ADDITIVE CONJUGACY AND THE BOHR COMPACTIFICATION OF ORTHOGONAL REPRESENTATIONS

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Abstract. We say that two unitary or orthogonal representations of a finitely generated group G are *additive conjugates* if they are intertwined by an additive map, which need not be continuous. We associate to each representation of G a topological action that is a complete additive conjugacy invariant: the action of G by group automorphisms on the Bohr compactification of the underlying Hilbert space. Using this construction we show that the property of having almost invariant vectors is an additive conjugacy invariant. As an application we show that G is amenable if and only if there is a nonzero homomorphism from $L^2(G)$ into $\mathbb{R}/\mathbb{Z}$ that is invariant to the G-action.

1. Introduction

Let G be a finitely generated group. To each unitary or orthogonal representation of G one can associate a probability measure preserving action—the so called Gaussian action. Conversely, to each probability measure preserving action of G one can associate the Koopman representation. These constructions have proven to be an important connection between ergodic theory and representation theory, with many interesting applications (see, e.g., [2, 3, 7, 8]).

In this paper we associate a *topological* action to representations, with the goal of establishing connections between the dynamical properties of the representation and the action. We use this construction to study properties of the representation that are "additive conjugacy" invariants; we define this notion below. As an application we derive a new characterization of amenability.

Let $G \curvearrowright^{\pi} \mathcal{H}$ be an orthogonal representation of a finitely generated group on a separable real Hilbert space.[1] We associate to the representation π the topological action of G on the *Bohr compactification* of $\mathcal{H}$. In general, the

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[1] Note that the class of orthogonal representations includes the unitary ones; we elaborate on this in §3.1. Thus all of our results apply to unitary representations on separable complex Hilbert spaces.

Bohr compactification bA of a topological abelian group A is the algebraic dual of $\hat{A}_d$, where the latter is the algebraic dual of A, equipped with the discrete topology. As we explain in §3.2, in our case of a real separable Hilbert space $\mathcal{H}$, an equivalent definition is to let $b\mathcal{H}$ be the set of all homomorphisms (i.e., additive maps) from $\mathcal{H}$ to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$b\mathcal{H} = \{\varphi\colon \mathcal{H} \to \mathbb{T} \mid \varphi(v+w) = \varphi(v) + \varphi(w)\}. \tag{1}$$

Importantly, $b\mathcal{H}$ includes maps that are not continuous. The Bohr compactification $b\mathcal{H}$ is indeed compact, when endowed with the subspace topology induced from the product space $\mathbb{T}^{\mathcal{H}}$. It also admits the obvious abelian group structure, which is compatible with this topology.

The group G acts on $b\mathcal{H}$ by precomposition:

$$[g\varphi](v) = \varphi(\pi_g^{-1} v).$$

It is straightforward to verify that this action is by automorphisms of $b\mathcal{H}$ as a topological group. Thus the action $G \curvearrowright b\mathcal{H}$ is a topological action—in fact, an algebraic action—that is associated to the representation π. This action will be useful in the study of the following notion of conjugacy:

Definition 1. *Two representations, $G \overset{\pi}{\curvearrowright} \mathcal{H}$ and $G \overset{\pi'}{\curvearrowright} \mathcal{H}'$, are* additive conjugates *if there exists a bijection $\xi\colon \mathcal{H} \to \mathcal{H}'$ such that for all $v, w \in \mathcal{H}$ and $g \in G$,*

$$\xi(v+w) = \xi(v) + \xi(w)$$

and

$$\xi(\pi_g v) = \pi'_g \xi(v).$$

That is, two representations are additive conjugates if they are intertwined by an additive bijection. Note that this bijection need not be continuous.

It is straightforward to check that the action on the Bohr compactification is a complete additive conjugacy invariant. That is, that π and π' are additive conjugates if and only if $G \curvearrowright b\mathcal{H}$ and $G \curvearrowright b\mathcal{H}'$ are conjugates, as topological algebraic actions (Claim 7).

