^2$. Therefore, we take $N = r^2 + \alpha$, where α is bounded as $r \to \infty$. With this N the right-hand side of (2.7) reads

$$\begin{aligned} &\frac{\exp\Big[-r^2\big(e^{2i\theta} + i(2\theta - \pi)\big) + (\alpha - 1)\pi i - (2\alpha + 1)\theta i\Big](2r^2 + \alpha)!}{\sqrt{\pi}(r^2 + \alpha - 1)!4^{r^2+\alpha-1}r^{2r^2+2\alpha+1}} \\ &\sim \frac{4r}{\sqrt{\pi}} e^{-i\alpha(2\theta-\pi)-i(\theta+\pi)} \exp\Big[-r^2\big(e^{2i\theta} + 1 + i(2\theta - \pi)\big)\Big], \end{aligned} \tag{2.8}$$

as $r \to \infty$. The dominant factor in the right-hand side of 2.8) is $\big|\exp\big[-r^2\big(e^{2i\theta} + 1 + i(2\theta - \pi)\big)\big]\big|$, and it is maximal at $\theta = \frac{1}{2}\pi$, where it is $\mathcal{O}(1)$, as $r \to \infty$. Noting that the minimal remainder is only of the same order as a constant at $\theta = \frac{1}{2}\pi$, we are going to use matched asymptotic expansions at this line, by writing

$$r = \frac{r_0}{\varepsilon}, \qquad \theta = \tfrac{1}{2}\pi + \delta\theta_0, \tag{2.9}$$

where ε and δ are the new small parameters. With the substitution of (2.9) into (2.7), we obtain

$$\begin{aligned} \frac{e^{-2i\delta\theta_0}}{r_0^2}\frac{\varepsilon^2}{\delta^2}\frac{d^2}{d\theta_0^2} R_N(z) - i\Big(\frac{e^{-2i\delta\theta_0}}{r_0^2}\frac{\varepsilon^2}{\delta} - \frac{1}{\delta}\Big)\frac{d}{d\theta_0} R_N(z) \\ \sim \frac{4r_0 i}{\varepsilon\sqrt{2\pi}} e^{-2i\delta\theta_0\alpha - i\delta\theta_0} \exp\Big[\frac{r_0^2}{\varepsilon^2}\big(e^{2i\delta\theta_0} - 1 - 2i\delta\theta_0\big)\Big] \\ \sim \frac{4r_0 i}{\varepsilon\sqrt{2\pi}} \exp\Big[-2r_0^2\theta_0^2\frac{\delta^2}{\varepsilon^2}\Big], \end{aligned} \tag{2.10}$$

as $\delta,\ \varepsilon \to 0$. From the final exponential we only obtain an interesting result when $\delta = \varepsilon$. Then, the dominant terms are the ε^{-1} terms, and we obtain

$$\frac{d}{d\theta_0} R_N(z) \sim \frac{4r_0}{\sqrt{2\pi}} \exp\bigl(-2r_0^2\theta_0^2\bigr), \tag{2.11}$$

with the solution

$$R_N(z) \sim A + \operatorname{erf}\bigl(\sqrt{2} r_0 \theta_0\bigr), \qquad z = \frac{r_0}{\varepsilon} e^{i(\frac{1}{2}\pi + \varepsilon\theta_0)}, \qquad \varepsilon \to 0, \tag{2.12}$$

where $\operatorname{erf}(z)$ is the error function. Matching as $\theta_0 \to -\infty$, the remainder $R_N(z)$ is exponentially small. Thence, $A = 1$. Thus all the change in the 'constant' term takes place in the neighbourhood of the Stokes line $\operatorname{ph} z = \frac{1}{2}\pi$, and the change reads

$$R_N(z) \sim 1 + \operatorname{erf}\bigl(\sqrt{2} r_0 \theta_0\bigr), \tag{2.13}$$

in agreement with [1]. Figure (2.2) shows the appearance of the 'constant' term in the asymptotic behaviour of $\operatorname{erfc}(z)$ in the sector $0 \le \operatorname{ph} z \le 2\pi$. Notice that the 'constant' term appears at the Stokes line $\operatorname{ph} z = \frac{1}{2}\pi$, it is dominant in the sector $\frac{3}{4}\pi < \operatorname{ph} z < \frac{5}{4}\pi$, and it disappears at the Stokes line $\operatorname{ph} z = \frac{3}{2}\pi$.

