GAUSSIAN ANALYTIC FUNCTIONS AND OPERATOR SYMBOLS OF DIRICHLET TYPE

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Abstract. Let $\mathcal{H}$ be a separable infinite-dimensional $\mathbb{C}$-linear Hilbert space, with sesquilinear inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Given any two orthonormal systems $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ in $\mathcal{H}$, we show that

$$\sum_{l=2}^{+\infty} s^l \Bigg| \sum_{j,k:j+k=l} (jk)^{-\frac{1}{2}} \langle x_j, y_k\rangle_{\mathcal{H}} \Bigg|^2 \le 2s \log \frac{\mathrm{e}^{1/2}}{1-s}, \qquad 0 \le s < 1. \tag{1}$$

In terms of the weighted sums

$$S(l) := \sum_{j,k:j+k=l} \left(\frac{l}{jk}\right)^{\frac{1}{2}} \langle x_j, y_k\rangle_{\mathcal{H}},$$

this means that

$$\sum_{l=2}^{+\infty} \frac{s^l}{l} |S(l)|^2 \le 2 \log \frac{\mathrm{e}^{1/2}}{1-s}, \qquad 0 \le s < 1.$$

Expressed more vaguely, $|S(l)|^2 \lessapprox 2$ holds in the sense of averages. Concerning the optimality of the bound (1), a construction due to Zachary Chase shows that the statement does not hold if the number 2 is replaced by the smaller number 1.72. In the construction, the system $y_1, y_2, y_3, \ldots$ is a permutation of the system $x_1, x_2, x_3, \ldots$. We interpret our bound in terms of the correlation $\mathbb{E}\Phi(z)\Psi(z)$ of two copies of a Gaussian analytic function with possibly intricate Gaussian correlation structure between them. The Gaussian analytic function we study arises in connection with the classical Dirichlet space, which is naturally Möbius invariant. The study of the correlations $\mathbb{E}\Phi(z)\Psi(z)$ leads us to introduce a new space, the *mock-Bloch space*, which is slightly bigger than the standard Bloch space. Our bound has an interpretation in terms of McMullen's asymptotic variance, originally considered for functions in the Bloch space. Finally, we show that the correlations $\mathbb{E}\Phi(z)\Psi(w)$ may be expressed as Dirichlet symbols of contractions on $L^2(\mathbb{D})$, and show that the Dirichlet symbols of Grunsky operators associated with univalent functions find a natural characterization in terms of a nonlinear wave equation.

1. Introduction

1.1. **Basic notation in the plane.** We write $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the positive integers, $\mathbb{R}$ for the real line, and $\mathbb{C}$ for the complex plane. Moreover, we write $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ for the extended complex plane (the Riemann sphere). For a complex variable $z = x + \mathrm{i}y \in \mathbb{C}$, let

$$\mathrm{d}s(z) := \frac{|\mathrm{d}z|}{2\pi}, \qquad \mathrm{d}A(z) := \frac{\mathrm{d}x\mathrm{d}y}{\pi},$$

denote the normalized arc length and area measures, as indicated. Moreover, we shall write

$$\varDelta_z := \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

for the normalized Laplacian, and

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

for the standard complex derivatives; then $\varDelta$ factors as $\varDelta_z = \partial_z\bar\partial_z$. Often we will drop the subscript for these differential operators when it is obvious from the context with respect to which variable they

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apply. We let $\mathbb{D}$ denote the open unit disk, $\mathbb{T} := \partial\mathbb{D}$ the unit circle, and $\mathbb{D}_e$ the exterior disk:

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}, \qquad \mathbb{D}_e := \{z \in \mathbb{C}_\infty : \ |z| > 1\}.$$

We will find it useful to introduce the sesquilinear forms $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, as given by

$$\langle f, g\rangle_{\mathbb{C}} := \int_{\mathbb{C}} f(z)\bar{g}(z)\mathrm{d}A(z), \qquad \langle f, g\rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)\bar{g}(z)\mathrm{d}A(z),$$

where we need $f\bar{g} \in L^1(\mathbb{C})$ in the first instance and $f\bar{g} \in L^1(\mathbb{D})$ in the second. These are standard Lebesgue spaces with respect to normalized area measure $\mathrm{d}A$. Here, generally, for a given complex-valued function f, we denote by $\bar{f}$ the function whose values are the complex conjugates of f. To simplify the notation further, we write

$$\langle f\rangle_{\mathbb{C}} = \langle f, 1\rangle_{\mathbb{C}}, \quad \langle f\rangle_{\mathbb{D}} = \langle f, 1\rangle_{\mathbb{D}}.$$

As for operators $\mathbf{T}$ on a Hilbert function space, we let $\mathbf{T}^*$ denote the adjoint, while $\bar{\mathbf{T}}$ means the operator defined by

$$\bar{\mathbf{T}}f = \overline{\mathbf{T}\bar{f}}.$$

1.2. **Complex Gaussian Hilbert space.** A *Gaussian Hilbert space* is a closed linear subspace $\mathfrak{G}$ of $L^2(\Omega) = L^2(\Omega, \mathrm{d}P)$, where $(\Omega, \mathrm{d}P)$ is a probability space with a given σ-algebra, with the property that each element $\gamma \in \mathfrak{G}$ has a Gaussian distribution with mean 0. Since we will be working with the complex field $\mathbb{C}$, this means that the real and imaginary parts of γ are jointly Gaussian, and that the mean is 0 of each one. Here, the *expectation* (or *mean*) operation $\mathbb{E}$ is just given by $\mathbb{E}\gamma := \langle\gamma\rangle_\Omega = \int_\Omega \gamma \mathrm{d}P$. We say that γ is *symmetric* if $\mathbb{E}(\gamma^2) = 0$. Moreover, γ is a *standard complex Gaussian* variable if it has mean 0, is symmetric and has $\mathbb{E}(|\gamma|^2) = 1$. In other words, the values of γ are distributed according to the density $\mathrm{e}^{-|z|^2}\mathrm{d}A(z)$ in the plane. We will assume for convenience that $\mathfrak{G}$ is *conjugation-invariant*, that is, $\gamma \in \mathfrak{G} \iff \bar{\gamma} \in \mathfrak{G}$. We refer to [14] for an exposition on Gaussian Hilbert spaces. We will write $\langle\gamma, \gamma'\rangle_\Omega = \langle\gamma\bar{\gamma}'\rangle_\Omega = \mathbb{E}\gamma\bar{\gamma}'$ for the inner product of $\mathfrak{G}$. We shall need the following observation. If $\mathfrak{G}$ is separable and infinite-dimensional, then there exists a sequence $\gamma_1, \gamma_2, \gamma_3, \ldots$ in $\mathfrak{G}$ consisting of i i d standard complex Gaussians, such that the sequence $\gamma_1, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_2, \ldots$ forms an orthonormal basis in $\mathfrak{G}$. In particular, $\mathfrak{G}$ splits as an orthogonal sum $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{H}_*$, where $\mathfrak{H}$ is the closed subspace spanned by $\gamma_1, \gamma_2, \gamma_3, \ldots$, while $\mathfrak{H}_*$ is spanned by $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \ldots$.

1.3. **Gaussian analytic functions associated with the Dirichlet space.** We now outline a more direct approach to the analytic part of GFF outlined in the preceding subsection. Let $A^2(\mathbb{D})$ denote the subspace of $L^2(\mathbb{D})$ consisting of the holomorphic functions, which is a closed subspace and hence a Hilbert space in its own right, known as the *Bergman space*. The *Dirichlet space* is the space $\mathcal{D}(\mathbb{D})$ of analytic functions f with $f' \in A^2(\mathbb{D})$, equipped with the Dirichlet inner product

$$\langle f, g\rangle_\nabla := \langle f', g'\rangle_{\mathbb{D}},$$

The importance of the Dirichlet space comes from its conformal invariance property. For instance, if ϕ is a Möbius automorphism of the unit disk $\mathbb{D}$, we have that

$$\langle f \circ \phi, g \circ \phi\rangle_\nabla = \langle f, g\rangle_\nabla.$$

The Dirichlet inner product gives rise to a seminorm

$$\|f\|_\nabla^2 := \|f'\|^2_{A^2(\mathbb{D})} = \langle f', f'\rangle_{\mathbb{D}},$$

which vanishes on the constant functions. So, to make it a norm, we could add the requirement that the functions should vanish at a given point $\lambda \in \mathbb{D}$:

$$\mathcal{D}_\lambda(\mathbb{D}) := \{f \in \mathcal{D}(\mathbb{D}) : \ f(\lambda) = 0\}.$$

We will focus our attention to $\lambda = 0$, and study the space $\mathcal{D}_0(\mathbb{D})$. By the Möbius invariance of the seminorm, this choice is not restrictive as we may easily move any other point λ to the origin using a Möbius automorphism.

In recent years, *Gaussian analytic functions* has received increasing attention. For instance, see [19] and the book [12]. In the space $\mathcal{D}_0(\mathbb{D})$, we have a canonical orthogonal basis

$$e_j(z) := j^{-\frac{1}{2}} z^j, \qquad j = 1, 2, 3, \ldots,$$

and we form a $\mathcal{D}_0$-Gaussian analytic function ($\mathcal{D}_0$-GAF)

$$\Phi(z) := \sum_{j=1}^{+\infty} \alpha_j \, e_j(z) = \sum_{j=1}^{+\infty} \frac{\alpha_j}{\sqrt{j}} \, z^j, \tag{1.3.1}$$

where the α_j are i i d (independent identically distributed) standard complex Gaussian variables, taken from a Gaussian Hilbert space $\mathfrak{G}$. Then for two points in the disk $z, w \in \mathbb{D}$, we have the complex correlation structure

$$\mathbb{E}(\Phi(z)\Phi(w)) = 0, \quad \mathbb{E}(\Phi(z)\bar{\Phi}(w)) = \log \frac{1}{1 - z\bar{w}}. \tag{1.3.2}$$

Since Gaussian random variables are determined by their correlation structures, we may, depending on the point of view, take (1.3.2) as the defining property instead of the more explicit (1.3.1). On the right-hand side of (1.3.2), we recognize the reproducing kernel for the Dirichlet space,

$$\mathrm{k}_{\mathcal{D}_0}(z, w) = \log \frac{1}{1 - z\bar{w}}, \tag{1.3.3}$$

with the point evaluation property

$$f(w) = \langle f, \mathrm{k}_{\mathcal{D}_0}(\cdot, w) \rangle_\nabla, \qquad f \in \mathcal{D}_0(\mathbb{D}).$$

It is appropriate to think of the correlation structure (1.3.2) in terms of the matrix-valued correlation structure

$$\mathbb{k}_{2\times 2}[\Phi](z, w) = \mathbb{E} \begin{pmatrix} \Phi(z) \\ \bar{\Phi}(z) \end{pmatrix} \begin{pmatrix} \bar{\Phi}(w) & \Phi(w) \end{pmatrix} = \begin{pmatrix} \mathbb{E}\Phi(z)\bar{\Phi}(w) & \mathbb{E}\Phi(z)\Phi(w) \\ \mathbb{E}\bar{\Phi}(z)\bar{\Phi}(w) & \mathbb{E}\bar{\Phi}(z)\Phi(w) \end{pmatrix} = \begin{pmatrix} \log \frac{1}{1 - z\bar{w}} & 0 \\ 0 & \log \frac{1}{1 - \bar{z}w} \end{pmatrix}, \tag{1.3.4}$$

and the associated 4×4 matrix

$$\begin{pmatrix} \mathbb{k}_{2\times 2}[\Phi](z, z) & \mathbb{k}_{2\times 2}[\Phi](z, w) \\ \mathbb{k}_{2\times 2}[\Phi](z, w)^* & \mathbb{k}_{2\times 2}[\Phi](w, w) \end{pmatrix} \tag{1.3.5}$$

is positive semidefinite (the asterisque $*$ stands for the operation of taking the adjoint of the matrix). The real part of $\Phi(z)$ may be understood, up to an additive constant, as the restriction of the Gaussian free field (GFF) on $\mathbb{C}$ conditioned to be harmonic in $\mathbb{D}$. For some background on GFF, we refer to the survey paper [**?**] as well as to [8]. Alternatively, the process $\Phi(z)$ may be identified as the limit of the logarithm of the characteristic polynomial for random unitary matrices as the size of the matrices tends to infinity (see below).