2. RESULTS

In all of our results below, G is a finitely generated group, Hilbert spaces are separable and either real or complex, and representations are, respectively, either orthogonal or unitary—unless otherwise specified.

Our main result ties an important property of a representation with a dynamical property of its associated topological action. Recall that π is said to have *almost invariant vectors* if there exists a sequence of unit vectors $(u_n)_n$ in $\mathcal{H}$ such that $\lim_n \|\pi_g u_n - u_n\| = 0$ for each $g \in G$. A *fixed point* x of a topological action $G \curvearrowright X$ is one that satisfies $gx = x$ for all $g \in G$.

Theorem 1. *$G \overset{\pi}{\curvearrowright} \mathcal{H}$ has almost invariant vectors if and only if the associated action $G \curvearrowright b\mathcal{H}$ has a nonzero fixed point.*

Since the action on the Bohr compactification is a complete additive conjugacy invariant, this theorem has an immediate corollary.

Corollary 2. *Let π_1 and π_2 be additive conjugates. Then π_1 has almost invariant vectors if and only if π_2 has almost invariant vectors.*

We note that, as far as we know, this is not known even for the case of $G = \mathbb{Z}$. This corollary may be a-priori surprising, since almost invariant vectors are defined using the topology of the Hilbert space, whereas this topology does not appear in the definition of additive conjugacy.

By the Hulanicki-Reiter Theorem (see, e.g., [1, Theorem G.3.2]), G is amenable if and only if the left regular real representation $G \overset{\lambda}{\curvearrowright} L^2(G)$ has almost invariant vectors. Hence the following is another corollary of Theorem 1:

Corollary 3. *G is amenable if and only if there exists a nonzero homomorphism $\varphi \colon L^2(G) \to \mathbb{T}$ that is invariant to the G action: $\varphi(f) = \varphi(\lambda_g f)$ for all $f \in L^2(G)$ and $g \in G$.*

Note that this homomorphism may not be continuous.

2.1. **Proof sketch.** Both directions of the proof of Theorem 1 require some work. An important tool is the natural homomorphism $\sigma \colon \mathcal{H} \to b\mathcal{H}$, which, for a real Hilbert space $\mathcal{H}$, is given by

$$[\sigma_v](w) = \langle v, w \rangle + \mathbb{Z}.$$

When π has almost invariant vectors $(v_n)_n$, it is straightforward to show that any limit point of $(\sigma_{v_n})_n$ is a fixed point of $b\mathcal{H}$. However, this fixed point might be zero. To overcome this, we construct from $(v_n)_n$ a modified sequence of almost invariant vectors $(w_n)_n$ such that all limit points of $(\sigma_{w_n})_n$ are nonzero.

When π does not have almost invariant vectors, we in fact prove a stronger statement. Given a symmetric probability measure μ on G whose support is equal to a finite generating set of G containing the identity, we say that $\varphi \in b\mathcal{H}$ is μ-harmonic if

$$\varphi\left(\sum_g \mu(g)\pi_g v\right) = \varphi(v) \tag{2}$$

for every $v \in \mathcal{H}$. We prove the following claim, which implies the corresponding direction of Theorem 1.

Proposition 4. *Suppose π does not have almost invariant vectors and μ is a symmetric generating measure for G. Then $b\mathcal{H}$ has no nonzero μ-harmonic points, and in particular has no nonzero fixed points.*

2.2. **Open questions and additional results.** This paper leaves unanswered the larger question of what properties of a representation are reflected in its Bohr compactification, or, equivalently, what properties are additive conjugacy invariants.

It is impossible that additive conjugacy of representations is equivalent to (the usual notion of) conjugacy: a simple obstruction is provided by the example of the trivial representations on $\mathbb{R}$ and $\mathbb{R}^2$. These representations are additive conjugates since there exists an additive bijection between $\mathbb{R}$ and $\mathbb{R}^2$; but they are not conjugates, since such a map cannot be continuous. Nevertheless, we have no example of two irreducible (or even cyclic) representations that are not conjugates and yet are additive conjugates.

