A model for variable thickness superconducting thin films

S. J. Chapman *
Mathematical Institute
University of Oxford
24-29 St. Giles'
Oxford OX1 3LB, UK

Qiang Du †
Department of Mathematics
Michigan State University
East Lansing MI 48824-1027, USA

M. D. Gunzburger ‡
Interdisciplinary Center for Applied Mathematics
Virginia Tech
Blacksburg VA 24061-0531, USA

Abstract

A model for superconductivity in thin films having variable thickness is derived through an averaging process across the film. When the film is of uniform thickness the model is identical to a model for superconducting cylinders as the Ginzburg-Landau parameter tends to ∞. This means that all superconducting materials, whether type I

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or type II in bulk, behave as type-II superconductors when made into sufficiently thin films. When the film is of non-uniform thickness the variations in thickness appear as spatially varying coefficients in the thin-film differential equations. After providing a formal derivation of the model, some results about solutions of the variable thickness model are given. In particular, it is shown that solutions obtained from the new model are an appropriate limit of a sequence of averages of solutions of the three-dimensional Ginzburg-Landau model as the thickness of the film tends to zero. An application of the variable thickness thin film model to flux pinning is then provided. In particular, the results of a numerical calculation are given that show that the vortex-like structures present in superconductors are attracted to relatively thin regions.

1 Introduction

Type-II superconductors are characterised by the appearance of vortex-like structures of superconducting current. When a transport current is applied to (or induced in) the superconductor these vortices experience a force which causes them to move. This movement dissipates energy and leads to an electrical resistance, and is therefore undesireable in applications. Thus the immobilization of vortices at “pinning sites” has been the subject of much recent research. Many methods of pinning vortices have been proposed, including introducing impurities, dislocations, grain boundaries — in fact any inhomogeneity — into the material.

Many experiments are performed on thin film superconductors in applied magnetic fields which have a component perpendicular to the film, and it is this configuration which is the subject of the present paper. By taking advantage of the fact that the film is very thin we are able to derive a simplified model in which the variations in film thickness appear as spatially varying coefficients. It is then hoped to use this model to study the effect that these spatial variations have on vortex motion and vortex pinning.

We will find that all superconductors, even those that are type-I materials when in bulk form, behave as extreme type-II materials when the film is thin enough (which will be quantified later); see [9, 11, 12, 13, 14, 15]. In Section 2, we introduce the Ginzburg-Landau theory of superconductivity and derive the thin-film model using the methods of formal asymptotics. In Section 3, we justify the formal procedure rigorously. In Section 4, we present the results of some preliminary numerical calculations. In Section 5, we discuss the time dependent version of the variable thickness thin film model. Finally, in Section 6, we present our conclusions.

2 Formal derivation of the thin-film model

Our analysis is based in the Ginzburg-Landau theory of superconductivity. For a more complete introduction to this theory the reader is referred to [3, 4] and the references therein. Here we merely state the dimensionless Ginzburg-Landau equations as

$$\begin{aligned}
\left(\frac{1}{\kappa}\nabla - i\boldsymbol{A}\right)^2\Psi &= \left(|\Psi|^2-1\right)\Psi \quad \text{in } \Omega\,, &(1)\\
(\mathrm{curl})^2\boldsymbol{A} &= \frac{i}{2\kappa}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right) - |\Psi|^2\boldsymbol{A} \quad \text{in } \Omega\,, &(2)\\
(\mathrm{curl})^2\boldsymbol{A} &= \boldsymbol{0} \quad \text{in } \mathrm{I\!R}^3\backslash\Omega\,, &(3)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
[\mathrm{curl}\boldsymbol{A}] &= \boldsymbol{0}\,, &(4)\\
[\boldsymbol{A}] &= 0\,, &(5)\\
\boldsymbol{n}\cdot\left(\frac{1}{\kappa}\nabla - i\boldsymbol{A}\right)\Psi &= 0 \quad \text{on } \partial\Omega\,, &(6)\\
\mathrm{curl}\ \boldsymbol{A} &\to \boldsymbol{H}_{ext} \quad \text{as } |\boldsymbol{x}|\to\infty\,. &(7)
\end{aligned}$$

Here Ψ is the complex superconducting order parameter such that $|\Psi|^2$ represents the number density of superconducting electrons ($|\Psi| = 1$ corresponds to the superconducting state, $|\Psi| = 0$ corresponds to the normal state), $\boldsymbol{A}$ is the magnetic vector potential which is such that the magnetic field $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = \text{curl } \boldsymbol{A}\,, \tag{8}$$

Ω is the domain occupied by the superconducting material, $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, $[\,\cdot\,]$ denotes the jump in the enclosed quantity across $\partial\Omega$, * denotes complex conjugation, and κ is a material parameter which determines the type of superconducting material; $\kappa < 1/\sqrt{2}$ describes a type-I superconductor, $\kappa > 1/\sqrt{2}$ describes a type-II superconductor.

Equations (1)-(7) are gauge invariant in the sense that they are invariant under transformations of the form

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla w\,, \qquad \Psi \to \Psi e^{i\kappa w}.$$

We may rewrite the equations in terms of real variables by introducing the gauge invariant potential

$$\boldsymbol{Q} = \boldsymbol{A} - \frac{\nabla\chi}{\kappa}\,, \tag{9}$$

where $\Psi = fe^{i\chi}$ with f and χ real (and χ is taken to be zero in $\mathrm{I\!R}^3\backslash\Omega$). This leads to the following set of equations:

$$\begin{aligned}
\frac{1}{\kappa^2}\nabla^2 f &= f^3 - f + f\,|\boldsymbol{Q}|^2 \qquad \text{in } \Omega\,, && (10)\\
\text{div } \left(f^2\boldsymbol{Q}\right) &= 0 \qquad \text{in } \Omega\,, && (11)\\
(\text{curl})^2\boldsymbol{Q} &= -f^2\boldsymbol{Q} \qquad \text{in } \Omega\,, && (12)\\
(\text{curl})^2\boldsymbol{Q} &= \boldsymbol{0} \qquad \text{in } \mathrm{I\!R}^3\backslash\Omega\,, && (13)\\
\boldsymbol{H} &= \text{curl } \boldsymbol{Q}\,, && (14)\\
[\text{curl } \boldsymbol{Q}] &= \boldsymbol{0}\,, && (15)\\
\boldsymbol{n}\cdot\boldsymbol{Q} &= 0 \qquad \text{on } \partial\Omega\,, && (16)\\
\frac{\partial f}{\partial n} &= 0 \qquad \text{on } \partial\Omega\,, && (17)\\
\boldsymbol{H} &\to \boldsymbol{H}_{ext} \qquad \text{as } |\boldsymbol{x}| \to \infty\,. && (18)
\end{aligned}$$

Note that (11), which was derived from (1), now follows immediately from (12).

We are now in a position to begin our derivation. We consider a three-dimensional thin film that is symmetric with respect to the (x, y)-plane. (Nonsymmetric thickness variations can be treated in an analogous manner.) Thus, the superconducting domain is given by

$$\Omega_\epsilon = \Big\{ \, (x, y, z) \in \mathrm{I\!R}^3 \; : \; (x, y) \in \Omega_0 \,, \;\; -\epsilon d(x, y) < z < \epsilon d(x, y) \, \Big\} \,,$$

where $d(x, y) \geq d_0 > 0$ for all $(x, y) \in \Omega_0$ and Ω_0 is a bounded domain in $\mathrm{I\!R}^2$. We consider the formal asymptotic limit as $\epsilon \to 0$ with κ fixed. The existence of the various limits and the validity of the limiting processes that arise in this derivation will be justified in the next section.