In analogy with [17], it might be of interest to study the random zeros of the function $\Phi(z)$, but since one of them is deterministic (the origin), we should not expect full Möbius automorphism invariance. By the Edelman-Kostlan formula (see [19]) the density of zeros is given by

$$\varDelta \log \mathrm{k}_{\mathcal{D}_0}(z, z) \, \mathrm{d}A(z) = \varDelta \log \log \frac{1}{1 - |z|^2} \mathrm{d}A(z), \tag{1.3.6}$$

which has a unit point mass at the origin due to the deterministic zero there. Here, one might also be interested in the process for the critical points. We will not pursue any of these directions here. A rather interesting object appears to be the random curve (or tree) structure we obtain by following the gradient flow for the random harmonic function $\mathrm{Re}\,\Phi(z)$ which stops at critical points. At each critical point we would instead choose among the possible directions, for instance by maximizing the second directional derivative (perhaps after precomposing with a Möbius mapping to put the critical point at the origin). Although quite promising, We will not pursue this matter further here. A related setting of gradient flow for the plane defined in terms of the Bargmann-Fock space was studied by Nazarov, Sodin, and Volberg [16].

1.4. **$\mathcal{D}_0$-Gaussian analytic functions and random unitary matrices.** Let M_n be a random $n\times n$ unitary matrix with distribution given by Haar measure. Let

$$\chi_{M_n}(\lambda)=\det(\lambda I_n-M_n)$$

be the associated random characteristic polynomial, where I_n is the $n\times n$ identity matrix. Diaconis and Evans [6] found an interesting relationship connecting the characteristic polynomial of M_n with the process given by (1.3.1). They showed that

$$\operatorname{tr}\log(I_n-zM_n^*)=\log\det(I_n-zM_n^*)=\log\frac{\chi_{M_n}(z)}{\chi_{M_n}(0)}$$

converges, as $n\to+\infty$, in distribution, to the $\mathcal{D}_0$-Gaussian analytic function $\Phi(z)$ given by (1.3.1). The details are supplied in Example 5.6 of [6]. For the convenience of the reader, we mention that the master relationship between their random function $F_n(z)$ and $\chi_{M_n}(z)$ has a typo, and should be replaced by

$$F_n(z)=\frac{n}{2\pi}-\frac{z}{\pi}\frac{\chi'_{M_n}(z)}{\chi_{M_n}(z)}.$$

Remark 1.4.1. The matters considered here, the possible correlation structure of two jointly Gaussian $\mathcal{D}_0$-GAFs, have their (finite-dimensional) counterpart for random matrices. Let M_n and M_n' be two copies of the random $n\times n$ unitary matrix enemble, with possibly complicated correlation structure between M_n and M_n', but at least all their entries are jointly (complex) Gaussian variables. What could we say about the structure of the $\mathbb{C}^2$-valued process of normalized random characteristic polynomials

$$\Big(\frac{\chi_{M_n}(z)}{\chi_{M_n}(0)},\frac{\chi_{M_n'}(z)}{\chi_{M_n'}(0)}\Big)?$$

1.5. **Two interacting copies of the $\mathcal{D}_0$-Gaussian analytic function process.** The topic here involves two copies of the process (1.3.1),

$$\Phi(z):=\sum_{j=1}^{+\infty}\frac{\alpha_j}{\sqrt{j}}z^j,\quad \Psi(z):=\sum_{j=1}^{+\infty}\frac{\beta_j}{\sqrt{j}}z^j, \tag{1.5.1}$$

where $\Phi(z)$ is as before and the β_j are i i d from $N_{\mathbb{C}}(0,1)$, taken from the same Gaussian Hilbert space $\mathfrak{G}\subset L^2(\Omega)$. We will refer to $(\Phi(z),\Psi(z))$ as a *pair of jointly Gaussian $\mathcal{D}_0$-GAFs*. Consisting of jointly Gaussian variables with zero mean, the vector-valued process $(\Phi(z),\Psi(z))$ is governed by the correlation matrix

$$\begin{aligned}
\mathbb{k}_{4\times4}[\Phi,\Psi](z,w)&:=\mathbb{E}\begin{pmatrix}\Phi(z)\\ \bar\Phi(z)\\ \Psi(z)\\ \bar\Psi(z)\end{pmatrix}\begin{pmatrix}\bar\Phi(w) & \Phi(w) & \bar\Psi(w) & \Psi(w)\end{pmatrix}\\
&=\begin{pmatrix}\mathbb{E}\Phi(z)\bar\Phi(w) & \mathbb{E}\Phi(z)\Phi(w) & \mathbb{E}\Phi(z)\bar\Psi(w) & \mathbb{E}\Phi(z)\Psi(w)\\ \mathbb{E}\bar\Phi(z)\bar\Phi(w) & \mathbb{E}\bar\Phi(z)\Phi(w) & \mathbb{E}\bar\Phi(z)\bar\Psi(w) & \mathbb{E}\bar\Phi(z)\Psi(w)\\ \mathbb{E}\Psi(z)\bar\Phi(w) & \mathbb{E}\Psi(z)\Phi(w) & \mathbb{E}\Psi(z)\bar\Psi(w) & \mathbb{E}\Psi(z)\Psi(w)\\ \mathbb{E}\bar\Psi(z)\bar\Phi(w) & \mathbb{E}\bar\Psi(z)\Phi(w) & \mathbb{E}\bar\Psi(z)\bar\Psi(w) & \mathbb{E}\bar\Psi(z)\Psi(w)\end{pmatrix}\\
&=\begin{pmatrix}\log\frac{1}{1-z\bar w} & 0 & \mathbb{E}\Phi(z)\bar\Psi(w) & \mathbb{E}\Phi(z)\Psi(w)\\ 0 & \log\frac{1}{1-\bar z w} & \mathbb{E}\bar\Phi(z)\bar\Psi(w) & \mathbb{E}\bar\Phi(z)\Psi(w)\\ \mathbb{E}\Psi(z)\bar\Phi(w) & \mathbb{E}\Psi(z)\Phi(w) & \log\frac{1}{1-z\bar w} & 0\\ \mathbb{E}\bar\Psi(z)\bar\Phi(w) & \mathbb{E}\bar\Psi(z)\Phi(w) & 0 & \log\frac{1}{1-\bar z w}\end{pmatrix},
\end{aligned} \tag{1.5.2}$$

and the associated 8×8 matrix

$$\begin{pmatrix}\mathbb{k}_{4\times4}[\Phi](z,z) & \mathbb{k}_{4\times4}[\Phi](z,w)\\ \mathbb{k}_{4\times4}[\Phi](z,w)^* & \mathbb{k}_{4\times4}[\Phi](w,w)\end{pmatrix} \tag{1.5.3}$$

is positive semidefinite. Note that although there are eight unknown entries in (1.5.2), in fact only two are needed, as clearly,

$$\mathbb{E}(\bar{\Phi}(z)\bar{\Psi}(w)) = \overline{\mathbb{E}(\Phi(z)\Psi(w))}, \quad \mathbb{E}(\bar{\Phi}(z)\Psi(w)) = \overline{\mathbb{E}(\Phi(z)\bar{\Psi}(w))},$$

and the remaining four only involve exchanging the variables z and w.

So we need only be concerned with the quantities

$$\mathbb{E}(\Phi(z)\bar{\Psi}(w)) \quad \text{and} \quad \mathbb{E}(\Phi(z)\Psi(w)). \tag{1.5.4}$$

In a sense they complement each other, as we see below.

Proposition 1.5.1. *We have that*

$$|\mathbb{E}\Phi(z)\bar{\Psi}(w)| + |\mathbb{E}\Phi(z)\Psi(w)| \le \Big(\log\frac{1}{1-|z|^2}\Big)^{\frac12}\Big(\log\frac{1}{1-|w|^2}\Big)^{\frac12}, \qquad z, w \in \mathbb{D}.$$

Since for a given point with $|z| = |w|$ each of the two terms on the left-hand side may reach up to the right-hand side bound, the estimate tells us they cannot do so simultaneously. The proof of this estimate is presented in Subsection 3.2.

1.6. **The fundamental integral estimate.** The following is our basic estimate of the correlations.

Theorem 1.6.1. *For $a, b \in \mathbb{C}$, we have the estimate*

$$\int_{\mathbb{D}} \Big|aw\mathbb{E}\Phi(z)\Psi'(w) + b\bar{w}\mathbb{E}\Phi(z)\bar{\Psi}'(w)\Big|^2 \frac{\mathrm{d}A(w)}{|w|^2} \le (|a|^2 + |b|^2)\log\frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

This may be interpreted as an estimate of the radial derivative (with respect to w) of the harmonic function

$$a\mathbb{E}\Phi(z)\Psi(w) + b\mathbb{E}\Phi(z)\bar{\Psi}(w).$$

Indeed, if F is holomorphic in $\mathbb{D}$, then its radial derivative is

$$\partial_r F(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}F'(r\mathrm{e}^{\mathrm{i}\theta}),$$

so that the estimate of Theorem 1.6.1 asserts that ($\partial_{r(w)}$ is the radial derivative in the w variable)

$$\int_{\mathbb{D}} \Big|\partial_{r(w)}\Big(a\mathbb{E}\Phi(z)\Psi(w) + b\mathbb{E}\Phi(z)\bar{\Psi}(w)\Big)\Big|^2 \mathrm{d}A(w) \le (|a|^2 + |b|^2)\log\frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{1.6.1}$$

Interesting estimates are obtained for instance when $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$. We shall mainly focus on the first of these, when $(a, b) = (1, 0)$. We defer the proof of this result to Section 5.