This leaves two possibilities, both of which we find intriguing. The first is that—for irreducible representations—additive conjugacy coincides with conjugacy, and thus the action on the Bohr compactification somehow contains all of the data of the representation. The second is that these notions are not the same for irreducible representations, in which case it would be interesting to understand what properties of a representation are additive conjugacy invariants.

One may imagine that there is some connection between weak containment and additive conjugacy. Indeed, this is perhaps suggested by Corollary 2. We prove an additional result in this direction. This result can be interpreted to imply that the Bohr compactification records the data of the weakly contained irreducible representations.

Proposition 5. *Let G be a finitely generated group. Let $G \curvearrowright^{\pi} \mathcal{H}$ be an orthogonal representation that weakly contains the irreducible orthogonal representation $G \curvearrowright^{\pi'} \mathcal{H}'$. Then for every $v' \in \mathcal{H}'$ there are $\varphi \in b\mathcal{H}$ and $v \in \mathcal{H}$ such that for all $g \in G$,*

$$[g\varphi](v) = \langle gv', v' \rangle + \mathbb{Z}.$$

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3. DEFINITIONS

3.1. **Orthogonal and unitary representations.** Let $\mathcal{H}$ be a separable real Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$. An *orthogonal representation* π of a discrete group G is a homomorphism $\pi\colon G \to O(\mathcal{H})$, where $O(\mathcal{H})$ is the group of orthogonal (i.e., linear and inner product preserving) bijections from $\mathcal{H}$ to $\mathcal{H}$. That is, π is a group homomorphism of G into the group of linear transformations of $\mathcal{H}$ that preserve the inner product of $\mathcal{H}$. We henceforth omit π from our notation and write the image of $w \in \mathcal{H}$ under π_g simply as gw.

As the next lemma shows, every *unitary* representation on a complex Hilbert space $\mathcal{H}$ is also an orthogonal representation of the associated real Hilbert space.

Lemma 6. *Let $G \curvearrowright^{\pi} \mathcal{H}$ be a unitary representation of G on a complex Hilbert space $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ denote the realification of $\mathcal{H}$, with inner product $\langle u,v\rangle_{\widetilde{\mathcal{H}}} := \Re\langle u,v\rangle_{\mathcal{H}}$. Let $\widetilde{\pi} : G \curvearrowright \widetilde{\mathcal{H}}$ be the same action as π. Then $\widetilde{\pi}$ is an orthogonal representation, and π has almost invariant vectors if and only if $\widetilde{\pi}$ does.*

Proof. For any $v,w \in \widetilde{\mathcal{H}}$, $\langle gv,gw\rangle_{\widetilde{\mathcal{H}}} = \Re\langle gv,gw\rangle_{\mathcal{H}} = \Re\langle v,w\rangle_{\mathcal{H}} = \langle v,w\rangle_{\widetilde{\mathcal{H}}}$. The equivalence of having almost invariant vectors follows from the fact that the norms on $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are the same and that the actions are the same. □

It follows from Lemma 6 that it suffices to prove Theorem 1 for orthogonal representations to conclude that it also holds for unitary representations. The same holds for Corollary 2 and Proposition 4.

3.2. **Pontryagin duality and the Bohr compactification.** A homomorphism of an abelian group A into $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a map φ that satisfies $\varphi(v+w) = \varphi(v) + \varphi(w)$ for all $v,w \in A$. The set of all *continuous* such homomorphisms, equipped with the compact-open topology, is called the algebraic dual of A and is denoted by $\hat{A}$. The Bohr compactification bA of A is $\widehat{\hat{A}_d}$, where $\hat{A}_d$ is $\hat{A}$ equipped with the discrete topology. That is, bA is the set of *all* (i.e., not necessarily continuous) homomorphisms from $\hat{A}$ to $\mathbb{T}$.