We will find that it is more natural to work on the length scale $1/\kappa$, and we rescale accordingly:

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{\kappa} \,, \qquad \boldsymbol{H} = \kappa \boldsymbol{H}' \,, \qquad \boldsymbol{H}_{ext} = \kappa \boldsymbol{H}'_{ext} \,.$$

We write equations (10)-(18) in component form, dropping the primes:

$$\begin{aligned}
\nabla^2 f &= f^3 - f + f \, |\boldsymbol{Q}|^2 \qquad \text{in } \Omega_\epsilon \,, && (19)\\
\nabla \cdot \Big(f^2 \boldsymbol{Q} \Big) &= 0 \qquad \text{in } \Omega_\epsilon \,, && (20)\\
\kappa^2 \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} \right) &= f^2 Q_1 \qquad \text{in } \Omega_\epsilon \,, && (21)\\
\kappa^2 \left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} \right) &= f^2 Q_2 \qquad \text{in } \Omega_\epsilon \,, && (22)\\
\kappa^2 \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) &= f^2 Q_3 \qquad \text{in } \Omega_\epsilon \,, && (23)\\
\text{curl } \boldsymbol{H} &= \boldsymbol{0} \qquad \text{in } \mathrm{I\!R}^3 \backslash \Omega_\epsilon \,, && (24)\\
\text{curl } \boldsymbol{Q} &= \boldsymbol{H} \,, && (25)
\end{aligned}$$

where $\boldsymbol{H} = (H_1, H_2, H_3)$ and $\boldsymbol{Q} = (Q_1, Q_2, Q_3)$, with boundary conditions

$$\begin{aligned} \frac{\partial f}{\partial n} &= 0 \qquad \text{on } \partial\Omega_\epsilon\,, & (26)\\ \boldsymbol{Q}\cdot\boldsymbol{n} &= 0 \qquad \text{on } \partial\Omega_\epsilon\,, & (27)\\ [\boldsymbol{H}] &= \boldsymbol{0}\,, & (28)\\ \boldsymbol{H} &\to \boldsymbol{H}_{ext} \quad \text{as } \boldsymbol{x}\to\infty\,. & (29) \end{aligned}$$

Defining

$$\overline{g} := \frac{1}{2\epsilon d}\int_{-\epsilon d}^{\epsilon d} g\,dz\,,$$

we integrate equations (21)-(23) from $-\epsilon d$ to ϵd to obtain

$$\begin{aligned} \frac{\kappa^2}{2\epsilon d}\left[H_2\right]_{-\epsilon d}^{\epsilon d} &= \frac{\partial \overline{H}_3}{\partial y} - \overline{f^2 Q_1}\,, & (30)\\ \frac{\kappa^2}{2\epsilon d}\left[H_1\right]_{-\epsilon d}^{\epsilon d} &= \frac{\partial \overline{H}_3}{\partial x} + \overline{f^2 Q_2}\,, & (31)\\ \kappa^2\left(\frac{\partial \overline{H}_2}{\partial x} - \frac{\partial \overline{H}_1}{\partial y}\right) &= \overline{f^2 Q_3}\,. & (32) \end{aligned}$$

Since $\boldsymbol{H}$ is divergence-free we also have

$$\frac{1}{2\epsilon d}\left[H_3\right]_{-\epsilon d}^{\epsilon d} = -\frac{\partial \overline{H}_1}{\partial x} - \frac{\partial \overline{H}_2}{\partial y}\,. \qquad (33)$$

Denoting the leading order behaviour by the superscript (0) we find that, on letting $\epsilon \to 0$ in (30)-(33) (assuming that f, $\boldsymbol{Q}$ and $\boldsymbol{H}$ are bounded) we have

$$\begin{aligned} \left[H_1^{(0)}\right]_{0-}^{0+} &= 0\,, & (34)\\ \left[H_2^{(0)}\right]_{0-}^{0+} &= 0\,, & (35)\\ \left[H_3^{(0)}\right]_{0-}^{0+} &= 0\,. & (36) \end{aligned}$$

Thus the current in the film is insufficient to affect the applied field at leading order, and the solution for $\boldsymbol{H}^{(0)}$ is simply

$$\boldsymbol{H}^{(0)} = \boldsymbol{H}_{ext}\,.$$

This was to be expected since if the current density is bounded then the total current is of order ϵ.

We may now use equation (25) to solve for $\boldsymbol{Q}^{(0)}$ up to the addition of a gradient:

$$\boldsymbol{Q}^{(0)} = (zH_{ext,2}, xH_{ext,3} - zH_{ext,1}, 0) + \nabla w\,, \tag{37}$$

where $\boldsymbol{H}_{ext} = (H_{ext,1}, H_{ext,2}, H_{ext,3})$. Integrating (19) from $-\epsilon d$ to ϵd yields

$$\frac{1}{2\epsilon d}\left[\frac{\partial f}{\partial z}\right]_{-\epsilon d}^{\epsilon d} + \overline{\nabla}^2\overline{f} = \overline{f^3} - \overline{f} + \overline{f\,|\boldsymbol{Q}|^2}, \tag{38}$$

where $\overline{\nabla}$ is the two-dimensional Laplacian operator. By (26) we have

$$\frac{\partial f}{\partial z}(\pm\epsilon d) = \pm\epsilon\nabla d\cdot\nabla f(\pm\epsilon d)\,;$$

hence

$$\frac{1}{2\epsilon d}\left[\frac{\partial f}{\partial z}\right]_{-\epsilon d}^{\epsilon d} = \frac{1}{2}\left(\frac{1}{d}\nabla d\cdot\nabla f(\epsilon d) + \frac{1}{d}\nabla d\cdot\nabla f(-\epsilon d)\right)\,.$$

At this stage we let $\epsilon \to 0$. Now, in the limit $\epsilon \to 0$, $f(\pm\epsilon d) \to \overline{f}^{(0)}$, since both tend to $f(x,y,0)$, so that equation (38) becomes

$$\frac{1}{d}\overline{\nabla}\cdot\left(d\overline{\nabla}\,\overline{f}^{(0)}\right) = (\overline{f}^{(0)})^3 - \overline{f}^{(0)} + \overline{f}^{(0)}\,|\overline{\boldsymbol{Q}}^{(0)}|^2\ . \tag{39}$$

(Note that we have presumed the relations $(\overline{f}^{(0)})^3 = (\overline{f^3})^{(0)}$ and $\overline{f}^{(0)}|\overline{\boldsymbol{Q}}^{(0)}|^2 = (\overline{f|\boldsymbol{Q}|^2})^{(0)}$; these are justified in the next section.) A similar analysis on (20) gives

$$\frac{1}{d}\overline{\nabla}\cdot\left(d(\overline{f}^{(0)})^2\overline{\boldsymbol{Q}}^{(0)}\right) = 0\,. \tag{40}$$

The boundary conditions on (39), (40) are

$$\frac{\partial\overline{f}^{(0)}}{\partial n} = 0\,, \tag{41}$$

$$\overline{\boldsymbol{Q}}^{(0)}\cdot\boldsymbol{n} = 0\,, \tag{42}$$

where $\boldsymbol{n}$ is the unit outward normal to Ω_0.

Finally, since $\overline{\boldsymbol{Q}}^{(0)} = \boldsymbol{Q}^{(0)}(x, y, 0)$, $\overline{\boldsymbol{Q}}^{(0)}$ is known up to the addition of a gradient. Moreover, since curl $\boldsymbol{H}^{(0)} = \boldsymbol{0}$, equation (32) implies $\overline{Q}_3^{(0)} = 0$; thus we in fact know $\overline{\boldsymbol{Q}}^{(0)}$ up to the addition of the gradient of a function of x and y:

$$\overline{\boldsymbol{Q}}^{(0)} = (0, xH_{ext,3}, 0) + \nabla w(x, y). \tag{43}$$

Equations (39)-(42) are then enough to determine $\overline{f}^{(0)}$ and w.