1.7. **Growth of correlations in the mean along diagonals.** We are interested in the behavior of the correlations

$$\mathbb{E}\Phi(z)\Psi(w), \quad \mathbb{E}\Phi(z)\bar{\Psi}(w)$$

as $z, w \in \mathbb{D}$ approach the unit circle $\mathbb{T}$. The first one we will refer to as the *analytic correlation*, and the second the *sesquianalytic correlation*. We may study the growth behavior by looking along complex lines through the origin $w = \lambda z$ for some parameter $\lambda \in \mathbb{C}$ in which case our correlations are

$$\mathbb{E}\Phi(z)\Psi(\lambda z), \quad \mathbb{E}\Phi(z)\bar{\Psi}(\lambda z). \tag{1.7.1}$$

The alternative study of conjugate-linear lines $w = \mu\bar{z}$ with $\mu \in \mathbb{C}$ is completely analogous and essentially only corresponds to reversing the order of these correlations (in the sense that $w \mapsto \bar{\Psi}(\mu\bar{w})$ is a GAF). For this reason we will not consider such conjugate-linear lines further. When $|\lambda| < 1$ the process $\Phi(z)$ dominates in the correlations since $\Psi(\lambda z)$ is analytic in the disk $\mathbb{D}(0, |\lambda|^{-1})$, while if $|\lambda| > 1$ instead the process $\Psi(\lambda z)$ dominates. The most interesting instance seems to be the balanced case when $|\lambda| = 1$, in which case the line $w = \lambda z$ might be called a *generalized diagonal*. For $|\lambda| = 1$, the process $\Psi(\lambda z)$ is just another copy of the $\mathcal{D}_0$-GAF, so as long as λ is fixed we might as well consider $\lambda = 1$. So the study of (1.7.1) for fixed λ with $|\lambda| = 1$ reduces to the diagonal case

$$\mathbb{E}\Phi(z)\Psi(z), \quad \mathbb{E}\Phi(z)\bar{\Psi}(z). \tag{1.7.2}$$

We note that by Proposition 1.5.1,

$$|\mathbb{E}\Phi(z)\Psi(z)| + |\mathbb{E}\Phi(z)\bar{\Psi}(z)| \le \log\frac{1}{1-|z|^2}. \tag{1.7.3}$$

Some examples should elucidate which term, if any, may be dominant on the left-hand side.

Remark 1.7.1. We supply some examples which help us understand the size of the two contributions on the left-hand side of (1.7.3).
(a) If $\Psi = \Phi$, then

$$\mathbb{E}\Phi(z)\Psi(z) = \mathbb{E}(\Phi(z)^2) = 0, \quad \mathbb{E}\Phi(z)\bar{\Psi}(z) = \mathbb{E}|\Phi(z)|^2 = \log\frac{1}{1-|z|^2}.$$

In this case we have *equality* in (1.7.3), and on the left-hand side the first term vanishes, while the second is dominant.
(b) If $\Psi(z)$ and $\Phi(z)$ are stochastically independent, we have

$$\mathbb{E}\Phi(z)\bar{\Psi}(z) = 0, \quad \mathbb{E}\Phi(z)\Psi(z) = \mathbb{E}(\Phi(z)^2) = 0,$$

so that both contributions to the left-hand side (1.7.3) collapse.
(c) Consider $\Psi(z) = \bar{\Phi}(\bar{z})$, when

$$\mathbb{E}\Phi(z)\Psi(z) = \mathbb{E}\Phi(z)\bar{\Phi}(\bar{z}) = \log\frac{1}{1-z^2}, \quad \mathbb{E}\Phi(z)\bar{\Psi}(z) = \mathbb{E}\Phi(z)\Phi(\bar{z}) = 0.$$

So at least pointwise, $\mathbb{E}\Phi(z)\Psi(z)$ may be the dominant contribution in (1.7.3).

The example in Remark 1.7.1(a) shows that the sesquianalytic correlation $\mathbb{E}\Phi(z)\bar{\Psi}(z)$ may be maximally big in the sense of modulus *everywhere in the disk* $\mathbb{D}$. However, the example in Remark 1.7.1(c) only says that the analytic correlation $\mathbb{E}\Phi(z)\Psi(z)$ may be maximal in modulus along the radius $[0,1[$ emanating from the origin. This leaves open the possibility of bounding L^2 means along concentric circles. The fact that $\mathbb{E}\Phi(z)\Psi(z)$ represents a holomorphic function in $\mathbb{D}$ limits to some extent the possible growth of the function. However, from the work of Abakumov and Doubtsov [1], we see that this is not a very strong restriction, and effectively knowing that $\mathbb{E}\Phi(z)\Psi(z)$ is holomorphic does not add much to the growth control beyond the pointwise bound (1.7.3), which may be understood as belonging to a Korenblum-type growth space. For some other aspects on the growth behavior of functions in Korenblum-type spaces, see [4]. To measure growth of functions in the Bloch space, the asymptotic variance of a function in the Bloch space has been studied (see [15], [2], [13], [7]). We recall that the *Bloch space* $\mathcal{B}(\mathbb{D})$ consists of all complex-valued holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < +\infty.$$

Naturally, this defines a seminorm on $\mathcal{B}(\mathbb{D})$, as constants get seminorm value 0. The *asymptotic variance* of a function $f \in \mathcal{B}(\mathbb{D})$ is the quantity

$$\sigma(f)^2 := \limsup_{r\to1^-}\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{T}}|f(r\zeta)|^2\mathrm{d}s(\zeta). \tag{1.7.4}$$

At least in dynamical situations, it captures very well the boundary growth of the given function. From a probabilistic point of view, it is based on thinking of the evolution of the function $r \mapsto f(r\zeta)$ as a Brownian motion in time $\log\frac{1+r}{1-r} \sim \log\frac{1}{1-r^2}$. The analytic correlation $f(z) = \mathbb{E}\Phi(z)\Psi(z)$ need not be an element of the Bloch space $\mathcal{B}(\mathbb{D})$. However, it has a finite asymptotic variance nevertheless.

Theorem 1.7.2. *For all jointly Gaussian processes* (Φ, Ψ) *consisting of* $\mathcal{D}_0$*-GAFs, we have the estimate*

$$\int_{\mathbb{T}} |\mathbb{E}\Phi(r\zeta)\Psi(r\zeta)|^2 \mathrm{d}s(\zeta) \le 2r^2 \log \frac{1}{1-r^2} + r^2.$$

This means that in the L^2-average sense on concentric circles, the function $\mathbb{E}\Phi(z)\Psi(z)$ spends most of its time on $|z| = r$ with values bounded by a constant times *the square root of* $\log \frac{1}{1-r^2}$, which is of course much smaller than what the bound (1.7.3) would allow for. In terms of the random variables α_j, β_k, the left-hand side expresssion in the above theorem equals

$$\int_{\mathbb{T}} |\mathbb{E}\Phi(r\zeta)\Psi(r\zeta)|^2 \mathrm{d}s(\zeta) = \sum_{l=2}^{+\infty} r^{2l} \Bigg| \sum_{j,k:j+k=l} (jk)^{-\frac{1}{2}} \langle \alpha_j, \bar{\beta}_k \rangle_\Omega \Bigg|^2. \tag{1.7.5}$$

It is natural to wonder if the bound $\sigma(f)^2 \le 2$ for the asymptotic variance of the analytic correlation $f(z) = \mathbb{E}\Phi(z)\Psi(z)$ in Theorem 1.7.2 is optimal. By a construction due to Zachary Chase [5], we have the following.

Theorem 1.7.3. (Chase) *There is a permutation* $\pi : \mathbb{Z}_+ \to \mathbb{Z}_+$ *such that if* $\beta_j = \bar{\alpha}_{\pi(j)}$ *and* $f(z) = \mathbb{E}\Phi(z)\Psi(z)$*, we have* $\sigma(f)^2 \ge 1.72$.

So, it remains to investigate the universal quantity $\Sigma^2 := \sup_f \sigma(f)^2$, where f runs over all possible analytic correlations $\mathbb{E}\Phi(z)\Psi(z)$.

1.8. **Orthonormal systems in separable Hilbert space.** In terms of the inner products $\langle \alpha_j, \bar{\beta}_k \rangle_\Omega$, the condition that the elements belong to a Gaussian Hilbert space is inconsequential and may be removed.

Corollary 1.8.1. *Let* $\{x_1, x_2, x_3, \ldots\}$ *an* $\{y_1, y_2, y_3, \ldots\}$ *be orthonormal systems in a separable complex Hilbert space* $\mathcal{H}$*. Then, for* $0 \le r < 1$*, we have the estimate*

$$\sum_{l=2}^{+\infty} r^{2l} \Bigg| \sum_{j,k:j+k=l} (jk)^{-\frac{1}{2}} \langle x_j, y_k \rangle_{\mathcal{H}} \Bigg|^2 \le 2 \log \frac{\mathrm{e}}{1-r^2}.$$

One possible interpretation of the corollary is that *on average*, the sums

$$\Bigg| \sum_{j,k:j+k=l} \left(\frac{l}{jk} \right)^{\frac{1}{2}} \langle x_j, y_k \rangle_{\mathcal{H}} \Bigg|^2$$

are bounded by 2.

1.9. **The analytic correlation and Dirichlet operator symbols.** For $w \in \mathbb{D}$, let s_w denotes the Szegő kernel

$$\mathrm{s}_z(\zeta) := \frac{1}{1 - \bar{z}\zeta}. \tag{1.9.1}$$

For functions in the Bergman space $A^2(\mathbb{D})$, taking the inner product with s_ζ is the same as finding the average

$$\langle f, \mathrm{s}_z \rangle_{\mathbb{D}} = \int_0^1 f(zt) \mathrm{d}t, \qquad f \in A^2(\mathbb{D}). \tag{1.9.2}$$

Definition 1.9.1. Let $\mathbf{T}$ be a bounded $\mathbb{C}$-linear operator on $L^2(\mathbb{D})$. The *Dirichlet operator symbol* associated with $\mathbf{T}$ is the function

$$\mathcal{P}[\mathbf{T}](z, w) := \langle \mathbf{T}(\bar{\mathrm{s}}_z), \mathrm{s}_w \rangle_{\mathbb{D}}, \qquad z, w \in \mathbb{D},$$

which is holomorphic in $\mathbb{D}^2$, with diagonal restriction

$$\oslash\mathcal{P}[\mathbf{T}](z) = \langle \mathbf{T}(\bar{\mathrm{s}}_z), \mathrm{s}_z \rangle_{\mathbb{D}}, \qquad z \in \mathbb{D}.$$

Remark 1.9.2. If $\mathbf{T} = \mathbf{M}_\mu$, the operator of multiplication by $\mu \in L^\infty(\mathbb{D})$, then

$$\oslash\mathcal{P}[\mathbf{M}_\mu](z) = \langle \mathbf{M}_\mu(\bar{\mathrm{s}}_z), \mathrm{s}_z\rangle_{\mathbb{D}} = \int_{\mathbb{D}} \frac{\mu(\xi)\mathrm{d}A(\xi)}{(1-z\bar{\xi})^2}, \qquad z \in \mathbb{D}, \tag{1.9.3}$$

which shows that $\oslash\mathcal{P}[\mathbf{T}]$ is a generalization of the Bergman projection to the setting of general bounded operators. There is a way to write $\mathcal{P}[\mathbf{T}]$ which makes the analogy with (1.9.3) clearer:

$$\mathcal{P}[\mathbf{T}](z,w) = \langle \mathbf{T}, \mathrm{s}_z \otimes \mathrm{s}_w\rangle_{\mathrm{tr}}.$$

Here, we use the bilinear tensor product $(f\otimes g)(h) = \langle h, \bar{g}\rangle f$, and the notation $\langle A, \mathbf{B}\rangle_{\mathrm{tr}} = \mathrm{tr}(\mathbf{A}\bar{\mathbf{B}}) = \mathrm{tr}(\bar{\mathbf{B}}\mathbf{A})$ for the trace inner product.