The natural map $\tau\colon A \to bA$ given by

$$[\tau(v)](\gamma) = \gamma(v)$$

is known to be injective and continuous when A is locally compact, in which case its image is dense in bA.

Some groups, such as $\mathbb{R}$, can be (non-canonically) identified with their algebraic dual. In this case, bA is simply the set of all homomorphisms from A to $\mathbb{T}$. As we show (Proposition 13) this identification holds for a separable real Hilbert space $\mathcal{H}$. We hence define the Bohr compactification of $\mathcal{H}$ as in (1), by letting $b\mathcal{H}$ be the algebraic dual of $\mathcal{H}$ equipped with the discrete topology.

Since $b\mathcal{H}$ is compact, it follows from the Pontryagin Duality Theorem (see, e.g., [5]) that its algebraic dual $\widehat{b\mathcal{H}}$ can be canonically identified with $\mathcal{H}$ equipped with the discrete topology. This identification is realized by

$$v(\varphi) = \varphi(v). \tag{3}$$

3.3. **Generating measures and harmonic homomorphisms.** Let S be a finite, symmetric generating set for G containing the identity, and let μ be a symmetric probability measure whose support is equal to S. We call such μ "symmetric generating measures".

Let $P_\mu \colon \mathcal{H} \to \mathcal{H}$ be the continuous linear operator given by

$$P_\mu w = \sum_{h \in S} \mu(h) h w,$$

and let $D_u : \mathcal{H} \to \mathcal{H}$ be given by

$$D_\mu w = w - P_\mu w.$$

We say that $\varphi \in b\mathcal{H}$ is μ-harmonic if $\varphi(D_\mu w) = 0$ for all $w \in \mathcal{H}$. By additivity, this is equivalent to the definition above, in (2).

4. General properties of the action on the Bohr compactification

In this section we outline some simple, general properties of the action on the Bohr compactification and its relation to the representation.

Given a compact group A, an algebraic action $G \curvearrowright A$ is a homomorphism from G into the group of continuous group isomorphisms of A. Thus, two algebraic actions are conjugate if they are intertwined by a continuous group isomorphism.

Claim 7. *Two orthogonal representations of $G \curvearrowright^{\pi} \mathcal{H}$ and $G \curvearrowright^{\pi'} \mathcal{H}'$ are additive conjugates if and only if $G \curvearrowright b\mathcal{H}$ and $G \curvearrowright b\mathcal{H}'$ are conjugate algebraic actions.*

Proof. Assume first that $\xi \colon \mathcal{H} \to \mathcal{H}'$ witnesses the additive conjugacy of π and π'. Then $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic as *discrete* abelian groups, and hence their algebraic duals $b\mathcal{H}$ and $b\mathcal{H}'$ are isomorphic as topological groups; this is witnessed by $\xi_*\colon b\mathcal{H} \to b\mathcal{H}'$, defined by

$$[\xi_*\varphi](v') = \varphi(\xi^{-1}v').$$

It is straightforward to check that since ξ intertwines π and π', it holds that $\xi_* g\varphi = g\xi_*\varphi$, and hence the actions $G \curvearrowright b\mathcal{H}$ and $G \curvearrowright b\mathcal{H}'$ are conjugate.

Conversely, assume that $\xi_*\colon b\mathcal{H} \to b\mathcal{H}'$ witnesses the conjugacy of $G \curvearrowright b\mathcal{H}$ and $G \curvearrowright b\mathcal{H}'$. Then in particular $b\mathcal{H}$ and $b\mathcal{H}'$ are isomorphic as topological groups, and hence their algebraic duals $\mathcal{H}$ and $\mathcal{H}'$—endowed with the discrete topology—are conjugate (see the end of §3.2). This is witnessed by $\xi\colon \mathcal{H} \to \mathcal{H}'$, defined by

$$[\xi v](\varphi') = v(\xi_*^{-1}\varphi).$$

It is again straightforward to check that since ξ_* intertwines the actions on $b\mathcal{H}$ and $b\mathcal{H}'$, ξ intertwines π and π'. Likewise, ξ is immediately seen to be additive. $\square$

Let m be the unique Haar probability measure on the compact group $b\mathcal{H}$. Since G acts on $b\mathcal{H}$ by automorphisms, it preserves m, and so $G \curvearrowright (b\mathcal{H}, m)$ is a probability measure preserving action.