Having determined $\overline{f}^{(0)}$ and $\overline{\boldsymbol{Q}}^{(0)}$ we may determine the correction to the applied magnetic field by proceeding to the next order in equations (21)-(23). This gives

$$\boldsymbol{H} = \boldsymbol{H}_{ext} + \epsilon \boldsymbol{H}^{(1)} + \cdots \tag{44}$$

with $\boldsymbol{H}^{(1)}$ satisfying

$$\begin{aligned} \text{div } \boldsymbol{H}^{(1)} &= 0\,, & (45)\\ \text{curl } \boldsymbol{H}^{(1)} &= \boldsymbol{0}\,, & (46)\\ \left[\boldsymbol{H}^{(1)} \wedge \hat{\boldsymbol{z}}\right]_{0-}^{0+} &= d\overline{\boldsymbol{j}}_s^{(0)}\,, & (47) \end{aligned}$$

where $\overline{\boldsymbol{j}}_s^{(0)}$ is the superconducting surface current given by

$$\overline{\boldsymbol{j}}_s^{(0)} = -\frac{1}{\kappa^2}(\overline{f}^{(0)})^2 \overline{\boldsymbol{Q}}^{(0)}. \tag{48}$$

Complex formulation of the thin film model

If we set

$$\overline{\boldsymbol{Q}}^{(0)} = \overline{\boldsymbol{A}} - \nabla\chi$$

say, where $\overline{\boldsymbol{A}} = (0, xH_{ext,3}, 0)$ and χ is a function of x and y, then, writing $\overline{\Psi} = \overline{f}^{(0)} e^{i\chi}$ we obtain the following equations for $\overline{\Psi}$:

$$\frac{1}{d}\left(\nabla - i\overline{\boldsymbol{A}}\right) \cdot d\left(\nabla - i\overline{\boldsymbol{A}}\right)\overline{\Psi} = \left(|\,\overline{\Psi}\,|^2 - 1\right)\overline{\Psi}, \quad \text{in } \Omega_0, \tag{49}$$

$$\boldsymbol{n} \cdot \left(\nabla - i\overline{\boldsymbol{A}}\right)\overline{\Psi} = 0, \quad \text{on } \partial\Omega_0. \tag{50}$$

Before continuing, we make a remark about the form of equations (49)-(50). In the case in which the film is of uniform thickness, so that d is constant, the equations become

$$\begin{aligned} (\nabla - i\boldsymbol{A})^2 \Psi &= \left(|\Psi|^2 - 1\right) \Psi \qquad \text{in } \Omega_0, & (51)\\ \boldsymbol{n} \cdot (\nabla - i\boldsymbol{A}) \Psi &= 0 \qquad \text{on } \partial\Omega_0, & (52) \end{aligned}$$

with $\boldsymbol{A} = (0, xH_{ext,3}, 0)$. These are exactly the equations that were obtained in [2] for a superconducting cylinder in an axial applied magnetic field of strength $H_{ext,3}$, in the limit as the Ginzburg-Landau parameter $\kappa \to \infty$. This is because the current density is in that case order $1/\kappa^2$, and so again does not affect the magnetic field to leading order. In fact, we see from equations (30)-(31) that the important parameter is ϵ/κ^2, which measures the magnitude of the total current. Providing this parameter is small the leading order magnetic field will simply be the applied field. Thus, for all values of κ, when ϵ is small enough the film will behave as a type-II superconductor, since the equations that govern its behaviour are identical to those for a cylinder in the extreme type-II limit. This means that even type-I materials will form vortices when made into thin films; see [9, 11, 12, 13, 14, 15].

3 Rigorous proof of the convergence

We begin this section by introducing some notation that will be used below. Throughout, for any non-negative integer k and domain $\mathcal{D} \subset \mathbb{R}^n$, $n = 2$ or 3, $H^k(\mathcal{D})$ will denote the Sobolev space of real-valued functions having square integrable derivatives of order up to k. The corresponding spaces of complex-valued functions will be denoted by $\mathcal{H}^k(\mathcal{D})$. Corresponding spaces of vector-valued functions, each of whose components belong to $H^k(\mathcal{D})$, will be denoted by $\mathbf{H}^k(\mathcal{D})$, i.e., $\mathbf{H}^k(\mathcal{D}) = [H^k(\mathcal{D})]^n$. Norms of functions belonging to $H^k(\mathcal{D})$, $\mathbf{H}^k(\mathcal{D})$, and $\mathcal{H}^k(\mathcal{D})$ will all be denoted, without any possible ambiguity, by

$\|\cdot\|_{k,\mathcal{D}}$ or $\|\cdot\|_k$. The latter notation will be used when there is no chance of confusion. For details concerning these spaces, one may consult [1]. We will also use the

$$\begin{aligned}\mathbf{H}(\mathbb{R}^3) \quad &= \quad \text{the closure of the set } \mathbf{C}_0^\infty(\mathbb{R}^3) \\ &\qquad \text{under the norm} \left(\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx\, dy\, dz\right)^{1/2} .\end{aligned}$$

and

$$\mathbf{H}(\mathrm{div}, \mathbb{R}^3) = \{\, \mathbf{A} \in \mathbf{H}(\mathbb{R}^3) \quad : \quad \mathrm{div}\mathbf{A} = 0)\,\}\,.$$

3.1 The Ginzburg-Landau free energy

The nondimensional Gibbs free energy postulated by Ginzburg and Landau for a sample of superconducting material is, up to an unimportant additive constant, given by

$$\begin{aligned}\mathcal{G}_\epsilon(\Psi, \mathbf{A}) \quad &= \quad \int_{\Omega_\epsilon} \left(\frac{1}{2}(|\Psi|^2 - 1)^2 + \left|\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\Psi\right|^2\right) dxdydz \\ &\qquad + \int_{\mathbb{R}^3} |\boldsymbol{h} - \boldsymbol{H}_{ext}|^2 dxdydz\,, \end{aligned} \tag{53}$$

where $\boldsymbol{h} = \mathrm{curl}\mathbf{A}$ and $\boldsymbol{H}_{ext}$ is the constant external field. The physical state is thus described by $(\Psi, \mathbf{A})$ which is the minimizer of $\mathcal{G}$. Such a minimizer satisfies the Ginzburg-Landau equations (1)-(7).

In light of the discussion given in the previous section, we define $\overline{\boldsymbol{A}}_0 = (A_{0,1}, A_{0,2}, 0)$ by

$$\begin{aligned}&\mathrm{div}\, \overline{\boldsymbol{A}}_0 = 0\,, \\ &\mathrm{curl}\, \overline{\boldsymbol{A}}_0 = \boldsymbol{H}_{ext}\,.\end{aligned} \tag{54}$$

For example, for constant $\boldsymbol{H}_{ext}$, we may take $\overline{\boldsymbol{A}}_0(x, y, z) = (zH_{ext,2}, xH_{ext,3} - zH_{ext,1}, 0)$ as before. We can then define an equivalent formulation of the free energy:

$$\widetilde{\mathcal{G}}_\epsilon(\Psi,\mathbf{A}) = \int_{\Omega_\epsilon} \frac{1}{2}(|\Psi|^2-1)^2 + \left|\frac{i}{\kappa}\nabla\Psi + \mathbf{A}\Psi + \overline{\mathbf{A}}_0\Psi\right|^2 dxdydz + \int_{\mathbb{R}^3} |\mathrm{curl}\mathbf{A}|^2 d\mathbb{R}^3 \,. \tag{55}$$

We consider the following variational problem

$$\text{(I)} \qquad \min_{\Psi\in\mathbf{H}^1(\Omega_\epsilon)\,,\,\mathbf{A}\in\mathbf{H}(\mathbb{R}^3)} \widetilde{\mathcal{G}}_\epsilon(\Psi,\mathbf{A})\,.$$

Using a gauge transformation, the above problem is equivalent to

$$\text{(II)} \qquad \min_{\Psi\in\mathbf{H}^1(\Omega_\epsilon)\,,\,\mathbf{A}\in\mathbf{H}(\mathrm{div},\mathbb{R}^3)} \widetilde{\mathcal{G}}_\epsilon(\Psi,\mathbf{A})\,.$$

Let $(\Psi_\epsilon, \boldsymbol{A}_\epsilon)$ denote a minimizer of $\widetilde{\mathcal{G}}_\epsilon(\Psi,\mathbf{A})$. Note that $(\Psi_\epsilon, \boldsymbol{A}_\epsilon + \overline{\boldsymbol{A}}_0)$ is then a minimizer of $\mathcal{G}_\epsilon(\Psi,\boldsymbol{A})$.