The next result characterizes the analytic correlations $\mathbb{E}\Phi(z)\Psi(w)$ as the Dirichlet symbols associated with contractions on $L^2(\mathbb{D})$.

Theorem 1.9.3. (a) *Given a pair of jointly Gaussian $\mathcal{D}_0$-GAFs* $(\Phi(z), \Psi(z))$ *there exists a norm contraction* $\mathbf{T} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ *such that*

$$\mathbb{E}\Phi(z)\Psi(w) = zw\langle \mathbf{T}\bar{\mathrm{s}}_z, \mathrm{s}_w\rangle_{\mathbb{D}} \qquad z,w \in \mathbb{D}, \tag{i}$$

(b) *Given a norm contraction* $\mathbf{T}$ *on* $L^2(\mathbb{D})$*, there exists a pair of jointly Gaussian $\mathcal{D}_0$-GAFs* $(\Phi(z), \Psi(z))$ *such that* (i) *holds.*

In particular, we see that in the sense of the theorem, the analytic correlations $\mathbb{E}\Phi(z)\Psi(w)$ may be identified with the Dirichlet operator symbols of contractions on $L^2\mathbb{D})$:

$$\mathbb{E}\Phi(z)\Psi(w) = zw\mathcal{P}[\mathbf{T}](z,w).$$

1.10. **Analytic correlations and the Bloch space.** The *Bloch space* $\mathcal{B}(\mathbb{D})$ consists of all complex-valued holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < +\infty.$$

This defines a seminorm on $\mathcal{B}(\mathbb{D})$, since constants get seminorm 0.

Definition 1.10.1. The *mock-Bloch space* $\mathcal{B}^{\mathrm{mock}}(\mathbb{D})$ is the space of functions

$$\Big\{\oslash\mathcal{P}[\mathbf{T}] : \ \mathbf{T} \text{ is a bounded operator on } L^2(\mathbb{D})\Big\}.$$

This mock-Bloch space is naturally endowed with a norm, which equals the infimum of $\|\mathbf{T}\|$ over all operators $\mathbf{T}$ representing the same symbol $\oslash\mathcal{P}[\mathbf{T}]$. All functions in $\mathcal{B}(\mathbb{D})$ are in $\mathcal{B}^{\mathrm{mock}}(\mathbb{D})$. This is well-known an easy to see using multiplication operators $\mathbf{M}_\mu$, as in [7] (compare with (1.9.3)). On the other hand, is $\mathcal{B}^{\mathrm{mock}}(\mathbb{D})$ contained in $\mathcal{B}(\mathbb{D})$? This is answered in the negative by the following.

Theorem 1.10.2. *There exists a function $f \in \mathcal{B}^{\mathrm{mock}}(\mathbb{D})$ which is not in $\mathcal{B}(\mathbb{D})$.*

It is known that $\mathcal{B}(\mathbb{D})$ is maximal among Möbius-invariant spaces [18], so $\mathcal{B}^{\mathrm{mock}}(\mathbb{D})$ cannot be Möbius-invariant in the standard sense. For a Möbius automorphism $\phi : \mathbb{D} \to \mathbb{D}$, let

$$\mathbf{U}_\phi f(z) := \phi'(z) f\circ\phi(z), \quad \bar{\mathbf{U}}_\phi f(z) := \bar{\phi}'(z) f\circ\phi(z), \tag{1.10.1}$$

be the associated unitary transformations of $L^2(\mathbb{D})$.

Theorem 1.10.3. *For a Möbius automorphism $\varphi : \mathbb{D} \to \mathbb{D}$, and a bounded operator* $\mathbf{T}$ *on* $L^2(\mathbb{D})$*, we write* $\mathbf{T}_\phi := \mathbf{U}_\phi\mathbf{T}\bar{\mathbf{U}}_\phi^*$*, which has the same norm as* $\mathbf{T}$*. If we write* $\mathcal{Q}[\mathbf{T}](z,w) := zw\mathcal{P}[\mathbf{T}](z,w)$ *and* $\oslash\mathcal{Q}[\mathbf{T}](z) := z^2\mathcal{P}[\mathbf{T}](z,z)$*, we then have the identity*

$$\oslash\mathcal{Q}[\mathbf{T}_\phi](z) = \oslash\mathcal{Q}[\mathbf{T}]\circ\phi(z) - \mathcal{Q}[\mathbf{T}](\phi(z),\phi(0)) - \mathcal{Q}[\mathbf{T}](\phi(0),\phi(z)) + \oslash\mathcal{Q}[\mathbf{T}](\phi(0)).$$

Typically, in Möbius-invariant spaces, the correction after a Möbius transform amounts to the subtraction of an appropriate constant. Here, we instead subtract a function in the Dirichlet space.

1.11. **Symbolds of Grunsky operators.** Let $\varphi:\mathbb{D}\to\mathbb{C}$ be a univalent function. In other words, φ is a conformal mapping onto a simply connected domain. The associated *Grunsky operator* $\mathbf{\Gamma}_\varphi$ is given by the expression

$$\mathbf{\Gamma}_\varphi f(z):=\int_{\mathbb{D}}\bigg(\frac{\varphi'(z)\varphi'(w)}{(\varphi(z)-\varphi(w))^2}-\frac{1}{(z-w)^2}\bigg)f(w)\mathrm{d}A(w),\qquad z\in\mathbb{D}. \tag{1.11.1}$$

It is well-known that $\mathbf{\Gamma}_\varphi$ is a norm contraction on $L^2(\mathbb{D})$, and it maps into the Bergman space $A^2(\mathbb{D})$. This contractiveness is called the *Grunsky inequalities*, and in this form it was studied in, e.g., [3]. For a given φ, we may consider instead the normalized mapping

$$\tilde\varphi(z)=\frac{\varphi(z)-\varphi(0)}{\varphi'(0)},$$

which has $\tilde\varphi(0)=0$ and $\tilde\varphi'(0)=1$. It is easy to see that $\mathbf{\Gamma}_{\tilde\varphi}=\mathbf{\Gamma}_\varphi$, so we might as well replace φ by its normalized variant $\tilde\varphi$, and require of φ that $\varphi(0)=0$ and $\varphi'(0)=1$. The Dirichlet symbol associated with $\mathbf{\Gamma}_\varphi$ is then

$$\boldsymbol{Q}[\mathbf{\Gamma}_\varphi](z,w)=zw\,\boldsymbol{\mathcal{P}}[\mathbf{\Gamma}_\varphi](z,w)=\log\frac{zw(\varphi(z)-\varphi(w))}{(z-w)\varphi(z)\varphi(w)},\qquad (z,w)\in\mathbb{D}^2, \tag{1.11.2}$$

with diagonal restriction

$$\oslash\boldsymbol{Q}[\mathbf{\Gamma}_\varphi](z)=z^2\oslash\boldsymbol{\mathcal{P}}[\mathbf{\Gamma}_\varphi](z)=\log\frac{z^2\varphi'(z)}{(\varphi(z))^2},\qquad z\in\mathbb{D}.$$

We want to characterize the Dirichlet symbols of the above form (1.11.2) among all Dirichlet symbols $\boldsymbol{Q}[\mathbf{T}](z,w)$ of norm contractions $\mathbf{T}$ on $L^2(\mathbb{D})$.

Theorem 1.11.1. *A function $Q=Q(z,w)$ which is holomorphic on $\mathbb{D}^2$ is of the form $\boldsymbol{Q}[\mathbf{\Gamma}_\varphi](z,w)$ for a normalized univalent function $\varphi:\mathbb{D}\to\mathbb{C}$ if and only if*
(a) *$Q(0,w)\equiv0$ and $Q(z,0)\equiv0$, and*
(b) *$Q=Q(z,w)$ solves the nonlinear wave equation*

$$\partial_z\partial_w Q+(\partial_z Q)(\partial_w Q)=\frac{z^2\partial_z Q-w^2\partial_w Q}{zw(z-w)}.$$

Remark 1.11.2. This result ties in nicely with deformation theory. Suppose we look for a solution $Q=Q_\lambda$ with an additional parameter added, $\lambda\in\mathbb{D}$, with respect to which Q_λ depends holomorphically, and that with $\lambda=0$, $Q_\lambda=Q_0=0$. Then we expand $Q=\sum_{j=1}^\infty\lambda^j\hat Q_j$ and see that the the nonlinear wave equation of Theorem 1.11.1 becomes a sequence of linear PDEs for the coefficient functions $\hat Q_j$. First, $\hat Q_1$ solves a homogeneous wave-type equation, and then for $j=2,3,4,\ldots$, $\hat Q_j$ solves an inhomogeneous wave-type equation, where the inhomogeneity involves the lower order coefficient functions $\hat Q_k$ for $1\le k<j$.

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2. The duality induced by the bilinear form of GAF

2.1. **The GAF as a duality.** Let us for the moment write $\Phi_{\boldsymbol{\alpha}}(z)$ for the $\mathcal{D}_0$-Gaussian analytic function given by (1.3.1), having in mind the notation $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3, \ldots)$ for the Gaussian vector of elements from $\mathfrak{G}$. The closure in $\mathfrak{G}$ of the linear span of the vectors α_j, $j = 1, 2, 3, \ldots$, will be denoted by $\mathfrak{A}$. We shall also need the closure in $\mathfrak{G}$ of the linear span of the vectors $\bar{\alpha}_j$, $j = 1, 2, 3, \ldots$, and we denote it by $\mathfrak{A}_*$. The independence and symmetry of of these random variables means that the vectors α_j form an orthonormal system in $\mathfrak{G}$, and that $\mathfrak{A}$ is orthogonal to $\mathfrak{A}_*$.

Continuing along the same line of thinking, we would write $\Phi_{\boldsymbol{\beta}}(z)$ for $\Psi(z)$, the second copy of the same Gaussian process. Now, if $\mathbf{M}$ is a bounded linear operator on $\mathfrak{A}$, then $\mathbf{M}\alpha_j \in \mathfrak{A}$ and hence has a convergent expansion in basis vectors:

$$\mathbf{M}\alpha_j = \sum_{k=1}^{+\infty} M_{j,k}\alpha_k,$$

where the sequence $k \mapsto M_{j,k}$ is in l^2. If we write $\mathbf{M}\boldsymbol{\alpha} = (\mathbf{M}\alpha_1, \mathbf{M}\alpha_2, \mathbf{M}\alpha_3, \ldots)$, we may speak of a Gaussian analytic function process

$$\Phi_{\mathbf{M}\boldsymbol{\alpha}}(z) = \sum_{j=1}^{+\infty} \mathbf{M}\alpha_j\, e_j(z) = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} M_{j,k}\alpha_k\, e_j(z) = \sum_{k=1}^{+\infty} \alpha_k \sum_{j=1}^{+\infty} M_{j,k}\, e_j(z) = \sum_{k=1}^{+\infty} \alpha_k\, \mathbf{M}^\dagger e_k(z), \tag{2.1.1}$$

where $e_j(z) = j^{-\frac{1}{2}} z^j$ as before. Moreover, the *GAF transpose of* $\mathbf{M}$, given by

$$\mathbf{M}^\dagger e_k(z) := \sum_{j=1}^{+\infty} M_{j,k} e_j(z) \tag{2.1.2}$$

defines a bounded linear mapping on $\mathcal{D}_0(\mathbb{D})$, as it just corresponds to the transpose of the matrix for $\mathbf{M}$, and shifting the basis from that of the Gaussian space $\mathfrak{A}$ to that of $\mathcal{D}_0(\mathbb{D})$. This way we have a natural transpose mapping $\mathbf{M} \to \mathbf{M}^\dagger$, and it is perhaps also natural to let its inverse be denoted the same way, so that $(\mathbf{M}^\dagger)^\dagger = \mathbf{M}$.