Claim 8. *The following are equivalent:*

1. *The action $G \curvearrowright (b\mathcal{H}, m)$ is ergodic.*
2. *The action $G \curvearrowright b\mathcal{H}$ is topologically transitive.*
3. *The orbit $\{gv \mid g \in G\}$ is infinite for every nonzero $v \in \mathcal{H}$.*

Proof. The first two conditions are equivalent by [9, Theorem 1.1] (in fact, this equivalence holds more generally for actions by automorphisms on compact groups). By a result of Halmos [4] for $\mathbb{Z}$ actions, which was extended by Kaplansky to finitely generated groups [6],[2] the negation of the first condition is equivalent to the existence of a nonzero $\chi \in \widehat{b\mathcal{H}}$ with a finite G-orbit. It follows from Pontryagin duality that the dual $\widehat{b\mathcal{H}}$ of $b\mathcal{H}$ can be identified with $\mathcal{H}$, equipped with the discrete topology (see the end of §3.2). Thus a nonzero character $\chi \in \widehat{b\mathcal{H}}$ with a finite orbit is simply equivalent to a nonzero vector $v \in \mathcal{H}$ with a finite orbit. $\square$

5. PROOFS

5.1. Preliminary claims.

Claim 9. *Every fixed point of $b\mathcal{H}$ is μ-harmonic for every symmetric generating measure μ.*

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[2] See also [9, Lemma 1.2 and remark (3) on page 9].

Proof. Suppose μ is a symmetric generating measure with support S and φ is a fixed point of $b\mathcal{H}$. Then for any $v \in H$,

$$\varphi(D_\mu v) = \varphi\left(v - \sum_{g\in S}\mu(g)gv\right) = \varphi(v) - \sum_{g\in S}\varphi(g(\mu(g)v)).$$

Because φ is a fixed point, $\varphi(g(\mu(g)v)) = \varphi(\mu(g)v)$. Hence,

$$\varphi(D_\mu v) = \varphi(v) - \sum_{g\in S}\varphi(\mu(g)v) = \varphi(v) - \varphi\left(\sum_{g\in S}\mu(g)v\right).$$

But μ is a probability measure, and so $\sum \mu(g)v = v$. We thus obtain

$$\varphi(D_\mu v) = \varphi(v) - \varphi(v) = 0.$$

□

5.2. **Proof of main theorem.**

Proposition 10. *If π has almost invariant vectors, then $b\mathcal{H}$ has a nonzero fixed point.*

Proof. Let S be a finite generating set for G. Take a sequence $(v_n)_n$ of almost invariant unit vectors. For each $n \geq 1$, let

$$\epsilon_n = \max_{h\in S}\|hv_n - v_n\|,$$

and let $w_n = \frac{v_n}{\sqrt{\epsilon_n}}$. As $b\mathcal{H}$ is compact, there is a subnet $(\sigma_{w_\alpha})_\alpha$ of $(\sigma_{w_n})_n$ that converges, say to φ, in $b\mathcal{H}$.