3.2 Uniform bounds

Here, we derive some uniform bounds, independent of ϵ, for the minimizer $(\Psi_\epsilon, \boldsymbol{A}_\epsilon)$. First, since $(\Psi,\mathbf{A}) = (0,\mathbf{0})$ is feasible for (II), we get:

Lemma 1 *For any $\epsilon > 0$,*

$$\frac{1}{\epsilon}\widetilde{\mathcal{G}}(\Psi_\epsilon,\boldsymbol{A}_\epsilon) \le 2\int_{\Omega_0} d(x,y)dxdy\,.$$

Thus, we get:

Corollary 1 *For any $\epsilon > 0$,*

$$\frac{1}{\sqrt{\epsilon}}\|\nabla\boldsymbol{A}_\epsilon\|_{\mathbf{L}^2(\mathbb{R}^3)} \le \sqrt{2}\left(\int_{\Omega_0} d(x,y)dxdy\right)^{1/2} \,.$$

Proof: Since for any $\mathbf{A} \in C_0^\infty(\mathrm{I\!R}^3)$ with $\mathrm{div}\mathbf{A} = 0$,

$$\begin{aligned}
\|\mathrm{curl}\mathbf{A}\|^2_{\mathbf{L}^2(\mathrm{I\!R}^3)} &= \sum_{i\neq j}\left[\int_{\mathrm{I\!R}^3}\left(\frac{\partial A_i}{\partial x_j}\right)^2 + \left(\frac{\partial A_i}{\partial x_j}\right)\left(\frac{\partial A_j}{\partial x_i}\right)\right] dx\,dy\,dz \\
&= \sum_{i\neq j}\left[\int_{\mathrm{I\!R}^3}\left(\frac{\partial A_i}{\partial x_j}\right)^2 + \left(\frac{\partial A_i}{\partial x_i}\right)\left(\frac{\partial A_j}{\partial x_j}\right)\right] dx\,dy\,dz \\
&= \int_{\mathrm{I\!R}^3}\left[\sum_{i\neq j}\left(\frac{\partial A_i}{\partial x_j}\right)^2 + \sum_i\left(\frac{\partial A_i}{\partial x_i}\right)^2\right] dx\,dy\,dz
\end{aligned}$$

$$= \|\nabla\mathbf{A}\|^2_{\mathbf{L}^2(\mathrm{I\!R}^3)}\,,$$

we have the norm equivalence

$$\|\mathrm{curl}\mathbf{A}\|^2_{\mathbf{L}^2(\mathrm{I\!R}^3)} = \|\nabla\mathbf{A}\|^2_{\mathbf{L}^2(\mathrm{I\!R}^3)} \qquad \forall\,\mathbf{A} \in \mathbf{H}(\mathrm{div}, \mathrm{I\!R}^3)\,. \quad \square$$

Now, we introduce the scaled variables $z = \epsilon\tilde{z}$,

$$\begin{aligned}
\widetilde{\Psi}_\epsilon(x, y, \tilde{z}) &= \Psi_\epsilon(x, y, \epsilon\tilde{z})\,, \\
\widetilde{\mathbf{A}}_\epsilon(x, y, \tilde{z}) &= \mathbf{A}_\epsilon(x, y, \epsilon\tilde{z})\,, \\
\widetilde{\mathbf{A}}_0(x, y, \tilde{z}) &= \mathbf{A}_0(x, y, \epsilon\tilde{z})\,,
\end{aligned}$$

The domain Ω_ϵ in terms of scaled variables is then given by

$$\widetilde{\Omega}_1 = \Big\{\ (x, y, \tilde{z}) \in \mathrm{I\!R}^3\ :\ (x, y) \in \Omega_0\,,\ \ -d(x, y) < \tilde{z} < d(x, y)\ \Big\}\,.$$

Also, we let

$$\overline{\nabla} = (\,\frac{\partial}{\partial x}\,, \frac{\partial}{\partial y}\,) \quad \text{and} \quad \widehat{\nabla} = \frac{\partial}{\partial \tilde{z}}\,.$$

Using the change of variables, we may restate the earlier bounds as:

Corollary 2 *For any $\epsilon > 0$,*

$$\|\overline{\nabla}\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^2(\mathrm{I\!R}^3)} \leq \sqrt{2}\left(\int_{\Omega_0} d(x, y)dxdy\right)^{1/2}\,,$$

$$\frac{1}{\epsilon}\|\widehat{\nabla}\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^2(\mathrm{I\!R}^3)} \leq \sqrt{2}\left(\int_{\Omega_0} d(x, y)dxdy\right)^{1/2}\,.$$

Using a standard interpolation inequality [10]

$$\|\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^6(\mathbb{R}^3)} \leq c\|\nabla\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

we get,

Corollary 3 *There exists a constant $c > 0$, such that for any $\epsilon > 0$,*

$$\|\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^6(\mathbb{R}^3)} \leq \ c\ .$$

Now, we consider the restriction on Ω_ϵ, or, in the scaled variables, to $\widetilde{\Omega}_1$. First, earlier estimates imply that:

Lemma 2 *There exists a constant $c > 0$, such that for any $\epsilon > 0$,*

$$\|\overline{\nabla}\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^2(\widetilde{\Omega}_1)} \leq \sqrt{2}\left(\int_{\Omega_0} d(x,y)dxdy\right)^{1/2} ,$$

$$\frac{1}{\epsilon}\|\widehat{\nabla}\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^2(\widetilde{\Omega}_1)} \leq \sqrt{2}\left(\int_{\Omega_0} d(x,y)dxdy\right)^{1/2} ,$$

and

$$\|\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{L}^6(\widetilde{\Omega}_1)} \leq \ c\ .$$

By the imbedding $\mathbf{L}^6(\widetilde{\Omega}_1) \hookrightarrow \mathbf{L}^2(\widetilde{\Omega}_1)$, we have the following result:

Proposition 1 *There exists a constant $c > 0$, independent of ϵ, such that:*

$$\|\widetilde{\mathbf{A}}_\epsilon\|_{\mathbf{H}^1(\widetilde{\Omega}_1)} \leq c\,.$$

Next, we consider the order parameter. By Lemma 1 and Proposition 1, we get:

Proposition 2 *There exists a constant $c > 0$, independent of ϵ, such that:*

$$\|\overline{\nabla}\widetilde{\Psi}_\epsilon\|_{\mathcal{L}^2(\widetilde{\Omega}_1)} \leq c\,,$$

$$\frac{1}{\epsilon}\|\widehat{\nabla}\widetilde{\Psi}_\epsilon\|_{\mathcal{L}^2(\widetilde{\Omega}_1)} \leq c\,.$$

Meanwhile, similar to [4], we can show that:

Lemma 3

$$\|\widetilde{\Psi}_\epsilon\|_{\mathcal{L}^\infty(\widetilde{\Omega}_1)} \leq 1\,.$$

Hence:

Corollary 4 *There exists a constant $c > 0$, independent of ϵ, such that*

$$\|\widetilde{\Psi}_\epsilon\|_{\mathcal{H}^1(\widetilde{\Omega}_1)} \leq c\,.$$

3.3 Compactness and convergence

By Corollary 1, we have

Corollary 5 *As $\epsilon \to 0$,*

$$\nabla \boldsymbol{A}_\epsilon \to \mathbf{0} \quad \textit{in } \mathbf{L}^2(\mathbb{R}^3)\,.$$

In particular, the above result implies $\mathrm{curl}\boldsymbol{A}_\epsilon \to \mathbf{0}$, which means that the magnetic field approaches the constant field $\boldsymbol{H}_{ext}$ as $\epsilon \to 0$.

For the scaled variables, we use the estimates in Corollary 2. The weak compactness of $\mathbf{L}^2(\mathbb{R}^3)$ implies that there exists a subsequence $\epsilon_k \to 0$ as $k \to \infty$ such that

$$\nabla \widetilde{\mathbf{A}}_{\epsilon_k} \rightharpoonup \boldsymbol{Q} = (Q_1, Q_2, Q_3) \quad \text{in } \mathbf{L}^2(\mathbb{R}^3)\,,$$

where Q_j is the weak limit of $\nabla \widetilde{\mathbf{A}}_{\epsilon_k, j}$. Now, we prove:

Proposition 3 *As $\epsilon_k \to 0$,*

$$\nabla \widetilde{\mathbf{A}}_{\epsilon_k} \rightharpoonup \mathbf{0} \quad \textit{in } \mathbf{L}^2(\mathbb{R}^3)\,.$$