Typically, (2.1.1) will define a Gaussian analytic function with a correlation kernel which is different from that of $\Phi_{\boldsymbol{\alpha}}(z)$. Indeed, while $\mathbb{E}\Phi_{\mathbf{M}\boldsymbol{\alpha}}(z)\Phi_{\mathbf{M}\boldsymbol{\alpha}}(w) = 0$ automatically since $\mathfrak{A}$ is orthogonal to $\mathfrak{A}_*$, we see that

$$\mathbb{E}\Phi_{\mathbf{M}\boldsymbol{\alpha}}(z)\bar{\Phi}_{\mathbf{M}\boldsymbol{\alpha}}(w) = \sum_{j,k=1}^{+\infty} \langle \mathbf{M}\alpha_j, \mathbf{M}\alpha_k\rangle_\Omega\, e_j(z)\bar{e}_k(w), \tag{2.1.3}$$

which need not coincide with the corresponding correlation for $\Phi_{\boldsymbol{\alpha}}$. However, in the special case when the restriction $\mathbf{M}|_{\mathfrak{A}} = \mathbf{U}$ is unitary on $\mathfrak{A}$, so that $\mathbf{U}^*\mathbf{U} = \mathbf{I}$ on $\mathfrak{A}$, (2.1.3) gives us

$$\mathbb{E}\Phi_{\mathbf{U}\boldsymbol{\alpha}}(z)\bar{\Phi}_{\mathbf{U}\boldsymbol{\alpha}}(w) = \sum_{j,k=1}^{+\infty} \langle \mathbf{U}^*\mathbf{U}\alpha_j, \alpha_k\rangle_\Omega\, e_j(z)\bar{e}_k(w) = \sum_{j=1}^{+\infty} e_j(z)\bar{e}_j(w) = \log\frac{1}{1 - z\bar{w}}, \tag{2.1.4}$$

that is, the same correlation structure as for $\Phi_{\boldsymbol{\alpha}}(z)$. In other words, $\Phi_{\mathbf{U}\boldsymbol{\alpha}}$ is another copy of the $\mathcal{D}_0$-GAF. When $\mathbf{U} : \mathfrak{A} \to \mathfrak{A}$ is unitary, its GAF transpose $\mathbf{U}^\dagger$ acts unitarily on $\mathcal{D}_0(\mathbb{D})$, and the functions $\mathbf{U}^\dagger e_j(z)$ form an orthonormal basis for $\mathcal{D}_0(\mathbb{D})$. Naturally, this goes the other way around as well, that is, if a unitary transformation $\mathbf{V}$ on $\mathcal{D}_0(\mathbb{D})$ is given, this defines another unitary transformation $\mathbf{V}^\dagger$ on $\mathfrak{A}$ via (2.1.1) with $\mathbf{V}$ in place of $\mathbf{M}^\dagger$. An important instance is when the unitary transformation on $\mathcal{D}_0(\mathbb{D})$ is generated by a Möbius automorphism ϕ of the disk $\mathbb{D}$. If $\phi : \mathbb{D} \to \mathbb{D}$ is a Möbius automorphism, then the operator $\mathbf{V}_\phi$ given by

$$\mathbf{V}_\phi f(z) := f \circ \phi(z) - f \circ \phi(0)$$

is unitary on $\mathcal{D}_0(\mathbb{D})$ and therefore corresponds to a unitary transformation $\mathbf{V}_\phi^\dagger$ acting on $\mathfrak{A}$ such that

$$\Phi_{\mathbf{V}_\phi^\dagger\boldsymbol{\alpha}}(z) = \sum_{j=1}^{+\infty} \mathbf{V}_\phi^\dagger\alpha_j\, e_j(z) = \sum_{j=1}^{+\infty} \alpha_j\, \mathbf{V}_\phi e_j(z) = \sum_{j=1}^{+\infty} \alpha_j\, j^{-\frac{1}{2}}(\phi(z)^j - \phi(0)^j). \tag{2.1.5}$$

2.2. **GAF and Hankel-type duality.** We describe a variation on the above-mentioned GAF duality theme. Suppose that instead $\mathbf{M}$ is a bounded linear operator $\mathfrak{A} \to \mathfrak{A}_*$ (like a Hankel operator). In the same fashion as before, we write

$$\mathbf{M}\alpha_j = \sum_{k=1}^{+\infty} M_{j,k}\bar{\alpha}_k,$$

and obtain that

$$\Phi_{\mathbf{M}\alpha}(z) = \sum_{j=1}^{+\infty} \mathbf{M}\alpha_j\, e_j(z) = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} M_{j,k}\bar{\alpha}_k\, e_j(z) = \sum_{k=1}^{+\infty} \bar{\alpha}_k \sum_{j=1}^{+\infty} M_{j,k}\, e_j(z) = \sum_{k=1}^{+\infty} \bar{\alpha}_k\, \mathbf{M}^{\ddagger} e_k(z), \tag{2.2.1}$$

with $\mathbf{M}^{\ddagger}$, *the GAF-Hankel transpose of* $\mathbf{M}$, given by the analogue of (2.1.2),

$$\mathbf{M}^{\ddagger} e_k(z) := \sum_{j=1}^{+\infty} M_{j,k} e_j(z). \tag{2.2.2}$$

As with the GAF transpose, we let it be its own inverse, so that $(\mathbf{M}^{\ddagger})^{\ddagger} = \mathbf{M}$. If $\mathbf{M} : \mathfrak{A} \to \mathfrak{A}_*$ is isometric and onto, then $\mathbf{M}^{\ddagger}$ acts unitarily on $\mathcal{D}_0(\mathbb{D})$. On the other hand, if $\mathbf{V}$ is unitary on $\mathcal{D}_0(\mathbb{D})$, the $\mathcal{D}_0$-GAF

$$\sum_{k=1}^{+\infty} \bar{\alpha}_k\, \mathbf{V} e_k(z) = \sum_{k=1}^{+\infty} \bar{\alpha}_k \sum_{j=1}^{+\infty} V_{k,j}\, e_j(z) = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} V_{k,j}\bar{\alpha}_k\, e_j(z) = \sum_{j=1}^{+\infty} \mathbf{V}^{\ddagger}\alpha_j\, e_j(z), \tag{2.2.3}$$

where

$$\mathbf{V}^{\ddagger}\alpha_j = \sum_{k=1}^{+\infty} V_{k,j}\bar{\alpha}_k. \tag{2.2.4}$$

2.3. **Representation of the correlations $\mathbb{E}\Phi(z)\Psi(w)$ and $\mathbb{E}\Phi(z)\bar{\Psi}(w)$.** In view of the definitions of $\Phi(z)$ and $\Psi(w)$, we have that

$$\Phi(z)\Psi(w) = \sum_{j,k=1}^{+\infty} \frac{\alpha_j\beta_k}{\sqrt{jk}} z^j w^k, \tag{2.3.1}$$

so that taking expectations, we obtain that

$$\mathbb{E}\Phi(z)\Psi(w) = \sum_{j,k=1}^{+\infty} (jk)^{-\frac{1}{2}} (\mathbb{E}\alpha_j\beta_k)\, z^j w^k = \sum_{j,k=1}^{+\infty} (jk)^{-\frac{1}{2}} \langle \alpha_j, \bar{\beta}_k\rangle_\Omega\, z^j w^k, \qquad z, w \in \mathbb{D}. \tag{2.3.2}$$

Next, let $\mathbf{S} : \mathfrak{G} \to \mathfrak{G}$ be the bounded linear operator which maps $\mathfrak{A}_* \to \mathfrak{B}_*$ according to $\mathbf{S}\bar{\alpha}_j = \bar{\beta}_j$ for $j = 1, 2, 3, \ldots$, while $\mathbf{S}\gamma = 0$ holds for all $\gamma \in \mathfrak{G} \ominus \mathfrak{X}_* = \mathfrak{A} \oplus \mathfrak{N}$. Then $\mathbf{S}$ is a partial isometry: it vanishes on $\mathfrak{A} \oplus \mathfrak{N}$, and acts isometrically on $\mathfrak{A}_*$. In terms of this operator, we may rewrite (2.3.2):

$$\mathbb{E}\Phi(z)\Psi(w) = \sum_{j,k=1}^{+\infty} (jk)^{-\frac{1}{2}} \langle \alpha_j, \bar{\beta}_k\rangle_\Omega\, z^j w^k = \sum_{j,k=1}^{+\infty} (jk)^{-\frac{1}{2}} \langle \alpha_j, \mathbf{S}\bar{\alpha}_k\rangle\, z^j w^k, \qquad z \in \mathbb{D}. \tag{2.3.3}$$

While the representation (2.3.3) has some good properties, it is not too convenient to give useful estimates. We split

$$\bar{\beta}_j = \mathbf{S}\bar{\alpha}_j = \mathbf{P}_{\mathfrak{A}}\mathbf{S}\bar{\alpha}_j + \mathbf{P}_{\mathfrak{A}}^{\perp}\mathbf{S}\bar{\alpha}_j \quad \Longleftrightarrow \quad \beta_j = \bar{\mathbf{S}}\alpha_j = \mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}}\alpha_j + \mathbf{P}_{\mathfrak{A}_*}^{\perp}\bar{\mathbf{S}}\alpha_j,$$

so that the process $\Psi(w)$ takes the form

$$\Psi(w) = \sum_{j=1}^{+\infty} \beta_j\, e_j(w) = \sum_{j=1}^{+\infty} \mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}}\alpha_j\, e_j(w) + \sum_{j=1}^{+\infty} \mathbf{P}_{\mathfrak{A}_*}^{\perp}\bar{\mathbf{S}}\alpha_j\, e_j(w) =: \Psi_1(w) + \Psi_2(w),$$

with the obvious splitting of the process in two. Since

$$\mathbb{E}\Phi(z)\Psi_2(w) = \langle \Phi(z), \bar{\Psi}_2(w)\rangle_\Omega = 0$$

as a consequence of the properties of the projections, we see that

$$\mathbb{E}\Phi(z)\Psi(w)=\mathbb{E}\Phi(z)\Psi_1(w),$$

and from the GAF-Hankel duality of (2.2.1),

$$\Psi_1(w)=\sum_{j=1}^{+\infty}(\mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}}\alpha_j)\,e_j(w)=\sum_{j=1}^{+\infty}\bar{\alpha}_j\,(\mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}})^{\ddagger}e_j(w).$$