By Proposition 13, if $\sigma_{w_n} \to \sigma_w$, then $w_n \rightharpoonup w$. Since $\|w_n\| = \frac{1}{\sqrt{\epsilon_n}}$ is unbounded, the sequence $(w_n)_n$ has no weakly convergent subnet, and thus φ is not of the form σ_w for $w \in \mathcal{H}$. In particular, $\varphi \neq 0$. We finish by showing that φ is a fixed point. For any $h \in S$ and $v \in \mathcal{H}$, we have that

$$\begin{aligned}|\varphi(h^{-1}v) - \varphi(v)| &= \lim_\alpha |\sigma_{w_\alpha}(h^{-1}v) - \sigma_{w_\alpha}(v)| \\ &= \lim_\alpha |\langle v, hw_\alpha\rangle - \langle v, w_\alpha\rangle| \\ &= \lim_\alpha |\langle v, hw_\alpha - w_\alpha\rangle| \\ &\leq \lim_\alpha \|v\|\|hw_\alpha - w_\alpha\|.\end{aligned}$$

Since for any $h \in S$,

$$\|hw_n - w_n\| \leq \sqrt{\epsilon_n} \to 0,$$

it holds that

$$\|hw_\alpha - w_\alpha\| \to 0.$$

Therefore, $\varphi(h^{-1}v) = \varphi(v)$ for each $h \in S$, and consequently, since S is a symmetric generating set, φ is a fixed point of $b\mathcal{H}$. □

We now state an elementary lemma, which will be used to show D_μ is surjective.

Lemma 11. *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Suppose there exists some $c > 0$ so that $\|Tx\| \geq c\|x\|$ for all $x \in H$. Then the range of T is closed.*

Proof. If $y_n \to y$ with $Tx_n = y_n$, then

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| = \|y_n - y_m\| \to 0,$$

so $(x_n)_n$ is Cauchy. Say $x_n \to x$; then, since T is bounded, $Tx = \lim_n Tx_n = \lim_n y_n = y$, as desired. □

Proposition 12. *If π does not have almost invariant vectors, then D_μ is surjective for any symmetric generating measure μ.*

Proof. First observe that D_μ is self-adjoint since μ is symmetric. Furthermore, D_μ is injective since there are no nonzero invariant vectors. Indeed, note that if $D_\mu(w) = 0$ for some unit vector w, then

$$w = \sum_h \mu(h)hw$$

and so

$$1 = \sum_h \mu(h)\langle hw, w\rangle.$$

The right hand side is the average of numbers that are at most 1. Since this average is equal to 1 they all have to equal 1, and so (since μ is generating) w is invariant.

Since D_μ is self-adjoint and injective, D_μ has a dense image. Hence, by Lemma 11, it suffices to show the lower bound inequality. Since, for $\|v\| = 1$,

$$\|D_\mu(v)\| = \|v - P_\mu(v)\| \geq \|v\| - \|P_\mu(v)\| = 1 - \|P_\mu(v)\|,$$

it suffices to bound $\|P_\mu(v)\|$ away from 1. Let S denote the support of μ. Since there are no almost invariant vectors, there is an $\epsilon > 0$ so that for all $\|v\| = 1$ there exists an $h \in S$ such that $\|hv - v\| \geq \epsilon$ (see, e.g., [1, Proposition F.1.7]). Note for such a v and h, we have

$$\epsilon^2 \leq \|hv - v\|^2 = \langle hv - v, hv - v\rangle = 1 - 2\langle v, hv\rangle + 1$$

and thus

$$\langle v, hv\rangle \leq 1 - \frac{1}{2}\epsilon^2.$$

Therefore, for any unit vector $v \in \mathcal{H}$, taking again $h_0 \in S$ so that $\|h_0 v - v\| \geq \epsilon$ gives

$$\begin{aligned}\|P_\mu v\|^2 &= \left\langle \sum_{h \in S} \mu(h) h v, \sum_{h' \in S} \mu(h') h' v \right\rangle \\ &= \mu(h_0)\mu(e)\langle h_0 v, v\rangle + \sum_{(h,h') \neq (h_0, e)} \mu(h)\mu(h')\langle hv, h'v\rangle \\ &\leq \mu(h_0)\mu(e)(1 - \frac{1}{2}\epsilon^2) + \sum_{(h,h') \neq (h_0, e)} \mu(h)\mu(h') \\ &= 1 - \frac{1}{2}\mu(h_0)\mu(e)\epsilon^2.\end{aligned}$$

Consequently, for each $\|v\| = 1$,

$$\|P_\mu v\|^2 \leq 1 - \frac{1}{2}\epsilon^2 \mu(e) \inf_{h \in S} \mu(h).$$

Since S is finite, we are done. □

We now have the tools to prove Proposition 4.