Proof: Since $\frac{1}{\epsilon}\|\widetilde{\nabla\widetilde{\mathbf{A}}_\epsilon}\|_{\mathbf{L}^2(\mathbb{R}^3)}$ is bounded, independent of ϵ, we get

$$\widetilde{\nabla\widetilde{\mathbf{A}}_{\epsilon_k}} \rightharpoonup \mathbf{0} \ \text{ in } \mathbf{L}^2(\mathbb{R}^3)$$

as $\epsilon_k \to 0$. So $Q_j^3 = 0$ for $j = 1, 2, 3$. On the other hand, since

$$\operatorname{curl} \nabla\widetilde{\mathbf{A}}_{\epsilon_k} = \mathbf{0}$$

in the sense of distributions, we have

$$\operatorname{curl} Q_j = \mathbf{0}\,, \quad j = 1, 2, 3.$$

This implies that $Q_j = \nabla g_j$ for some scalar functions g_j, $j = 1, 2, 3$. However, since $Q_j^3 = 0$, we have $g_j = g_j(x, y)$. Meanwhile, we must have $Q_j = \nabla g_j \in \mathbf{L}^2(\mathbb{R}^3)$. This implies that for any j, g_j must be a constant. Hence, $Q_j = 0$, $j = 1, 2, 3$. $\square$

Using Proposition 1, we get

Theorem 1 *There exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ as $k \to \infty$ such that*

$$\widetilde{\mathbf{A}}_{\epsilon_k} \rightharpoonup \mathbf{0} \quad \textit{in } \mathbf{H}^1(\tilde{\Omega}_1)\,.$$

Proof: By Proposition 1 and 3, we have

$$\widetilde{\mathbf{A}}_{\epsilon_k} \rightharpoonup \boldsymbol{V} \quad \text{in } \mathbf{H}^1(\tilde{\Omega}_1)$$

for some constant vector $\boldsymbol{V}$. Using the boundary condition, we get $\boldsymbol{V} = \mathbf{0}$, so

$$\widetilde{\mathbf{A}}_{\epsilon_k} \rightharpoonup \mathbf{0} \quad \text{in } \mathbf{H}^1(\tilde{\Omega}_1)\,. \quad \square$$

After further extracting a subsequence, we have

Corollary 6 *As $k \to 0$,*

$$\widetilde{\mathbf{A}}_{\epsilon_k} \to \mathbf{0} \quad in\ \mathbf{L}^2(\widetilde{\Omega}_1)\,,$$

$$\widehat{\nabla}\widetilde{\mathbf{A}}_{\epsilon_k} \to \mathbf{0} \quad in\ \mathbf{L}^2(\widetilde{\Omega}_1)\,,$$

$$\int_{-1}^{1} \widetilde{\mathbf{A}}_{\epsilon_k}(x,y,d(x,y)\tau)d\tau \to \mathbf{0} \quad in\ \mathbf{L}^2(\Omega_0)\,.$$

Now, we turn to the order parameter. Using the bound on $\|\widetilde{\Psi}_\epsilon\|_{\mathcal{H}^1(\widetilde{\Omega}_1)}$, we get that there exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ as $k \to \infty$ such that

$$\widetilde{\Psi}_{\epsilon_k} \rightharpoonup \overline{\Psi} \quad \text{in } \mathcal{H}^1(\widetilde{\Omega}_1)\,.$$

Here, we have used the same notation for the subsequence as for the magnetic potential. From Proposition 2, we have $\|\widehat{\nabla}\widetilde{\Psi}_\epsilon\|_{\mathcal{L}^2(\widetilde{\Omega}_1)} \to 0$, so $\overline{\Psi} = \overline{\Psi}(x,y)$. Thus, it follows that:

Theorem 2 *There exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ as $k \to \infty$ such that*

$$\widetilde{\Psi}_{\epsilon_k} \rightharpoonup \overline{\Psi} = \overline{\Psi}(x,y) \quad in\ \mathcal{H}^1(\widetilde{\Omega}_1)\,,$$

$$\|\widehat{\nabla}\widetilde{\Psi}_{\epsilon_k}\|_{\mathcal{L}^2(\widetilde{\Omega}_1)} \to 0\,,$$

$$\widetilde{\Psi}_{\epsilon_k} \to \overline{\Psi} = \overline{\Psi}(x,y) \quad in\ \mathcal{L}^2(\widetilde{\Omega}_1)\,.$$

The last result in fact implies the almost everywhere convergence:

Corollary 7 *As $k \to \infty$,*

$$\frac{1}{2}\int_{-1}^{1} \widetilde{\Psi}_{\epsilon_k}(x,y,d(x,y)\tau)d\tau \rightharpoonup \overline{\Psi} \quad in\ \mathcal{H}^1(\Omega_0)\,,$$

$$\frac{1}{2}\int_{-1}^{1} \widetilde{\Psi}_{\epsilon_k}(x,y,d(x,y)\tau)d\tau \to \overline{\Psi} \quad in\ \mathcal{L}^2(\Omega_0)\,.$$

3.4 Passing to the limit

In order to obtain partial differential equations satisfied by the limiting function $\overline{\Psi}$, we examine the weak form for Ψ_ϵ:

$$\int_{\Omega_\epsilon} \Re\left\{(\frac{i}{\kappa}\nabla + \boldsymbol{A}_\epsilon + \overline{\boldsymbol{A}}_0)\Psi_\epsilon \cdot (-\frac{i}{\kappa}\nabla + \boldsymbol{A}_\epsilon + \overline{\boldsymbol{A}}_0)\phi^*\right\} dxdydz$$
$$+ \int_{\Omega_\epsilon} \Re\left\{(|\Psi_\epsilon|^2 - 1)\Psi_\epsilon\phi^*\right\} dxdydz = 0 \quad \forall\, \phi \in \mathcal{H}^1(\Omega_\epsilon)\,. \tag{56}$$

Setting $\phi = \phi(x,y) \in \mathcal{H}^1(\Omega_0)$ and letting $\epsilon_k \to 0$, we now pass to the limit in each of the terms in the above weak form. For simplicity, we will use ϵ to denote ϵ_k.

i) Using Theorem 2 and Corollary 7,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon} (|\Psi_\epsilon|^2 - 1)\Psi_\epsilon\phi^* dxdydz =$$
$$\int_{\Omega_0} \frac{1}{2}\left(\int_{-1}^{1} (|\Psi_\epsilon(x,y,d(x,y)\tau)|^2 - 1)\cdot \Psi_\epsilon(x,y,d(x,y)\tau)d\tau\right)$$
$$\cdot d(x,y)\phi^*(x,y)dxdy$$
$$\to \int_{\Omega_0} d(x,y)(|\overline{\Psi}(x,y)|^2 - 1)\overline{\Psi}(x,y)\phi^*(x,y)dxdy\,. \tag{57}$$

ii) By Corollary 7,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon} \overline{\mathbf{A}}_0\Psi_\epsilon \cdot \overline{\mathbf{A}}_0\phi^* dxdydz$$
$$= \int_{\Omega_0} \frac{1}{2}\,[\int_{-1}^{1} \Big((H_2^2\epsilon^2 d(x,y)^2\tau^2 + (H_3x - H_1\epsilon d(x,y)\tau)^2)\Big)$$
$$\cdot\Psi_\epsilon(x,y,d(x,y)\tau)d\tau\ ]\ \cdot d(x,y)\phi^*(x,y)dxdy$$
$$\to \int_{\Omega_0} d(x,y)H_3^2x^2\overline{\Psi}(x,y)\phi^*(x,y)dxdy\,. \tag{58}$$

iii) By Corollary 6,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon} \boldsymbol{A}_\epsilon\Psi_\epsilon \cdot \overline{\mathbf{A}}_0\phi^* dxdydz = \frac{1}{2}\int_{\widetilde{\Omega}_1} \widetilde{\mathbf{A}}_\epsilon\widetilde{\Psi}_\epsilon \cdot \overline{\mathbf{A}}_0\phi^* dxdydz \to 0\,.$$