It is now immediate that

$$\mathbb{E}\Phi(z)\Psi(w)=\mathbb{E}\Phi(z)\Psi_1(w)=\sum_{j=1}^{+\infty}e_j(z)\,(\mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}})^{\ddagger}e_j(w),\qquad z\in\mathbb{D}. \tag{2.3.4}$$

Turning our attention to the other correlation $\mathbb{E}\Phi(z)\bar{\Psi}(w)$, we split

$$\bar{\beta}_j=\mathbf{S}\bar{\alpha}_j=\mathbf{P}_{\mathfrak{A}_*}\mathbf{S}\bar{\alpha}_j+\mathbf{P}_{\mathfrak{A}_*}^{\perp}\mathbf{S}\bar{\alpha}_j\quad\Longleftrightarrow\quad\beta_j=\bar{\mathbf{S}}\alpha_j=\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}}\alpha_j+\mathbf{P}_{\mathfrak{A}}^{\perp}\bar{\mathbf{S}}\alpha_j,$$

so that the process $\Psi(w)$ takes the form

$$\Psi(w)=\sum_{j=1}^{+\infty}\beta_j\,e_j(w)=\sum_{j=1}^{+\infty}\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}}\alpha_j\,e_j(w)+\sum_{j=1}^{+\infty}\mathbf{P}_{\mathfrak{A}}^{\perp}\bar{\mathbf{S}}\alpha_j\,e_j(w)=:\Psi_3(w)+\Psi_4(w),$$

with the obvious splitting of the process in two. Since

$$\mathbb{E}\Phi(z)\bar{\Psi}_4(w)=\langle\Phi(z),\Psi_4(w)\rangle_{\Omega}=0$$

as a consequence of the properties of the projections, we find that

$$\mathbb{E}\Phi(z)\bar{\Psi}(w)=\mathbb{E}\Phi(z)\bar{\Psi}_3(w).$$

In addition, by the duality of (2.1.2),

$$\Psi_3(w)=\sum_{j=1}^{+\infty}(\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}}\alpha_j)\,e_j(w)=\sum_{j=1}^{+\infty}\alpha_j\,(\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}})^{\dagger}e_j(w).$$

which gives the equality

$$\mathbb{E}\Phi(z)\bar{\Psi}(w)=\sum_{j=1}^{+\infty}e_j(z)\,\overline{(\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}})^{\dagger}e_j(w)},\qquad z,w\in\mathbb{D}. \tag{2.3.5}$$

To simplify the notation, we write $\mathbf{Q}=(\mathbf{P}_{\mathfrak{A}_*}\bar{\mathbf{S}})^{\ddagger}$ and $\mathbf{R}=(\mathbf{P}_{\mathfrak{A}}\bar{\mathbf{S}})^{\dagger}$ which are both contractions on $\mathcal{D}_0(\mathbb{D})$. Then our main formulas become, for $z,w\in\mathbb{D}$:

$$\mathbb{E}\Phi(z)\Psi(w)=\sum_{j=1}^{+\infty}e_j(z)\,\mathbf{Q}e_j(w),\qquad\mathbb{E}\Phi(z)\bar{\Psi}(w)=\sum_{j=1}^{+\infty}e_j(z)\,\overline{\mathbf{R}e_j(w)}. \tag{2.3.6}$$

3. Proofs of the fundamental bounds

3.1. **The joint pointwise bound of correlations.**

Proof of Proposition 1.5.1. Essentially, we just need to use the property that the 8×8 matrix (1.5.3) is positive semidefinite. Since for complex constants a,b,c,d,

$$0\le\left|a\Phi(z)+b\bar{\Phi}(z)-c\Psi(w)-d\bar{\Psi}(w)\right|^2=(|a|^2+|b|^2)|\Phi(z)|^2+(|c|^2+|d|^2)|\Psi(w)|^2+2\,\mathrm{Re}(a\bar{b}(\Phi(z))^2)$$
$$-2\,\mathrm{Re}(a\bar{c}\Phi(z)\bar{\Psi}(w))-2\,\mathrm{Re}(a\bar{d}\Phi(z)\Psi(w))-2\,\mathrm{Re}(\bar{b}c\Phi(z)\Psi(w))-2\,\mathrm{Re}(\bar{b}d\Phi(z)\bar{\Psi}(w))+2\,\mathrm{Re}(c\bar{d}(\Psi(w))^2),$$

the inequality survives after taking the expectation:

$$0\le\mathbb{E}\left|a\Phi(z)+b\bar{\Phi}(z)-c\Psi(w)-d\bar{\Psi}(w)\right|^2=(|a|^2+|b|^2)\log\frac{1}{1-|z|^2}+(|c|^2+|d|^2)\log\frac{1}{1-|w|^2}$$
$$-2\,\mathrm{Re}((a\bar{c}+\bar{b}d)\mathbb{E}\Phi(z)\bar{\Psi}(w))-2\,\mathrm{Re}((a\bar{d}+\bar{b}c)\mathbb{E}\Phi(z)\Psi(w)).$$

In other words, we have the inequality

$$2\,\mathrm{Re}((a\bar{d}+\bar{b}c)\mathbb{E}\Phi(z)\Psi(w))+2\,\mathrm{Re}((a\bar{c}+\bar{b}d)\mathbb{E}\Phi(z)\bar{\Psi}(w))\le(|a|^2+|b|^2)\log\frac{1}{1-|z|^2}+(|c|^2+|d|^2)\log\frac{1}{1-|w|^2}.$$

We now restrict the values of our parameters, and assume that $b=\bar{a}$ and $d=\bar{c}$. The above inequality then gives that

$$2\,\mathrm{Re}(ac\mathbb{E}\Phi(z)\Psi(w))+2\,\mathrm{Re}(a\bar{c}\mathbb{E}\Phi(z)\bar{\Psi}(w))\le|a|^2\log\frac{1}{1-|z|^2}+|c|^2\log\frac{1}{1-|w|^2}.$$

We write $ac=|ac|\omega_1$ and $a\bar{c}=|ac|\omega_2$, where $|\omega_1|=|\omega_2|=1$. Then

$$2\,\mathrm{Re}(\omega_1\mathbb{E}\Phi(z)\Psi(w))+2\,\mathrm{Re}(\omega_2\mathbb{E}\Phi(z)\bar{\Psi}(w))\le\frac{|a|}{|c|}\log\frac{1}{1-|z|^2}+\frac{|c|}{|a|}\log\frac{1}{1-|w|^2}.$$

On the right-hand side, we are free to minimize over $|a|$ and $|c|$, while on the left-hand side, we are free to maximize over the (freely choosable) unit vectors ω_1 and ω_2. After optimization, we arrive at the asserted estimate. □

3.2. **The proof of the fundamental integral estimate.**

Proof of Theorem 1.6.1. The first observation is that by $L^2(\mathbb{D})$-orthogonality,

$$\int_{\mathbb{D}}\left|aw\mathbb{E}\Phi(z)\Psi'(w)+b\bar{w}\mathbb{E}\Phi(z)\bar{\Psi}'(w)\right|^2\frac{\mathrm{d}A(w)}{|w|^2}=|a|^2\int_{\mathbb{D}}\left|\mathbb{E}\Phi(z)\Psi'(w)\right|^2\mathrm{d}A(w)+|b|^2\int_{\mathbb{D}}\left|\mathbb{E}\Phi(z)\bar{\Psi}'(w)\right|^2\mathrm{d}A(w).$$

Next, we observe that by the representation (2.3.6) and the norm contractive property of $\mathbf{Q}$,

$$\int_{\mathbb{D}}\left|\mathbb{E}\Phi(z)\Psi'(w)\right|^2\mathrm{d}A(w)=\Big\|\sum_{j=1}^{+\infty}e_j(z)\mathbf{Q}e_j\Big\|_{\nabla}^2\le\Big\|\sum_{j=1}^{+\infty}e_j(z)e_j\Big\|_{\nabla}^2=\sum_{j=1}^{+\infty}|e_j(z)|^2=\log\frac{1}{1-|z|^2},$$

and, that analogously, by the norm contractive property of $\mathbf{R}$,

$$\int_{\mathbb{D}}\left|\mathbb{E}\Phi(z)\bar{\Psi}'(w)\right|^2\mathrm{d}A(w)=\Big\|\sum_{j=1}^{+\infty}\bar{e}_j(z)\mathbf{R}e_j\Big\|_{\nabla}^2\le\Big\|\sum_{j=1}^{+\infty}\bar{e}_j(z)e_j\Big\|_{\nabla}^2=\sum_{j=1}^{+\infty}|e_j(z)|^2=\log\frac{1}{1-|z|^2}.$$

The proof is complete. □

4. Dirichlet symbols of contractions on $L^2(\mathbb{D})$ and analytic correlations of GAFs

4.1. **The correspondence between Dirichlet symbols and the analytic correlation.** We show the indicated relationship between the analytic correlation $\mathbb{E}\Phi(z)\Psi(w)$ and the Dirichlet symbols $\mathcal{P}[\mathbf{T}](z,w)$ for contractions $\mathbf{T}$ on $L^2(\mathbb{D})$.

Proof of Theorem 1.9.3. We begin with part (a), so we are given the orthonormal systems $\{\alpha_j\}_j$ and $\{\beta_j\}_j$ in the Gaussian Hilbert space $\mathfrak{G}$, and need to construct the norm contractive operator $\mathbf{T}$ on $L^2(\mathbb{D})$ with the indicated property. We let $\mathbf{S}:\mathfrak{G}\to\mathfrak{G}$ be the bounded linear operator with $\mathbf{S}\bar{\alpha}_j=\bar{\beta}_j$ for $j=1,2,3,\ldots$ while $\mathbf{S}\gamma=0$ for all $\gamma\in\mathfrak{G}\ominus\mathfrak{A}_*$. Given that $\mathbf{S}$ is a contraction, the product $\mathbf{P}_{\mathfrak{A}}\mathbf{S}$ is a contraction as well, and we may decompose

$$\mathbf{P}_{\mathfrak{A}}\bar{\beta}_k=\mathbf{P}_{\mathfrak{A}}\mathbf{S}\bar{\alpha}_k=\sum_{j=1}^{+\infty}A_{k,j}\alpha_j,$$

where $\sum_j|A_{k,j}|^2\le1$. For $j=1,2,3,\ldots$, we write $f_j(z)=e_j'(z)=j^{\frac12}z^{j-1}$, which constitutes an orthonormal basis in $A^2(\mathbb{D})$, and put

$$\mathbf{T}^*f_k=\sum_{l=1}^{+\infty}A_{k,l}\bar{f}_l,\qquad k=1,2,3,\ldots.$$

By linearity and norm boundedness of the matrix $(A_{j,k})_{j,k}$, this defines $\mathbf{T}^*$ on $A^2(\mathbb{D})$. Then