Proof of Proposition 4. Suppose π does not have almost invariant vectors, μ is a symmetric generating measure, and φ is a μ-harmonic point of $b\mathcal{H}$. Then $\varphi(D_\mu(\mathcal{H})) = 0$. Now, by Proposition 12, since π does not have almost invariant vectors, D_μ is surjective. Hence, $\varphi(\mathcal{H}) = 0$, and so $\varphi = 0$. Thus, the only μ-harmonic point of $b\mathcal{H}$ is 0. Further, since every fixed point of $b\mathcal{H}$ is μ-harmonic (Claim 9), it follows that the only fixed point of $b\mathcal{H}$ is 0. □

We are now ready to prove our main theorem.

Proof of Theorem 1. First, suppose that π has almost invariant vectors. Then by Proposition 10, $b\mathcal{H}$ has a nonzero fixed point. Now, suppose that $b\mathcal{H}$ has a nonzero fixed point. Then by Proposition 4, π must have almost invariant vectors. □

5.3. **The Bohr compactification of a separable Hilbert space.** Let $\mathcal{H}$ be a separable real Hilbert space. Recall that the *algberaic dual* of $\mathcal{H}$ is given by

$$\widehat{\mathcal{H}} = \{\phi : \mathcal{H} \to \mathbb{T} \mid \phi \text{ is a continuous homomorphism}\}$$

As above, for each $v \in \mathcal{H}$, let $\sigma_v : \mathcal{H} \to \mathbb{T}$ be given by

$$\sigma_v(w) = \langle v, w\rangle + \mathbb{Z}.$$

Proposition 13. *A real separable Hilbert space $\mathcal{H}$, equipped with the weak topology, can be identified as a topological group with its algebraic dual $\widehat{\mathcal{H}}$ via $v \mapsto \sigma_v$.*

Proof of Proposition 13. We first note that every σ_v is an element of $\widehat{\mathcal{H}}$, i.e., is a continuous homomorphism from $\mathcal{H}$ to $\mathbb{T}$. This follows from the fact that $w \mapsto \langle v, w\rangle$ is weakly continuous, and that the projection $\mathbb{R} \to \mathbb{T}$ is also continuous. We now show that every continuous homomorphism $\phi : \mathcal{H} \to \mathbb{T}$ is equal to some σ_w.

Given ϕ, let $\psi \colon \mathcal{H} \to \mathbb{R}$ be the unique lift of ϕ satisfying $\psi(0) = 0$. Then ψ is continuous and is easily seen to furthermore be linear. Hence it must be of the form $\psi(v) = \langle v, w\rangle$ for some $w \in \mathcal{H}$. Since ψ is a lift of ϕ, $\phi(v) = \langle v, w\rangle + \mathbb{Z}$. We have thus shown that $\phi = \sigma_w$.

Finally, we argue that $v \mapsto \sigma_v$ is continuous and has a continuous inverse. It follows immediately from the definition that if a net $(v_\alpha)_\alpha$ converges weakly to v then $(\sigma_{v_\alpha})_\alpha$ converges to σ_v. Hence $v \mapsto \sigma_v$ is continuous. Since the unit ball in $\mathcal{H}$ is weakly compact, the restriction of this map to this ball has an inverse that is also continuous. By the additivity of the map $v \mapsto \sigma_v$, it follows that this map has a continuous inverse. □