iv) Using Theorem 2 and Corollary 7,

$$\begin{aligned}
\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\frac{i}{\kappa}\nabla\Psi_\epsilon\cdot\overline{\mathbf{A}}_0\phi^* dxdydz &= \int_{\Omega_0}\frac{1}{2}\int_{-1}^{1}\frac{i}{\kappa}\overline{\nabla}\widetilde{\Psi}_\epsilon(x,y,d(x,y)\tau) \\
&\quad\cdot\begin{pmatrix} H_2\epsilon d(x,y)\tau \\ H_3x - H_1\epsilon d(x,y)\tau \end{pmatrix} d(x,y)\phi^* d\tau dxdy \\
&= \int_{\Omega_0}\frac{1}{2}\int_{-1}^{1}\frac{i}{\kappa}\overline{\nabla}\widetilde{\Psi}_\epsilon(x,y,d(x,y)\tau)\cdot\begin{pmatrix} H^2\epsilon d(x,y)\tau \\ -H_1\epsilon d(x,y)\tau \end{pmatrix} d(x,y)\phi^* d\tau dxdy \\
&\quad + \int_{\Omega_0}\frac{i}{2\kappa}\left(\overline{\nabla}\int_{-1}^{1}\widetilde{\Psi}_\epsilon(x,y,d(x,y)\tau)d\tau - \int_{-1}^{1}\widehat{\nabla}\widetilde{\Psi}_\epsilon(x,y,d(x,y)\tau)\overline{\nabla}d(x,y)\tau d\tau\right) \\
&\quad\cdot\begin{pmatrix} 0 \\ H_3x \end{pmatrix} d(x,y)\phi^* d\tau dxdy \\
&\to 0 + \int_{\Omega_0}\frac{i}{\kappa}\overline{\nabla}\,\overline{\Psi}\cdot\begin{pmatrix} 0 \\ H_3x \end{pmatrix} d(x,y)\phi^* dxdy + 0 \\
&\to \int_{\Omega_0} d(x,y)\frac{i}{\kappa}\overline{\nabla}\,\overline{\Psi}\cdot\begin{pmatrix} 0 \\ H_3x \end{pmatrix}\phi^* dxdy
\end{aligned} \tag{59}$$

v) Similar to iii), by Corollary 6, we have

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\overline{\mathbf{A}}_0\Psi_\epsilon\cdot\boldsymbol{A}_\epsilon\phi^* dxdydz \to 0\,.$$

vi) By Corollary 7,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\overline{\mathbf{A}}_0\Psi_\epsilon\cdot(-\frac{i}{\kappa})\nabla\phi^* dxdydz \to \int_{\Omega_0} d(x,y)\begin{pmatrix} 0 \\ H_3x \end{pmatrix}\overline{\Psi}\cdot(-\frac{i}{\kappa}\overline{\nabla}\phi^*)dxdy$$

vii) By Corollary 6,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\boldsymbol{A}_\epsilon\Psi_\epsilon\cdot\boldsymbol{A}_\epsilon\phi^* dxdydz \to 0\,.$$

viii) By Corollary 6,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\boldsymbol{A}_\epsilon\Psi_\epsilon\cdot(-\frac{i}{\kappa}\nabla\phi^*)dxdydz \to 0\,.$$

ix) By Corollary 6 and Theorem 2,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\frac{i}{\kappa}\nabla\Psi_\epsilon\cdot\boldsymbol{A}_\epsilon\phi^* dxdydz \to 0\,.$$

x) By Corollary 7 and Theorem 2,

$$\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\frac{i}{\kappa}\nabla\Psi_\epsilon\cdot(-\frac{i}{\kappa}\nabla\phi^*)dxdydz \to \int_{\Omega_0}\frac{i}{\kappa}\overline{\nabla}\,\overline{\Psi}\cdot(-\frac{i}{\kappa}\overline{\nabla}\phi^*)dxdy\,.$$

Hence, $\overline{\Psi}$ satisfies the weak form:

$$\int_{\Omega_0} d(x,y)\Re\left\{(\frac{i}{\kappa}\overline{\nabla}+\overline{\boldsymbol{B}}_0)\overline{\Psi}\cdot(-\frac{i}{\kappa}\overline{\nabla}+\overline{\boldsymbol{B}}_0)\phi^*\right\}dxdydz$$
$$\int_{\Omega_0} d(x,y)\Re\left\{(|\overline{\Psi}|^2-1)\overline{\Psi}\phi^*\right\}dxdy = 0 \quad \forall\,\phi\in\mathcal{H}^1(\Omega_0)\,, \quad (60)$$

where

$$\overline{\boldsymbol{A}}_0 = \left(\begin{array}{c}\overline{\boldsymbol{B}}_0\\ 0\end{array}\right) \quad \text{and} \quad \overline{\boldsymbol{B}}_0 = \left(\begin{array}{c}0\\ H_3x\end{array}\right).$$

Thus, we obtain the following equations for $\overline{\Psi}$:

$$\frac{1}{d}\left(\overline{\nabla}-i\overline{\boldsymbol{B}}_0\right)\cdot d\left(\overline{\nabla}-i\overline{\boldsymbol{B}}_0\right)\overline{\Psi} = \left(|\,\overline{\Psi}|^2-1\right)\overline{\Psi} \quad \text{in } \Omega_0\,, \quad (61)$$

$$\boldsymbol{n}\cdot\left(\overline{\nabla}-i\overline{\boldsymbol{B}}_0\right)\overline{\Psi} = 0 \quad \text{on } \partial\Omega_0\,. \quad (62)$$

Hence, we have shown that the weak limit of the average of solution of the three dimensional Ginzburg-Landau equations is, as the film thickness tends to zero, the solution of the thin film equation (49)-(50). We state the result as follows:

Theorem 3 *Let $\{\Psi_\epsilon, \mathbf{A}_\epsilon\}$ be a sequence of minimizers for the three dimensional Ginzburg-Landau functional* (55). *Then, there exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k\to 0$ as $k\to\infty$ such that*

$$\frac{1}{2\epsilon d}\int_{-\epsilon d}^{\epsilon d}\overline{\mathbf{A}}_{\epsilon_k}dz \to \mathbf{0} \quad \textit{in } \mathbf{L}^2(\tilde{\Omega}_1)\,,$$

and

$$\frac{1}{2\epsilon d}\int_{-\epsilon d}^{\epsilon d}\Psi_{\epsilon_k}dz \rightharpoonup \overline{\Psi} \quad \textit{in } \mathcal{H}^1(\tilde{\Omega}_1)\,,$$

where $\overline{\Psi}$ is a solution of the thin film equations (49)-(50).

4 Numerical results

Approximate solutions of the thin film equations (49)-(50) have been obtained using a finite element method for both the constant thickness and variable thickness cases. Details concerning the finite element discretization are similar to those found in [4, 5, 6] for the full Ginzburg-Landau equations (1)-(3) and therefore we omit them here.

Due to the necessity of resolving phenomena at the scale of the vortices one can only compute with material samples of very small size. Figure 1 shows the distribution of vortices in a square thin film of constant thickness and having sides equal to 30 coherence lengths. To be more specific, Figure 1 shows the level curves of the magnitude of the order parameter; for the sake of clarity, i.e., to display the vortical structure, we only plot the level curves for values between 0 and 0.5. The applied field is perpendicular to the film and has magnitude given by 0.5. (Recall that we have scaled the magnetic field by κ and that the Ginzburg-Landau parameter does not enter into the leading order thin film equations (49)-(50). In terms of the variables appearing in (1)-(3), the applied field we use has magnitude given by 0.5κ.) Away from the boundaries, the hexagonal structure of the vortex lattice is clearly evident.

Figure 2a shows the distribution of vortices in a square thin film of constant thickness and having sides equal to 20 coherence lengths. Again, the magnitude of the applied filed is 0.5. We next introduce some thinner regions into the sample. Figure 2b shows a square sample having sides equal to 20 coherence lengths in which there are ten thin square regions each having sides equal to one coherence length. In Figure 2c, we show the vortex distribution for the case where the thickness function $d = 0.5$ in the ten regions, with $d = 1$ elsewhere. Comparing Figures 2a and 2c, one sees that vortices are attracted, i.e., pinned, to the thin regions; this becomes more evident in Figure 2d where we have superimposed Figures 2b and 2c. Figures 2e and 2f are figures analogous to Figures 2c and 2d for the case of $d = 0.1$ in the ten thin regions. Again, the thin regions clearly attract vortices.

Correction to the magnetic field

To leading order, the magnetic field is given by the external field. The first-order correction to the magnetic is determined from leading order superconducting surface current $\boldsymbol{j}_s^{(0)}$ by (45)-(47). Figure 3 gives a vector plot of this current, which is determined from (48), for the sample and field configuration that led to Figure 2a. The shielding current running along the boundary of the sample, and the current circulating around each vortex core are cleary depicted.