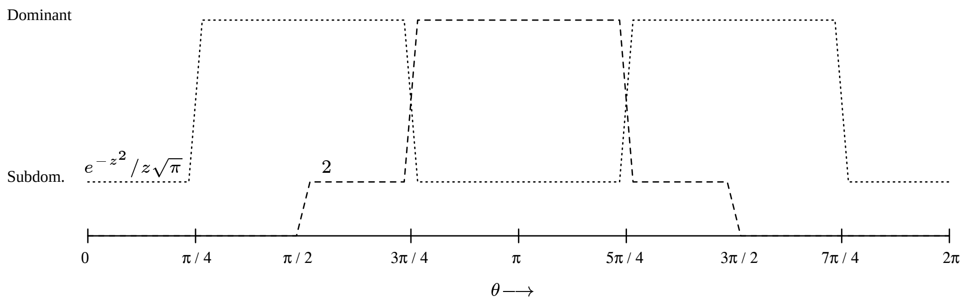

FIG. 2.2. *The appearance of the 'constant' term in the asymptotic behaviour of* $\operatorname{erfc}(z)$.

3. The Stokes phenomenon for solutions of a class of ordinary differential equations. The general homogeneous linear differential equation of the second order is given by

$$\frac{d^2}{dz^2} w(z) + \tilde{f}(z) \frac{d}{dz} w(z) + \tilde{g}(z) w(z) = 0. \tag{3.1}$$

If, as we shall suppose, the point at infinity is an irregular singularity of rank one, then the functions $\tilde{f}(z)$ and $\tilde{g}(z)$ can be expanded in power series

$$\tilde{f}(z) = \sum_{s=0}^{\infty} \frac{\tilde{f}_s}{z^s}, \qquad \tilde{g}(z) = \sum_{s=0}^{\infty} \frac{\tilde{g}_s}{z^s}, \tag{3.2}$$

which converge for $|z| \ge \tilde{\rho}$. Not all of the coefficients $\tilde{f}_0$, $\tilde{g}_0$ and $\tilde{g}_1$ vanish, otherwise infinity would be a regular singularity.

The asymptotic theory of solutions of (3.1) in these circumstances is well-known and will be found, for example, in Olver [6], Chap. 7, §§1-2. Following the notations used in that reference, we shall assume, without loss of generality, that the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 + \tilde{f}_0\lambda + \tilde{g}_0 = 0 \tag{3.3}$$

are distinct. (The case $\lambda_1 = \lambda_2$ can always be handled by application of a preliminary transformation due to Fabry (ibid., Chap. 7, §1.3).) Then equation (3.1) has unique solutions $w_1(z)$ and $w_2(z)$ such that

$$\begin{array}{lll} w_1(z) \sim e^{\lambda_1 z} z^{\mu_1} & |\mathrm{ph}\{(\lambda_2 - \lambda_1)z\}| \le \frac{3}{2}\pi - \nu, & \\ w_2(z) \sim e^{\lambda_2 z} z^{\mu_2} & |\mathrm{ph}\{(\lambda_1 - \lambda_2)z\}| \le \frac{3}{2}\pi - \nu, & \end{array} \quad \text{as } |z| \to \infty. \tag{3.4}$$

Here (and elsewhere in this paper) ν denotes an arbitrary small positive constant. The exponents μ_1 and μ_2 are given by

$$\mu_1 = \tilde{f}_1\lambda_1 + \tilde{g}_1, \qquad \mu_2 = -(\tilde{f}_1\lambda_2 + \tilde{g}_1). \tag{3.5}$$

Any branches may be taken for z^{μ_1} and z^{μ_2}, provided that they are continuous and used consistently throughout.