5.4. **Proof of Proposition 5.**

Lemma 14. *Let $\rho : G \to \mathbb{T}$ be a function. Suppose there exist vectors $(v_n)_n$ that weakly converge to 0, and that for each $g \in G$, $\lim_n \langle gv_n, v_n\rangle + \mathbb{Z} = \rho(g)$. Then there exist $\varphi \in b\mathcal{H}$ and $v \in \mathcal{H}$ so that $\varphi(g^{-1}v) = \rho(g)$ for each $g \in G$.*

Proof. Let $\{g_1, g_2, \dots\}$ be an enumeration of (the countable group) G. Let $w_1 = v_1$. With $w_1, \dots, w_{n-1}$ chosen, let $w_n = v_{k_n}$ be such that $|\langle v_{k_n}, gw_j\rangle| < 2^{-n^2}$ for each $1 \le j \le n-1$ and $g \in \{g_1, g_1^{-1}, \dots, g_{n-1}, g_{n-1}^{-1}\}$, which is possible since $v_n \rightharpoonup 0$. Let $v = \sum_n 2^{-n} w_n$. As $b\mathcal{H}$ is compact, we may take φ, a limit of some subnet of $(2^n \sigma_{w_n})_n$. Fix $g \in G$. Note, for each $n \ge 1$,

$$2^n \sigma_{w_n}(g^{-1}v) = \langle gw_n, w_n\rangle + \sum_{k<n} 2^n 2^{-k} \langle g^{-1}w_k, w_n\rangle + \sum_{k>n} 2^n 2^{-k} \langle w_k, gw_n\rangle.$$

By construction, if $g = g_l$ and $n > l$, then for $k < n$, $|\langle g^{-1}w_k, w_n\rangle| \le 2^{-n^2}$ and for $k > n$, $|\langle w_k, gw_n\rangle| \le 2^{-k^2}$. So, we obtain that

$$\begin{aligned}\lim_n 2^n \sigma_{w_n}(g^{-1}v) &= \lim_n \left(\langle gw_n, w_n\rangle + O\left(n2^n 2^{-n^2} + \sum_{k>n} 2^{n-k^2} \right) \right) \\ &= \rho(g).\end{aligned}$$

We conclude $\varphi(g^{-1}v) = \rho(g)$. □

Proof of Proposition 5. Since π weakly contains π' and π' is irreducible, for any $v' \in \mathcal{H}'$, there exists a sequence of vectors $(v_n)_n$ in $\mathcal{H}$ such that $\lim \langle gv_n, v_n \rangle = \langle gv', v' \rangle$ for each $g \in G$ (see, e.g., [1, Proposition F.1.4]). Since the norm of v_n converges to the norm of v', we may assume that $(v_n)_n$ has a weak limit, say v. If $v = 0$, then the result follows immediately from Lemma 14. So we may assume $v \neq 0$.

For each n, let $u_n = v_n - v$. For any $g \in G$, it follows from $u_n \rightharpoonup 0$ that

$$\begin{aligned}
\langle gv', v' \rangle &= \lim_n \langle gv_n, v_n \rangle \\
&= \lim_n \langle g(u_n + v), (u_n + v) \rangle \\
&= \lim_n \langle gu_n, u_n \rangle + \lim_n \langle gu_n, v \rangle + \lim_n \langle gv, u_n \rangle + \lim_n \langle gv, v \rangle \\
&= \lim_n \langle gu_n, u_n \rangle + \langle gv, v \rangle.
\end{aligned}$$

Since π' is irreducible, positive functions associated to it are extreme points in the cone of positive functions, a cone which is closed with respect to pointwise convergence (see, e.g., [1, Proposition C.5.2]). Therefore, $\lim_n \langle gu_n, u_n \rangle = t\langle gv', v' \rangle$ for some $t \in \mathbb{R}$. In particular, $\langle gv', v' \rangle = \frac{1}{1-t}\langle gv, v \rangle$ ($t \neq 1$ since $v \neq 0$). The result follows by taking $\varphi = \sigma_{v/(t-1)}$. $\square$

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