The solution of (45)-(47) for the corrected field $\boldsymbol{H}^{(1)}$ is given by

$$\boldsymbol{H}^{(1)}(x, y, z) = \int_{\Omega_0} \nabla_x \left(\frac{1}{r}\right) \times \boldsymbol{j}_s^{(0)}(x', y')\, dx'dy' \,, \tag{63}$$

where

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}\,.$$

Figure 4a shows the correction to the field $\boldsymbol{H}^{(1)}$ in an portion of the (x, z)-plane above the horizontal mid-line of the sample, i.e., for a fixed value of y half-way up the sample. The sample and applied field configuration are those that led to Figure 2a. The field correction was obtained from (63) by a Reimann sum approximation to the integral. The midline is depicted at the bottom of the figure so that the portion of the plane displayed extends beyond the sample on each side. Due to symmetry, the y-component of the field vanishes on that plane. In Figure 4b, we display the complete field $\boldsymbol{H}^{(0)} + \epsilon \boldsymbol{H}^{(1)}$ for $\epsilon = 0.1$. We see the pinching of the field lines into the four vortices that intersect with the displayed region.

5 Time dependent model

It is relatively easy to extend the preceeding analysis to a time-dependent setting.

In the time-dependent case we have to allow for a non-zero electric field, and we therefore introduce the electric scalar potential, ϕ, which is such that

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \tag{64}$$

We also choose here to fix the gauge of $\boldsymbol{A}$ by requiring that

$$\mathrm{div}\ \boldsymbol{A} = 0\,.$$

The following time-dependent version of the Ginzburg-Landau equations has been derived by Gor'kov and Éliashberg [8]: for $t > 0$,

$$\alpha\frac{\partial \Psi}{\partial t} + \alpha i \Psi\phi = (\nabla - i\boldsymbol{A})^2\,\Psi + \Psi(1- \mid \Psi\mid^2) \quad \text{in } \Omega\,, \tag{65}$$

$$\begin{aligned} -\kappa^2(\mathrm{curl})^2\boldsymbol{A} - \kappa^2\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right) \\ = \frac{i}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + \mid\Psi\mid^2\,\boldsymbol{A} \quad \text{in } \Omega\,. \end{aligned} \tag{66}$$

We wish to allow for the possibility of passing an electric current through the sample, and therefore the external region will comprise a region D of normally conducting material representing the electrical contacts (which we also assume to be of thickness order ϵ), as well as the vacuum region $\mathrm{I\!R}^3 \backslash D \cup \Omega$. In D we have normal conductivity, giving

$$\begin{aligned} \nabla^2\phi &= 0\,, & (67) \\ (\mathrm{curl})^2\boldsymbol{A} + \sigma\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right) &= 0 & (68) \end{aligned}$$

In the vacuum region $\mathbb{R}^3 \backslash D \cup \Omega$ we simply have

$$(\mathrm{curl})^2 \boldsymbol{A} = 0\,. \tag{69}$$

The boundary conditions on the interface between Ω and D will be the continuity of $\boldsymbol{A}$, $\boldsymbol{H}$, ϕ, $\sigma\partial\phi/\partial n$, where σ is the conductivity of the normal region (scaled with the normal conductivity of the superconducting region, so that $\sigma = 1$ in Ω). The boundary conditions on the other boundaries of Ω are as before, with the extra conditions that $\partial\phi/\partial n = 0$ on $\partial\Omega\backslash\partial\Omega \cap \partial D$, $\sigma\partial\phi/\partial n = I_{ext}$ on $\partial D\backslash\partial\Omega\cap\partial D$, and $[\sigma\partial\phi/\partial n] = 0$ on $\partial\Omega\cap\partial D$ on the electric potential, where I_{ext} is an externally imposed current and $[\cdot]$ represents the jump in the enclosed quantitiy across the indicated surface.

The derivation of the thin film model proceeds in a similar way to Section 2, and we omit the details. We assume that the current density in the electrical contacts is also order one, in which case the magnetic field is again equal to the applied field to leading order. We find that (49), (50) become

$$-\alpha\frac{\partial\overline{\Psi}}{\partial t} - \alpha i\overline{\Psi\phi} + \frac{1}{d}\left(\nabla - i\overline{\boldsymbol{A}}\right)\cdot d\left(\nabla - i\overline{\boldsymbol{A}}\right)\overline{\Psi} = \left(|\,\overline{\Psi}|^2 - 1\right)\overline{\Psi} \quad \text{in } \Omega_0, \tag{70}$$

$$\boldsymbol{n}\cdot\left(\nabla - i\overline{\boldsymbol{A}}\right)\overline{\Psi} = 0 \qquad \text{on } \partial\Omega_0\,. \tag{71}$$

We are left with the problem of determining $\overline{\phi}$. Taking the divergence of (66) gives

$$\nabla^2\phi = \mathrm{div}\ \boldsymbol{j}_s \qquad \text{in } \Omega, \tag{72}$$

where $\boldsymbol{j}_s$ is the superconducting current

$$\boldsymbol{j}_s = \frac{i}{2\kappa^2}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right) - |\,\Psi\,|^2\ \boldsymbol{A}\,.$$

Integrating from $-\epsilon d$ to ϵd gives

$$\nabla\cdot\left(d\nabla\overline{\phi}\right) = \nabla\cdot\left(d\overline{\boldsymbol{j}}_s\right) \qquad \text{in } \Omega_0\,. \tag{73}$$

Similarly

$$\sigma\nabla\cdot\left(d\nabla\overline{\phi}\right)=0 \qquad \text{in } D_0\,, \tag{74}$$

where D_0 denotes the projection of contact region D onto the (x, y)-plane. The boundary conditions on these equations are

$$\begin{aligned}
\frac{\partial\overline{\phi}}{\partial n} &= 0 \qquad \text{on } \partial\Omega_0\backslash\partial D_0\cap\partial\Omega_0\,, &(75)\\
\sigma\frac{\partial\overline{\phi}}{\partial n} &= I_{ext} \qquad \text{on } \partial D_0\backslash\partial D_0\cap\partial\Omega_0\,, &(76)\\
\left[\sigma\frac{\partial\overline{\phi}}{\partial n}\right] &= 0 \qquad \text{on } \partial\Omega_0\cap\partial D_0\,, &(77)
\end{aligned}$$

where $[\cdot]$ represents the jump in the enclosed quantity across the indicated surface. Hence we have the coupled system (70)-(71), (73)-(77) to solve for Ψ and ϕ.

6 Conclusion

We have analysed the behaviour of the Ginzburg-Landau model for a superconducting film in the limit as the film thickness tends to zero. We began with a formal asymptotic analysis to derive a thin-film model, and then justified this formal procedure with a rigorous study of the Ginzburg-Landau free-energy.

When the film is of uniform thickness the model is identical to the model recently derived in [2] for superconducting cylinders as the Ginzburg-Landau parameter $\kappa\to\infty$. This means that all superconducting materials, whether type I or type II in bulk, behave as type-II superconductors when made into sufficiently thin films.

When the film is of non-uniform thickness the variations in thickness appear as spatially varying coefficients in the thin-film differential equations. It has been suggested that such inhomogeneities would serve to "pin" vortices, i.e. that vortices would preferentially be sited at places of relatively low thickness. Our preliminary numerical calculations, presented in Section 4, show that this is indeed the case.

References

[1] R. A. Adams; *Sobolev Spaces*, Academic, New York, 1975.

[2] S. J. Chapman, Q. Du, M. D. Gunzburger and J. S. Peterson; Simplified Ginzburg-Landau type models of superconductivity in the high kappa, high field limit, to appear in *Adv. Math. Sciences Appl.*, 1995.

[3] S. J. Chapman, S. D. Howison, and J. R. Ockendon; Macroscopic models of superconductivity, *SIAM Review* **34**, 1992, 529-560.

[4] Q. Du, M. D. Gunzburger, and J. S. Peterson; Analysis and approximation of the Ginzburg-Landau model of superconductivity. *SIAM Review* **34**, 1992, 54-81.

[5] Q. Du, M. D. Gunzburger, and J. S. Peterson; Solving the Ginzburg-Landau equations by finite element methods, *Phys. Rev. B* **46**, 1992, 9027-9034.

[6] Q. Du, M. D. Gunzburger, and J. S. Peterson; Finite element approximation of a periodic Ginzburg-Landau model for type-II superconductivity, *Num. Math.* **64**, 1993, 85-114.