If we replace the independent variable z by $z/(\lambda_2 - \lambda_1)$, then the essential form (3.1) is unchanged, but the new characteristic values satisfy

$$\lambda_2 - \lambda_1 = 1. \tag{3.6}$$

Also, if we replace $w(z)$ by $v(z)e^{\lambda_2 z} z^{\mu_2}$, and if we take $\mu = \mu_1 - \mu_2$, then we obtain for $v(z)$ the following differential equation:

$$\frac{d^2}{dz^2}v(z) + f(z)\frac{d}{dz}v(z) + g(z)v(z) = 0, \tag{3.7}$$

where $f(z)$ and $g(z)$ can be expanded in the power series

$$f(z) = 1 - \frac{\mu}{z} + \sum_{s=2}^{\infty} \frac{f_s}{z^s}, \qquad g(z) = \sum_{s=2}^{\infty} \frac{g_s}{z^s}, \tag{3.8}$$

which converge for $|z| \ge \rho$. The two unique solutions have the following asymptotic behaviour:

$$v_1(z) \sim e^{-z} z^{\mu} \sum_{s=0}^{\infty} \frac{a_s}{z^s}, \qquad |\mathrm{ph}(z)| \le \tfrac{3}{2}\pi - \nu, \tag{3.9.a}$$

$$v_2(z) \sim 1, \qquad |\mathrm{ph}(z) - \pi| \le \tfrac{3}{2}\pi - \nu. \tag{3.9.b}$$

We choose $a_0 = 1$, and the other coefficients are determined by
(3.10)

$$-sa_s = s(s-1-\mu)a_{s-1} + \sum_{m=0}^{s-1}(g_{s+1-m} - f_{s+1-m})a_m - \sum_{m=0}^{s-2}(m-\mu)f_{s-m}a_m, \qquad s \ge 1.$$

A direct consequence of (3.2) (or (3.8)) is that $v_1(ze^{-2\pi i})$ is also a solution of (3.7), and note that $v_1(z)$ and $e^{2\pi i\mu}v_1(ze^{-2\pi i})$ are dominant solutions in the sector $\frac{1}{2}\pi+\nu \le \mathrm{ph} z \le \frac{3}{2}\pi-\nu$, and have exactly the same asymptotic expansion there. Thus there is a "Stokes multiplier" C such that

$$v_1(z) = e^{2\pi i\mu}v_1(ze^{-2\pi i}) + Cv_2(z). \tag{3.11}$$

With this connection formula we obtain the following asymptotic behaviour for $v_1(z)$:

(3.12.a) $\quad v_1(z) \sim e^{-z}z^\mu, \qquad -\frac{1}{2}\pi < \mathrm{ph} z < \frac{1}{2}\pi, \qquad$ subdominant,

(3.12.b) $\quad v_1(z) \sim e^{-z}z^\mu, \qquad \frac{1}{2}\pi < \mathrm{ph} z < \frac{3}{2}\pi, \qquad$ dominant,

(3.12.c) $\quad v_1(z) \sim C, \qquad \frac{3}{2}\pi < \mathrm{ph} z < \frac{5}{2}\pi, \qquad$ dominant.

Again, somewhere in the sector $\frac{1}{2}\pi < \mathrm{ph} z < \frac{3}{2}\pi$, new exponentially small terms appear. To obtain more information concerning the change in this sector, we truncate (3.9.a) after N terms

$$v_1(z) = e^{-z}z^\mu \sum_{s=0}^{N-1} \frac{a_s}{z^s} + R_N(z). \tag{3.13}$$

In the appendix we shall prove that the remainder is a solution of the following inhomogeneous differential equation:

$$\begin{aligned}\frac{d^2}{dz^2}R_N(z) + f(z)\frac{d}{dz}R_N(z) + g(z)R_N(z) &= e^{-z}z^{\mu-N-1}Na_N \\ &- e^{-z}z^\mu \sum_{k=N+2}^{\infty} z^{-k}\sum_{m=0}^{N-1} a_m\Big[g_{k-m} - f_{k-m} - (m-\mu)f_{k-1-m}\Big].\end{aligned} \tag{3.14}$$