$$\langle \bar{f}_j, \mathbf{T}^* f_k\rangle_{\mathbb{D}} = \sum_{l=1}^{+\infty} A_{k,l}\langle \bar{f}_j, \bar{f}_l\rangle_{\mathbb{D}} = A_{k,j} = \sum_{l=1}^{+\infty} A_{k,l}\langle \alpha_j, \alpha_l\rangle_{\Omega} = \langle \alpha_j, \mathbf{P}_{\mathfrak{A}}\mathbf{S}\bar{\alpha}_k\rangle_{\Omega} = \langle \alpha_j, \mathbf{S}\bar{\alpha}_k\rangle_{\Omega} = \langle \alpha_j, \bar{\beta}_k\rangle_{\Omega},$$

and since

$$\bar{z}\mathrm{s}_z(\zeta) = \frac{\bar{z}}{1-\bar{z}\zeta} = \sum_{j=1}^{+\infty} \bar{z}^j\zeta^{j-1} = \sum_{j=1}^{+\infty} \bar{e}_j(z) f_j(\zeta), \tag{4.1.1}$$

it now follows that

$$zw\,\langle \bar{\mathrm{s}}_z, \mathbf{T}^*\bar{\mathrm{s}}_w\rangle_{\mathbb{D}} = \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\langle \bar{f}_j, \mathbf{T}^* f_k\rangle_{\mathbb{D}} = \sum_{j,k=1}^{+\infty}\langle \alpha_j, \bar{\beta}_k\rangle_{\Omega} e_j(z)e_k(w) = \mathbb{E}\Phi(z)\Psi(w),$$

so that condition (i) holds if $\mathbf{T}$ is the adjoint of $\mathbf{T}^*$. But to properly define $\mathbf{T}$, we need to extend $\mathbf{T}^*$ to all of $L^2(\mathbb{D})$. To this end, we simply declare that $\mathbf{T}^* f = 0$ holds for $f \in L^2(\mathbb{D}) \ominus A^2(\mathbb{D})$. It remains to check that so constructed, $\mathbf{T}^*$ is a contraction on $L^2(\mathbb{D})$, for then the adjoint $\mathbf{T}$ is contractive as well. For a polynomial $f \in A^2(\mathbb{D})$, we decompose it as a finite sum $f = \sum_k b_k f_k$ where $\|f\|^2_{L^2(\mathbb{D})} = \sum_k |b_k|^2$, and since $\mathbf{T}^* f = \sum_{l,k} A_{k,l} b_k \bar{f}_l$, we find that

$$\|\mathbf{T}^* f\|^2_{L^2(\mathbb{D})} = \sum_l \left|\sum_k A_{k,l} b_k\right|^2 = \left\|\mathbf{P}_{\mathfrak{A}}\mathbf{S}\sum_k b_k\bar{\alpha}_k\right\|^2 \le \left\|\sum_k b_k\bar{\alpha}_k\right\|^2 = \sum_k |b_k|^2 = \|f\|^2_{L^2(\mathbb{D})},$$

and it follows that $\mathbf{T}^*$ defines a contraction on $A^2(\mathbb{D})$ and hence in a second step on all of $L^2(\mathbb{D})$. This concludes the demonstration of part (a).

We proceed with the remaining task of obtaining part (b), which amounts to constructing the Gaussian Hilbert space $\mathfrak{G}$ and the sequence β_j and associated partial isometry $\mathbf{S}$ for a given contraction $\mathbf{T}$ on $L^2(\mathbb{D})$. We recall that $\mathfrak{A}$ and $\mathfrak{A}_*$ are two orthogonal subspaces in $\mathfrak{G}$. However, the sum $\mathfrak{A} \oplus \mathfrak{A}_*$ need not be all of $\mathfrak{G}$. We will assume that $\mathfrak{N} := \mathfrak{G} \ominus (\mathfrak{A} \oplus \mathfrak{A}_*)$ is *separable and infinite-dimensional* which just amounts to considering a sufficiently big (separable) Gaussian Hilbert space $\mathfrak{G}$. We split $\mathfrak{N} = \mathfrak{M} \oplus \mathfrak{M}_*$, where $\mathfrak{M}$ is the closed linear span of certain elements $\nu_1, \nu_2, \nu_3, \ldots$ of $\mathfrak{N}$, which are all i i d standard complex Gaussian variables (see Subsection 1.2). The space $\mathfrak{M}_*$ is then the closed linear span of the complex conjugates $\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3, \ldots$. As for notation, we will need the orthogonal (Bergman) projection $\mathbf{P}_{A^2} : L^2(\mathbb{D}) \to A^2(\mathbb{D})$, and its conjugate $\bar{\mathbf{P}}_{A^2}$ defined by

$$\bar{\mathbf{P}}_{A^2}(f) = \overline{\mathbf{P}_{A^2}(\bar{f})}.$$

We begin with the observation that

$$\langle \mathbf{T}\bar{f}_j, f_k\rangle_{\mathbb{D}} = \langle \bar{f}_j, \mathbf{T}^* f_k\rangle_{\mathbb{D}} = \langle \bar{f}_j, \bar{\mathbf{P}}_{A^2}\mathbf{T}^* f_k\rangle_{\mathbb{D}}. \qquad j,k = 1,2,3,\ldots,$$

We need to find i i d standard Gaussian vectors $\beta_1, \beta_2, \beta_3, \ldots$ in the Gaussian Hilbert space $\mathfrak{G}$ such that

$$\mathbb{E}\alpha_j\beta_k = \langle \alpha_j, \bar{\beta}_k\rangle_{\Omega} = \langle \mathbf{T}\bar{f}_j, f_k\rangle_{\mathbb{D}} = \langle \bar{f}_j, \bar{\mathbf{P}}_{A^2}\mathbf{T}^* f_k\rangle_{\mathbb{D}}, \qquad j,k = 1,2,3,\ldots,$$

since by summing over j,k we arrive at

$$\begin{aligned}\mathbb{E}\Phi(z)\Psi(z) &= \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\,\mathbb{E}\alpha_j\beta_k = \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\,\langle \mathbf{T}\bar{f}_j, f_k\rangle_{\mathbb{D}} \\ &= \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\,\langle \bar{f}_j, \bar{\mathbf{P}}_{A^2}\mathbf{T}^* f_k\rangle_{\mathbb{D}} = \langle \bar{\mathrm{s}}_z, \bar{\mathbf{P}}_{A^2}\mathbf{T}^*\mathrm{s}_w\rangle_{\mathbb{D}} = \langle \bar{\mathbf{P}}_{A^2}\bar{\mathrm{s}}_z, \mathbf{T}^*\mathrm{s}_w\rangle_{\mathbb{D}} = \langle \bar{\mathrm{s}}_z, \mathbf{T}^*\mathrm{s}_w\rangle_{\mathbb{D}} = \langle \mathbf{T}\bar{\mathrm{s}}_z, \mathrm{s}_w\rangle_{\mathbb{D}},\end{aligned}$$

where we used (4.1.1).

The element $\bar{\mathbf{P}}_{A^2}\mathbf{T}^* f_k$ is in the space of complex conjugates of $A^2(\mathbb{D})$, and as such it has an expansion

$$\bar{\mathbf{P}}_{A^2}\mathbf{T}^* f_k = \sum_{l=1}^{+\infty} A_{k,l}\bar{f}_l,$$

where $\sum_j |A_{k,j}|^2 \le 1$. We need $\mathbf{S}$ to have the property that in terms of the above expansion,

$$\mathbf{P}_{\mathfrak{A}}\mathbf{S}\bar{\alpha}_k = \mathbf{A}\bar{\alpha}_k := \sum_{j=1}^{+\infty} A_{k,j}\alpha_j,$$

which defines $\mathbf{A}$ as an operator $\mathfrak{A}_* \to \mathfrak{A}$. As such, it is a contraction. Indeed, if $\gamma \in \mathfrak{A}_*$ has expansion $\gamma = \sum_k b_k\bar{\alpha}_k$, we obtain that

$$\|\mathbf{A}\gamma\|_\Omega^2 = \sum_j \Big|\sum_k A_{k,j}b_k\Big|^2 = \Big\|\bar{\mathbf{P}}_{A^2}\mathbf{T}\sum_k b_k\bar{e}_k\Big\|^2 \le \Big\|\sum_k b_k\bar{e}_k\Big\|^2 = \sum_k |b_k|^2 = \|\gamma\|^2,$$

which verifies the norm contractivity of $\mathbf{A}$. We proceed to define the operator $\mathbf{S}$ and hence the Gassian vectors $\bar{\beta}_j = \mathbf{S}\bar{\alpha}_j$. To do this, we appeal to a standard procedure in operator theory. Since $\mathbf{A}$ maps $\mathfrak{A}_* \to \mathfrak{A}$, it has an adjoint $\mathbf{A}^{\circledast}$ which maps $\mathfrak{A} \to \mathfrak{A}_*$. We now form the *defect operator*

$$\mathbf{D} := (\mathbf{I}_{\mathfrak{A}_*} - \mathbf{A}^{\circledast}\mathbf{A})^{1/2},$$

which maps $\mathfrak{A}_* \to \mathfrak{A}_*$. The square root is well-defined given that we are taking the square root of a positive (semidefinite) operator. We use this defect operator to define an associated operator $\tilde{\mathbf{D}}$ on $\mathfrak{M}$, by declaring that if $\mathbf{D}\bar{\alpha}_j = \sum_k D_{j,k}\bar{\alpha}_k$, then

$$\tilde{\mathbf{D}}\nu_j = \sum_k D_{j,k}\nu_k, \qquad j = 1, 2, 3, \ldots.$$

Then $\tilde{\mathbf{D}}$ becomes a contraction on $\mathfrak{M}$, and we may now define the operator $\mathbf{S}$. For $\gamma \in \mathfrak{G} \ominus \mathfrak{A}_*$, we declare $\mathbf{S}\gamma = 0$. For $\gamma \in \mathfrak{A}_*$, we expand in basis vectors $\gamma = \sum_k b_k\bar{\alpha}_k$, and define the Gaussian vectors

$$\bar{\beta}_k = \mathbf{S}\bar{\alpha}_k := \mathbf{A}\bar{\alpha}_k + \tilde{\mathbf{D}}\nu_k \in \mathfrak{A} \oplus \mathfrak{M}, \qquad k = 1, 2, 3, \ldots, \tag{4.1.2}$$

where $\mathbf{P}_{\mathfrak{A}}\mathbf{S}$ is as before. Since $\tilde{\mathbf{D}}\nu_k \in \mathfrak{M} \subset \mathfrak{N}$, we see that

$$\mathbf{P}_{\mathfrak{A}}\mathbf{S}\bar{\alpha}_k = \mathbf{P}_{\mathfrak{A}}\mathbf{A}\bar{\alpha}_k + \mathbf{P}_{\mathfrak{A}}\tilde{\mathbf{D}}\nu_k = \mathbf{A}\bar{\alpha}_k,$$

since $\mathbf{A}\bar{\alpha}_k \in \mathfrak{A}$ and we know that $\mathfrak{N}$ is orthogonal to $\mathfrak{A}$, so things are as they should be. Moreover, $\mathbf{S}$ acts isometrically on $\mathfrak{A}_*$, as we see from

$$\|\mathbf{S}\gamma\|_{L^2(\mathbb{D})}^2 = \|\mathbf{A}\gamma\|_{L^2(\mathbb{D})}^2 + \|\mathbf{D}\gamma\|^2 = \|\gamma\|^2.$$