[7] V. Girault and P.-A. Raviart; *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[8] L. P. Gor'kov and G. M. Éliashberg; Generalisation of the Ginzburg-Landau equations for non-stationary problems in the case of

alloys with paramagnetic impurities. *Soviet Phys. J.E.T.P.* **27**, 1968, 328-334.

[9] G. Lasher; Mixed state of type-I supercondcuting films in a perpendicular magnetic field, *Phys. Rev.* **154**, 1967, 345-348.

[10] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[11] J. D. Livingston and W. Desoro; The intermediate state in type I superconductors, in *Superconductivity*, Ed. by R. D. Parks, Marcel Dekker, New York, 1969, 1235-1281.

[12] K. Maki; Fluxoid structure in superconducting films, Ann. Phys. **34**, 1965, 363-376.

[13] M. Tinkham; Effect of fluxoid quantization on transitions of superconducting films, *Phys. Rev.* **129**, 1963, 2413-2422.

[14] M. Tinkham; Consequences of fluxoid quantization in the transitions of superconducting films, *Rev. Mod. Phys.* **36**, 1964, 268-276.

[15] M. Tinkham; *Introduction to Superconductivity*, McGraw-Hill, New York, 1975.

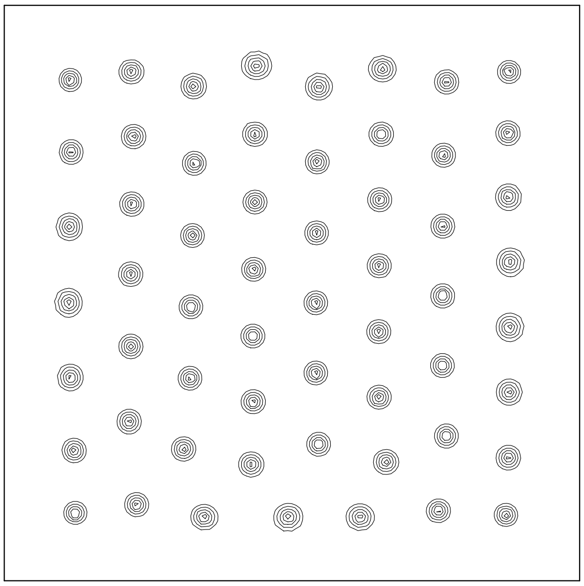
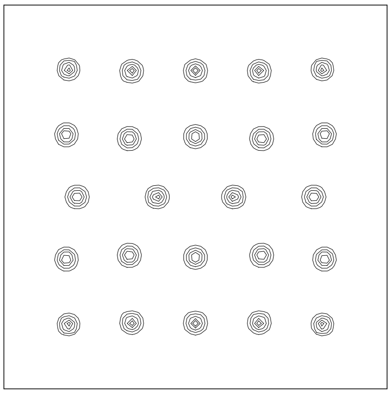
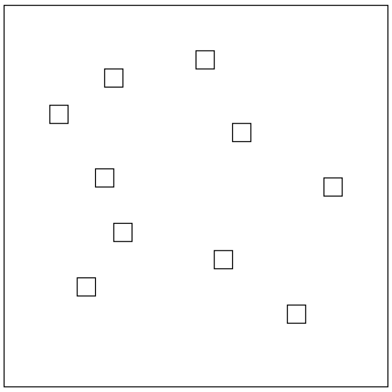


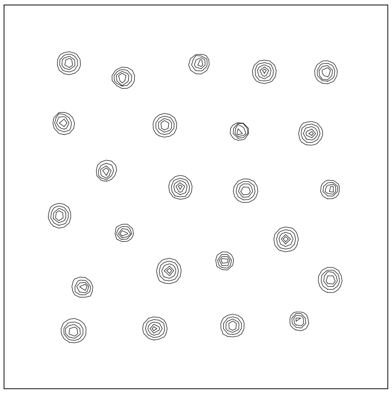
Figure 1. Level curves of the magnitude of the order parameter for a superconducting sample having sides equal to 30 coherence lengths.



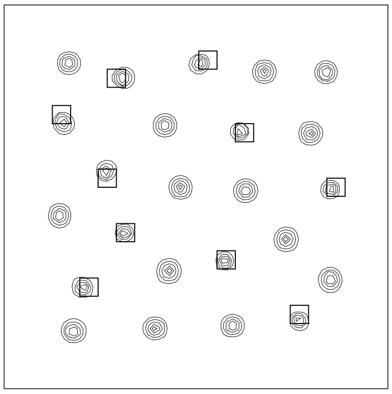
a. Constant thickness film.



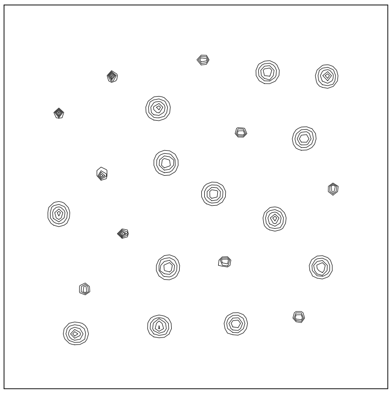
b. Distribution of thin regions.



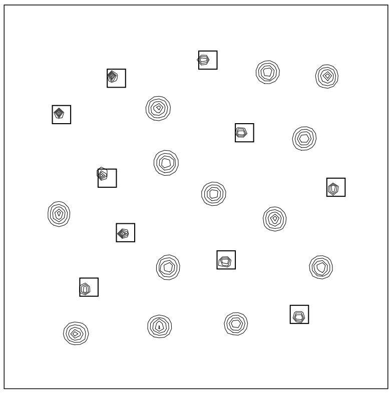
c. Variable thickness film ($d = 0.5$).



d. Figs. 2b and 2c superimposed.



e. Variable thickness film ($d = 0.1$).



f. Figs. 2b and 2e superimposed.

Figure 2. Level curves of the magnitude of the order parameter for superconducting samples having sides equal to 20 coherence lengths.

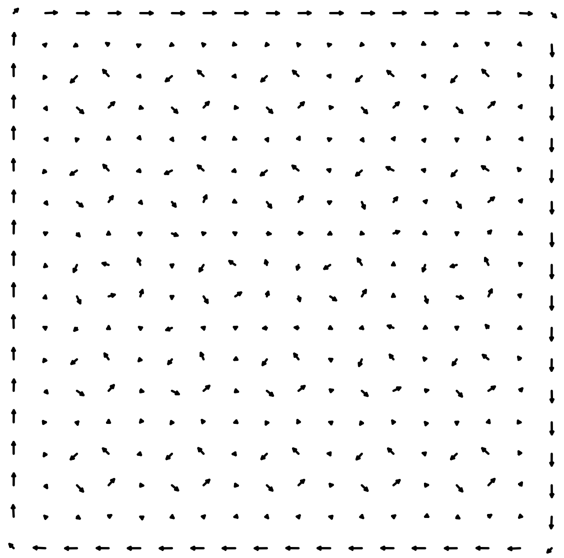
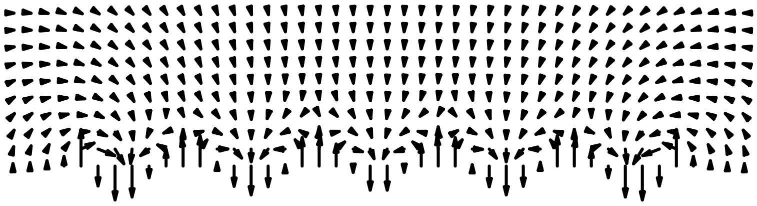
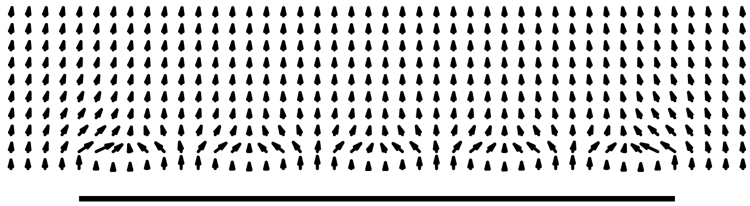


Figure 3. Leading-order supercurrent field.



a. First-order correction to the magnetic field.



b. Total magnetic field for $\epsilon/\kappa^2 = 0.1$.

Figure 4. Correction and total magnetic field above the vertical midline of the sample.