Olver shows in [8] that

$$a_N \sim \Lambda(-1)^N\Gamma(N-\mu), \qquad \text{as } N\to\infty, \tag{3.15}$$

where Λ is a constant. From this result we obtain that the first term of the right-hand side of (3.14) is minimal for $N \sim |z|$. We take $N = |z| + \beta$, with β bounded. In the appendix we shall show that with this N we can estimate the right-hand side of (3.14) by its first term;

$$\begin{aligned} r.h.s.\ of\ (3.14) &\sim e^{-z}z^{\mu-N-1}Na_N \\ &\sim \Lambda\sqrt{\frac{2\pi}{r}}e^{i\theta(\mu-1)-i\beta(\theta-\pi)}\exp\Big(-r\big[e^{i\theta}+1+i(\theta-\pi)\big]\Big), \\ &|z|\to\infty, \quad z = re^{i\theta}, \quad N = r+\beta, \end{aligned} \tag{3.16}$$

which is maximal at $\theta = \pi$. Again, we use matched asymptotic expansions, but now with $r = r_0/\varepsilon$ and $\theta = \pi + \delta\theta_0$.

$$\begin{aligned} -\frac{e^{-2i\delta\theta_0}}{r_0^2}\frac{\varepsilon^2}{\delta^2}\frac{d^2}{d\theta_0^2}R_N(z) &+ i\Big((1+\mu)\frac{e^{-2i\delta\theta_0}}{r_0^2}\frac{\varepsilon^2}{\delta} + \frac{e^{-i\delta\theta_0}}{r_0}\frac{\varepsilon}{\delta} + \cdots\Big)\frac{d}{d\theta_0}R_N(z) \\ &+ \Big(\frac{g_2 e^{-2i\delta\theta_0}}{r_0^2}\varepsilon^2 + \cdots\Big)R_N(z) \\ &\sim \Lambda\sqrt{\frac{2\pi\varepsilon}{r_0}}e^{i(\pi+\delta\theta_0)(\mu-1)-i\beta\delta\theta_0}\exp\Big(\frac{r_0}{\varepsilon}(e^{i\delta\theta_0}-1-i\delta\theta_0)\Big) \\ &\sim \Lambda\sqrt{\frac{2\pi\varepsilon}{r_0}}e^{i\pi(\mu-1)}\exp\Big(-\tfrac{1}{2}r_0\theta_0^2\frac{\delta^2}{\varepsilon}\Big), \end{aligned} \tag{3.17}$$

as $\delta,\ \varepsilon \to 0$. From the final exponential we obtain that we have to take $\delta^2 = \varepsilon$ in order to obtain an interesting balance. The dominant terms are the $\varepsilon^{\frac{1}{2}}$ terms, and we obtain

$$i\frac{d}{d\theta_0}R_N(z) \sim \Lambda\sqrt{2\pi}e^{i\pi(\mu-1)}r_0^{\frac{1}{2}}\exp\bigl(-\tfrac{1}{2}r_0\theta_0^2\bigr), \tag{3.18}$$

with the solution

$$R_N(z) \sim \Lambda 2\pi i e^{i\pi\mu}\bigl[A + \tfrac{1}{2}\mathrm{erf}(\theta_0\sqrt{\tfrac{1}{2}r_0})\bigr], \tag{3.19}$$

as $\varepsilon \to 0$, with $z = (r_0/\varepsilon)\exp\bigl(i(\pi + \sqrt{\varepsilon}\theta_0)\bigr)$. Matching as $\theta_0 \to -\infty$, the $R_N(z)$ is exponentially small. Thence, $A = \frac{1}{2}$. And from (3.12.c) we obtain that $\Lambda 2\pi i e^{i\pi\mu} = C$, such that the change in the neighbourhood of the Stokes line ph$z = \pi$ is given by

$$R_N(z) \sim C\Bigl[\tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}\bigl(\theta_0\sqrt{\tfrac{1}{2}r_0}\bigr)\Bigr], \tag{3.20}$$

in agreement with [5].

