Nucleation of vortices in type-II superconductors in increasing magnetic fields

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Abstract

The stability of the superconducting solution of a limiting form of the Ginzburg-Landau model is examined, to determine the point of first nucleation of vortices.

Keywords. superconductivity, vortices, instability, generation.

When a type-II superconductor below the critical temperature is placed in a sufficiently weak magnetic field, the field will be expelled from the interior of the sample except in thin boundary layers (this is known as the Meissner state). If the magnetic field is then increased there will be a point at which this superconducting state becomes unstable, and the magnetic field will begin to penetrate the sample in the form of quantised flux tubes, each circled by a vortex of superconducting current.

A popular model describing the response of a superconducting material to an applied magnetic field is the Ginzburg-Landau model [3, 6]. We consider a cylinder of superconducting material, unbounded in the z-direction, in an applied axial magnetic field $(0, 0, H_0)$. In [2] the following limiting form of the Ginzburg-Landau model was derived for large values of the Ginzburg-Landau parameter:

$$\boldsymbol{H} = \lambda \operatorname{curl} \boldsymbol{Q} \qquad \text{in } \Omega, \tag{1}$$

$$-\lambda^2 \operatorname{curl}^2 \boldsymbol{Q} = \left(1 - |\boldsymbol{Q}|^2\right) \boldsymbol{Q} \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{Q} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{3}$$

$$H = H_0 \qquad \text{on } \partial\Omega, \tag{4}$$

where $\boldsymbol{H} = (0, 0, H(x, y))$ is the magnetic field, $\boldsymbol{Q} = (Q_1(x, y), Q_2(x, y), 0)$ is the magnetic vector potential, $\Omega \in \mathbb{R}^2$ is the cross-section of the cylinder, and λ is known as the penetration depth of the magnetic field; typically $\lambda \ll 1$[1]. The solution of (1)-(4) was shown to be unstable whenever $|\boldsymbol{Q}| > 1/\sqrt{3}$, and it is conjectured that this instability will lead to the formation of vortices. In [1] it was shown that this instability will occur first on the boundary of the domain.

Many authors have examined the magnetic field at which the instability occurs under the assumption that Ω is large and $\partial\Omega$ is slowly varying by comparison to λ, in which case the superconductor is approximated by a halfspace [4, 5, 7]. The aim of this letter is to determine the point on the boundary at which the instability will occur; this will be the point of first nucleation of vortices.

[1] In [2] lengths were scaled with λ. Here they have been scaled with the size of the domain.

Let the boundary $\partial\Omega$ be given by $\Gamma(s)$, where s is arc length. We define

$$\boldsymbol{x} = \Gamma(s) + \lambda r \boldsymbol{n}, \qquad \lambda \ll 1. \tag{5}$$

Then r, s form a local orthogonal curvilinear coordinate system, with scaling factors

$$g_s = 1 - \lambda r\kappa, \qquad g_r = \lambda,$$

where κ is the curvature of Γ, positive if the centre of curvature lies in Ω. Equations (1)-(4) become

$$\begin{aligned}
-\frac{\partial H}{\partial r} &= \left(1 - Q_r^2 - Q_s^2\right) Q_s \qquad r > 0, & (6)\\
\frac{\lambda}{1-\lambda r\kappa}\frac{\partial H}{\partial s} &= \left(1 - Q_r^2 - Q_s^2\right) Q_r \qquad r > 0, & (7)\\
H &= \frac{\lambda}{1-\lambda r\kappa}\frac{\partial Q_r}{\partial s} - \frac{1}{(1-\lambda r\kappa)}\frac{\partial}{\partial r}\left((1+\lambda r\kappa)Q_s\right), \quad r > 0, & (8)\\
H(0,s) &= H_0, & (9)\\
Q_r(0,s) &= 0, & (10)\\
Q_r,\, Q_s,\, H &\to 0, \qquad \text{as } r \to \infty. & (11)
\end{aligned}$$

We expand H, Q_r and Q_s in powers of λ:

$$\begin{aligned}
H &= H^{(0)} + \lambda H^{(1)} + \cdots, & (12)\\
Q_r &= Q_r^{(0)} + \lambda Q_r^{(1)} + \cdots, & (13)\\
Q_s &= Q_s^{(0)} + \lambda Q_s^{(1)} + \cdots. & (14)
\end{aligned}$$

Equating coefficients of powers of λ in (6)-(11) gives at leading order

$$\begin{aligned}
Q_r^{(0)} &= 0, & (15)\\
\frac{\partial^2 Q_s^{(0)}}{\partial r^2} &= \left(1 - \left(Q_s^{(0)}\right)^2\right) Q_s^{(0)}, \qquad r > 0, & (16)\\
\frac{\partial Q_s^{(0)}}{\partial r}(0,s) &= -H_0. & (17)
\end{aligned}$$

Thus $Q_s^{(0)}$ is simply the solution corresponding to a halfspace, which for ease of notation we will denote simply by Q, and the slow variations in Ω are not noticed at leading order. Equating coefficients of powers of λ at the next order in (6)-(11) gives

$$\begin{aligned} Q_r^{(1)} &= 0, && (18)\\ \frac{\partial^2 Q_s^{(1)}}{\partial r^2} - \kappa\frac{dQ}{dr} &= \left(1-3Q^2\right)Q_s^{(1)}, \qquad r>0, && (19)\\ \frac{\partial Q_s^{(1)}}{\partial r}(0,s) &= \kappa Q(0), && (20)\\ Q_s^{(1)} &\to 0, \qquad \text{as } r\to\infty. && (21) \end{aligned}$$

Thus we see that the position on the boundary s appears only parametrically through the function $\kappa(s)$. We find the solution of (19)-(21) is given by

$$Q_s^{(1)}(r,s) = \kappa(s)A(r), \tag{22}$$

where

$$\begin{aligned} \frac{d^2A}{dr^2} - \frac{dQ}{dr} &= \left(1-3Q^2\right)A, \qquad r>0, && (23)\\ \frac{dA}{dr}(0) &= Q(0), && (24)\\ A &\to 0, \qquad \text{as } r\to\infty. && (25) \end{aligned}$$

The instability of the boundary current is controlled by the magnitude of $\boldsymbol{Q}$. We have

$$|\boldsymbol{Q}|^2 = Q^2 + 2\lambda\kappa QA + \cdots.$$

We take the leading order solution Q to be such that we are just on the point of instability, i.e., such that $Q(0) = 1/\sqrt{3}$. Then

$$|\boldsymbol{Q}(0,s)|^2 = \frac{1}{3} + \frac{2}{\sqrt{3}}\lambda\kappa(s)A(0) + \cdots,$$

and the instability will occur first at the point at which $\kappa A(0)$ is largest. Clearly then the nucleation site will be the point of largest curvature. In order to determine whether this will be positive or negative curvature we need to determine the sign of $A(0)$.

Since $dQ/dr < 0$ we see that A cannot have a negative minimum. Integrating (23) we find that

$$\int_0^\infty \left(1 - 3Q^2\right) A\, dr = 0.$$

Since $Q^2 < 1/3$ we conclude that A must take both signs. Then, in order to satisfy (25), we must have $A(0) < 0$. Hence the nucleation site will be the point of largest negative curvature.

Conclusion We have examined the stability of the superconducting solution to a limiting form of the Ginzburg-Landau model in a large and slowly varying domain. The solution first becomes unstable at the point of largest negative curvature of the domain boundary. This, we conjecture, will be the point of first nucleation of superconducting vortices.

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