4. The connection coefficient C. The result (3.15) describes the growth of the coefficients of asymptotic expansion (3.9.a), where Λ is a constant. At the end of the previous section we showed that $\Lambda = Ce^{-i\pi\mu}/2\pi i$, where C is the constant in connection formula (3.11). Thence

$$a_N \sim C\frac{(-1)^N}{2\pi i}e^{-i\pi\mu}\Gamma(N-\mu), \qquad \text{as } N \to \infty. \tag{4.1}$$

Since we have the recurrence relation (3.10), we can use (4.1) for the numerical computation of C, but, in general it is very difficult to obtain the value of C analytically. Indeed, only when (3.10) is simple enough for us to find a_N analytically, can (4.1) be used to obtain the exact value of C. For example, for the confluent hypergeometric functions (3.10) is a two term recurrence relation, but for these functions there are also different methods (e.g. by integral representations) that can be used to obtain the connection constants. There are however more complicated differential equations for which the result (4.1) can be used to obtain the exact value of C. Consider the equation

$$\frac{d^2}{dz^2}v(z) + (1 + \frac{1}{z} + \frac{\lambda}{z^2})\frac{d}{dz}v(z) = 0, \tag{4.2}$$

with the solutions

$$\begin{aligned} &v_1(z) \sim e^{-z}z^{-1}\sum_{s=0}^{\infty}\frac{a_s}{z^s}, \qquad |\mathrm{ph}z| \le \tfrac{3}{2}\pi - \nu, \\ &v_2(z) = 1. \end{aligned} \tag{4.3}$$

The recurrence relation for the coefficients can be written in the form: $a_0 = 1$, and for $s \ge 1$

$$s(a_s + sa_{s-1}) = \lambda\bigl(a_{s-1} + (s-1)a_{s-2}\bigr), \tag{4.4}$$

from which we obtain

$$a_s = (-1)^s s! \sum_{k=0}^{s} \frac{(-\lambda)^k}{(k!)^2}. \tag{4.5}$$

Hence, the constant C in

$$v_1(z) = v_1(ze^{-2\pi i}) + C,$$

is

$$C = \lim_{s\to\infty} \frac{-a_s}{(-1)^s s!} = -\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(k!)^2} = -J_0(2\sqrt{\lambda}). \tag{4.7}$$

There are also cases where the connection constants can be written down almost by inspection. An example is the differential equation for the Airy function $\mathrm{Ai}(z)$, namely

$$\frac{d^2}{dz^2}u(z) - zu(z) = 0. \tag{4.8}$$

This differential equation is not of the form (3.1), but the transformation $z \mapsto x^{\frac{2}{3}}$ leads to a differential equation of the form (3.1). From §3 we obtain that equation (4.8) has Stokes lines at $\mathrm{ph}z = 0,\ \frac{2}{3}\pi$, and $\frac{4}{3}\pi$. The Airy function $\mathrm{Ai}(z)$ is the subdominant solution at $\mathrm{ph}z = 0$. A second solution of (4.8) is $\mathrm{Ai}(ze^{-\frac{2}{3}\pi i})$, which is the subdominant solution at $\mathrm{ph}z = \frac{2}{3}\pi$. These two solutions have the following asymptotic behaviour:

$$\begin{aligned} \mathrm{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{\frac{3}{2}}} z^{-\frac{1}{4}}, && \left|\mathrm{ph}(z)\right| \le \pi - \nu, \\ \mathrm{Ai}(ze^{-\frac{2}{3}\pi i}) &\sim \frac{e^{\frac{1}{6}\pi i}}{2\sqrt{\pi}} e^{\frac{2}{3}z^{\frac{3}{2}}} z^{-\frac{1}{4}}, && \left|\mathrm{ph}(z) - \tfrac{2}{3}\pi\right| \le \pi - \nu, \end{aligned} \quad \text{as } |z| \to \infty. \tag{4.9}$$

It is now possible to obtain the connection formula for $\mathrm{Ai}(z)$ just from (3.11) and the fact that $\mathrm{Ai}(z)$ is an entire function. From (3.11) we obtain that

$$\mathrm{Ai}(z) = e^{-\frac{1}{3}\pi i}\mathrm{Ai}(ze^{-\frac{4}{3}\pi i}) + C_1\mathrm{Ai}(ze^{-\frac{2}{3}\pi i}). \tag{4.10}$$

A similar argument gives

$$\mathrm{Ai}(z) = C_2\mathrm{Ai}(ze^{\frac{2}{3}\pi i}) + e^{\frac{1}{3}\pi i}\mathrm{Ai}(ze^{\frac{4}{3}\pi i}). \tag{4.11}$$

Since the Airy function is an entire function, we know that $\mathrm{Ai}(ze^{-\frac{4}{3}\pi i}) = \mathrm{Ai}(ze^{\frac{2}{3}\pi i})$ and that $\mathrm{Ai}(ze^{-\frac{2}{3}\pi i}) = \mathrm{Ai}(ze^{\frac{4}{3}\pi i})$. Therefore, we obtain from (4.10) and (4.11) the connection formula

$$\mathrm{Ai}(z) = e^{-\frac{1}{3}\pi i}\mathrm{Ai}(ze^{\frac{2}{3}\pi i}) + e^{\frac{1}{3}\pi i}\mathrm{Ai}(ze^{-\frac{2}{3}\pi i}). \tag{4.12}$$

Figure (4.1) shows the appearance of exponentially small terms in the asymptotic behaviour of $\mathrm{Ai}(z)$ in the sector $0 \le \mathrm{ph}z \le 2\pi$.

Remark. In [6] the result (4.12) is obtained by the knowledge that there are constants $a,\ b,\ c$, not all zero, such that

$$a\mathrm{Ai}(z) + b\mathrm{Ai}(ze^{\frac{2}{3}\pi i}) + c\mathrm{Ai}(ze^{-\frac{2}{3}\pi i}) = 0. \tag{4.13}$$

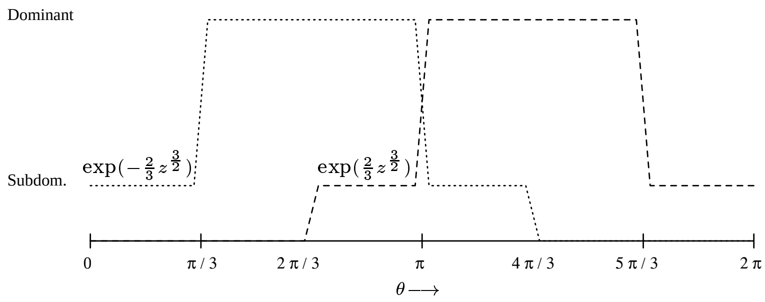

FIG. 4.1. *The appearance of exponentially small terms in the asymptotic behaviour of* Ai(z).

The values of a, b, and c are obtained by looking at the asymptotic behaviour of the left-hand side of (4.13) at the Stokes lines at ph$z = 0$, $\pm\frac{3}{2}\pi$. Alternatively, we can use arguments involving the deformation of contour integral representations of Ai(z) to derive (4.12).

5. Discussion and Conclusion. Using the format of matched asymptotic expansions, we have presented as unified an account as possible of the asymptotic behaviour of the solutions of certain ordinary differential equations in the vicinity of Stokes lines. The local behaviour of $R_N(z)$, the error in the optimally-truncated asymptotic expansion of the dominant solution, is affected by the dependence of N on z and the precise growth of a_n for $n > N$. For the problems we have considered, this behaviour is characterised by expressions such as (2.13) and (3.20), where both the argument of the error function and also the region over which $R_N(z)$ varies rapidly, depend crucially on the properties of N and a_n. This region is parabolic in the cases considered in [7], but in §2 it is wedge-shaped and other shapes are also possible.

For the class of problems considered in §3, we have presented a systematic procedure for determining the requisite information about $R_N(z)$, although in many cases this procedure involves numerical solution of the recurrence relation (3.10). Indeed, the idea of relating the characterisation of Stokes phenomenon to such recurrence relations has already been proposed in Immink [3].

The arguments we have put forward strongly suggest that a necessary and sufficient condition for the existence of an analytic expression for the Stokes multipliers is that (3.10) be explicitly solvable. It is an interesting conjecture as to how the explicit solvability of (3.10) reflects other global properties of the solutions.

6. Appendix. We use the notations of §3, and we define

$$\tilde{v}_1(z) = e^{-z} \sum_{s=0}^{N-1} \frac{a_s}{z^{s-\mu}}, \tag{6.1}$$

such that

$$v_1(z) = \tilde{v}_1(z) + R_N(z). \tag{6.2}$$

When we use (3.10) several times, we obtain

(6.3)
$$\begin{aligned}
&\frac{d^2}{dz^2}\tilde{v}_1(z)+f(z)\frac{d}{dz}\tilde{v}_1(z)+g(z)\tilde{v}_1(z)\\
&=e^{-z}z^{\mu}\Big[\frac{N(N-1-\mu)a_{N-1}}{z^{N+1}}-\sum_{k=N+1}^{\infty}z^{-k}\sum_{m=0}^{N-1}a_m(f_{k-m}-g_{k-m})\\
&\qquad -z^{-(N+1)}\sum_{m=0}^{N-2}(m-\mu)a_m f_{N-m}-\sum_{k=N+2}^{\infty}z^{-k}\sum_{m=0}^{N-1}(m-\mu)a_m f_{k-1-m}\Big]\\
&=_{*}e^{-z}z^{\mu}\Big[-\frac{Na_N}{z^{N+1}}+\sum_{k=N+2}^{\infty}z^{-k}\sum_{m=0}^{N-1}a_m\Big[g_{k-m}-f_{k-m}-(m-\mu)f_{k-1-m}\Big]\Big],
\end{aligned}$$

where in $*$ we use (3.10) with $s=N$. The combination of (6.2) with (6.3) leads to (3.14). To estimate the infinite sum in the final line of (6.3), we use

$$|a_m|\le K_1\big|\Gamma(m-\mu)\big|,\qquad m\ge 0,\tag{6.4.a}$$

$$|f_j|,\ |g_j|\le K_2\rho^j,\qquad j\ge 0,\tag{6.4.b}$$

where K_1 and K_2 are constants. To obtain (6.4.a), we use (3.15), and to obtain (6.4.b), we use the fact that the expansions of (3.8) converge for $|z|\ge\rho$. We substitute (6.4) into

(6.5)
$$\begin{aligned}
&\Big|\sum_{k=N+2}^{\infty}z^{-k}\sum_{m=0}^{N-1}a_m\Big[g_{k-m}-f_{k-m}-(m-\mu)f_{k-1-m}\Big]\Big|\\
&\qquad\le K_1K_2\sum_{k=N+2}^{\infty}\Big(\frac{\rho}{|z|}\Big)^k\sum_{m=0}^{N-1}\frac{\big|\Gamma(m-\mu)\big|}{\rho^m}\Big[2+\frac{m+|\mu|}{\rho}\Big]\\
&\qquad=K_1K_2\Big(\frac{\rho}{|z|}\Big)^{N+2}\frac{|z|}{|z|-\rho}\sum_{m=0}^{N-1}\frac{\big|\Gamma(m-\mu)\big|}{\rho^m}\Big[2+\frac{m+|\mu|}{\rho}\Big]\\
&\qquad\le K_1K_2\Big(\frac{2\rho^2+(N-1+|\mu|)\rho}{|z|^{N+2}}\Big)\frac{|z|}{|z|-\rho}\big|\Gamma(N-\mu)\big|\\
&\qquad\qquad\cdot\sum_{m=0}^{N-1}\frac{\rho^{N-m}}{\big|(m-\mu)(m+1-\mu)\cdots(N-1-\mu)\big|}\\
&\qquad\le_{*}K_1K_2\Big(\frac{2\rho^2+(N-1+|\mu|)\rho}{|z|^{N+2}}\Big)\frac{|z|}{|z|-\rho}\big|\Gamma(N-\mu)\big|K(\mu,\rho)\frac{N\rho}{|N-1-\mu|}\\
&\qquad=\frac{N}{|z|^{N+2}}\big|\Gamma(N-\mu)\big|\mathcal{O}(1),
\end{aligned}$$

as $|z|\to\infty$. In $*$ we use that

$$\frac{\rho^{N-m}}{\big|(m-\mu)(m+1-\mu)\cdots(N-1-\mu)\big|}\le K(\mu,\rho)\frac{\rho}{|N-1-\mu|},\tag{6.6}$$

$m=0,1,\cdots,N-1$, where $K(\mu,\rho)$ does not depend on N. From (3.15) we obtain that

$$\Big|\frac{Na_N}{z^{N+1}}\Big|\sim\Lambda\frac{N}{|z|^{N+1}}\big|\Gamma(N-\mu)\big|,\tag{6.7}$$

as $N \sim |z| \to \infty$. The combination of (6.7) and (6.5) leads to the first line of (3.16).

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