It follows that the functions $\bar{\beta}_k := \mathbf{S}\bar{\alpha}_k$ form an orthonormal system in $\mathfrak{G}$. It remains to verify that they are i i d standard complex Gaussians, which requires in addition to orthonormality that $\mathbb{E}\bar{\beta}_j\bar{\beta}_k = 0$ holds for all j and k. In view of (4.1.2),

$$\mathbb{E}\bar{\beta}_j\bar{\beta}_k = \langle\bar{\beta}_j, \beta_k\rangle_\Omega = 0,$$

given that $\bar{\beta}_j \in \mathfrak{A} \oplus \mathfrak{M}$ while $\beta_k \in \mathfrak{A}_* \oplus \mathfrak{M}_*$ and the subspaces $\mathfrak{A} \oplus \mathfrak{M}$ and $\mathfrak{A}_* \oplus \mathfrak{M}_*$ are orthogonal to one another in $\mathfrak{G}$. This tells us how to construct the sequence $\beta_1, \beta_2, \beta_3, \ldots$ stanrting from the contraction $\mathbf{T}$ on $L^2(\mathbb{D})$, and concludes the proof of part (b). □

4.2. **Orthonormal systems in Hilbert space and operator symbols.** We recall the setting of Corollary 1.8.1, where $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are two orthonormal systems in complex Hilbert space $\mathcal{H}$. Let $\mathcal{X}$ denote the closed linear span of the vectors $x_1, x_2, x_3, \ldots$, and $\mathbf{P}_{\mathcal{X}}$ the orthogonal projection $\mathcal{H} \to \mathcal{X}$.

Proof of Corollary 1.8.1. We recall the notation $f_j(z) = e_j'(z) = j^{1/2}z^{j-1}$, and let $\mathbf{T}^*$ be a linear operator with the property that

$$\mathbf{T}^* f_j = \sum_k \langle y_j, x_k\rangle_{\mathcal{H}}\bar{f}_k. \tag{4.2.1}$$

Then we have for scalars c_j (only finitely many nonzero) that

$$\Big\|\mathbf{T}^*\sum_j c_jf_j\Big\|_{\mathbb{D}}^2 = \Big\|\sum_{j,k} c_j\langle y_j, x_k\rangle_{\mathcal{H}}\bar{f}_k\Big\|_{\mathbb{D}}^2 = \Big\|\sum_{j,k} c_j\langle y_j, x_k\rangle_{\mathcal{H}}x_k\Big\|_{\mathcal{H}}^2 = \Big\|\mathbf{P}_{\mathcal{X}}\sum_j c_jy_j\Big\|_{\mathcal{H}}^2 \le \Big\|\sum_j c_jf_j\Big\|_{\mathcal{H}}^2$$

which shows that $\mathbf{T}^*$ defines a norm contraction $A^2(\mathbb{D}) \to \operatorname{conj} A^2(\mathbb{D})$. In a second step, we extend $\mathbf{T}^*$ to all of $A^2(\mathbb{D})$ by declaring that $\mathbf{T}^* f = 0$ for all $f \in L^2(\mathbb{D}) \ominus A^2(\mathbb{D})$, and we see that this defines a contraction on $L^2(\mathbb{D})$. The Dirichlet symbol of $\mathbf{T}$ is then, in view of (4.1.1),

$$zw\,\mathcal{P}[\mathbf{T}](z,w) = zw\langle \mathbf{T}\bar{\mathrm{s}}_z, \mathrm{s}_w\rangle_{\mathbb{D}} = zw\langle \bar{\mathrm{s}}_z, \mathbf{T}^*\mathrm{s}_w\rangle_{\mathbb{D}} = \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\langle \bar{f}_j, \mathbf{T}^* f_k\rangle_{\mathbb{D}} = \sum_{j,k=1}^{+\infty} e_j(z)e_k(w)\langle x_j, y_k\rangle_{\mathbb{D}}.$$

Taking the diagonal restriction, we have that

$$z^2\mathcal{P}[\mathbf{T}](z,z) = \sum_{l=2}^{+\infty} z^l \sum_{j,k:j+k=l} (jk)^{-\frac{1}{2}}\langle x_j, y_k\rangle_{\mathbb{D}},$$

and it follows that the claim is a direct consequence of Theorem 1.7.2. □

5. Hilbert spaces and diagonal restriction on the bidisk

5.1. **Weighted Bergman spaces on the disk and bidisk.** For real $\alpha > -1$, we write $A^2_\alpha(\mathbb{D})$ for the Hilbert space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ subject to the norm boundedness condition

$$\|f\|^2_{A^2_\alpha(\mathbb{D})} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \mathrm{d}A(z) < +\infty.$$

Moreover, we write $A^2_{-1,0}(\mathbb{D}^2)$ for the Hilbert space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ subject to the norm boundedness condition

$$\|f\|^2_{A^2_{-1,0}(\mathbb{D})} = \int_{\mathbb{D}}\int_{\mathbb{T}} |f(z,w)|^2 \mathrm{d}s(z)\mathrm{d}A(w) < +\infty.$$

For analytic functions f on the bidisk, we let $\oslash$ denote the operation of taking the diagonal restriction, $\oslash f(z) := f(z,z)$. We may for instance write $\partial_z^j \oslash (\partial_w^k f)$ to denote the function

$$\partial_z^j\Big(\partial_w^k f(z,w)\Big|_{w:=z}\Big).$$

In [9], the following diagonal norm expansion theorem was obtained.

Theorem 5.1.1. *For $f \in A^2_{-1,0}(\mathbb{D}^2)$, we have that*

$$\|f\|^2_{A^2_{-1,0}(\mathbb{D})} = \sum_{n=0}^{+\infty} \frac{(n+2)_n}{(n+1)!}\left\| \sum_{k=0}^{n} \frac{(-1)^k(k+2)_{n-k}}{k!(n-k)!(n+k+2)_{n-k}} \partial_z^{n-k} \oslash (\partial_w^k f)\right\|^2_{A^2_{2n+1}(\mathbb{D})}.$$

5.2. **The implementation of the fundamental estimate into the diagonal norm expansion.** Our starting point is the instance of $(a,b) = (1,0)$ in Theorem 1.6.1:

$$\int_{\mathbb{D}} \Big|a(z)\mathbb{E}\Phi(z)\Psi'(w)\Big|^2 \mathrm{d}A(w) \le |a(z)|^2 \log\frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

We dilate each variable using r, $0<r<1$, multiply by $|a(z)|^2$ for some $a \in H^2(\mathbb{D})$, and integrate over $\mathbb{T}\times\mathbb{D}$:

$$r^2\int_{\mathbb{T}}\int_{\mathbb{D}(0,\frac{1}{r})} \Big|a(z)\mathbb{E}\Phi(rz)\Psi'(rw)\Big|^2 \mathrm{d}A(w)\mathrm{d}s(z) \le \|a\|^2_{H^2}\,\log\frac{1}{1-r^2}.$$

We now throw away a part of the domain of integration (but, by monotonicity, we may remove the r^2 factor at the same time):

$$\int_{\mathbb{T}}\int_{\mathbb{D}} \Big|a(z)\mathbb{E}\Phi(rz)\Psi'(rw)\Big|^2 \mathrm{d}A(w)\mathrm{d}s(z) \le \|a\|^2_{H^2}\,\log\frac{1}{1-r^2}. \tag{5.2.1}$$

We recognize the left-hand side expression as the norm-square in the space $A^2_{-1,0}(\mathbb{D}^2)$ of the function $f(z,w) = a(z)\mathbb{E}\Phi(rz)\Psi'(rw)$. Clearly,

$$\oslash(\partial_w^k f)(z) = r^k a(z)\mathbb{E}\Phi(rz)\Psi^{(k+1)}(rz),$$

so an application of Theorem 5.1.1 gives that

$$\sum_{n=0}^{+\infty}\frac{2(n+2)_n}{n!}\int_{\mathbb{D}}\left|\sum_{k=0}^{n}\frac{(-1)^k(k+2)_{n-k}\,r^k}{k!(n-k)!(n+k+2)_{n-k}}\partial_z^{n-k}\Big(a(z)\mathbb{E}\Phi(rz)\Psi^{(k+1)}(rz)\Big)\right|^2(1-|z|^2)^{2n+1}\mathrm{d}A(z) \le \|a\|_{H^2}^2\,\log\frac{1}{1-r^2}. \tag{5.2.2}$$

We choose for simplicity $a(z)\equiv 1$, and expand the higher order derivative using the Leibniz rule

$$\partial_z^{n-k}\Big(\mathbb{E}\Phi(rz)\Psi^{(k+1)}(rz)\Big)=r^{n-k}\sum_{l=0}^{n-k}\frac{(n-k)!}{l!(n-k-l)!}\mathbb{E}\Phi^{(n-k-l)}(rz)\Psi^{(k+l+1)}(rz).$$

It follows that

$$\begin{aligned}\sum_{k=0}^{n}\sum_{l=0}^{n-k}&\frac{(-1)^k(k+2)_{n-k}\,r^k}{k!(n-k)!(n+k+2)_{n-k}}\partial_z^{n-k}\Big(\mathbb{E}\Phi(rz)\Psi^{(k+1)}(rz)\Big)\\ &=r^n\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-1)^k(k+2)_{n-k}}{k!l!(n-k-l)!(n+k+2)_{n-k}}\mathbb{E}\Phi^{(n-k-l)}(rz)\Psi^{(k+l+1)}(rz)\\ &=r^n\sum_{m=0}^{n}\frac{(-1)^m(n+1)[(n-m+1)_m]^2}{m!(m+1)!(n+2)_n}\Big(\mathbb{E}\Phi^{(n-m)}(rz)\Psi^{(m+1)}(rz)\Big)\end{aligned} \tag{5.2.3}$$

since it happens to be true for integers m with $0\le m\le n$ that

$$\sum_{k,l\ge 0:k+l=m}\frac{(-1)^k(k+2)_{n-k}}{k!l!(n-m)!(n+k+2)_{n-k}}=\frac{(-1)^m(n+1)[(n-m+1)_m]^2}{m!(m+1)!(n+2)_n}.$$

As we implement (5.2.3) into (5.2.2), we arrive at

$$\sum_{n=0}^{+\infty}\frac{2(n+1)^3\,r^{2n}}{(2n+1)!}\int_{\mathbb{D}}\left|\sum_{m=0}^{n}\frac{(-1)^m[(n-m+1)_m]^2}{m!(m+1)!}\Big(\mathbb{E}\Phi^{(n-m)}(rz)\Psi^{(m+1)}(rz)\Big)\right|^2(1-|z|^2)^{2n+1}\mathrm{d}A(z) \le \log\frac{1}{1-r^2}.$$

If we only keep the first term with $n=0$ on the left-hand side we are left with

$$2\int_{\mathbb{D}}\Big|\mathbb{E}\Phi(rz)\Psi'(rz)\Big|^2(1-|z|^2)\mathrm{d}A(z)\le\log\frac{1}{1-r^2}. \tag{5.2.4}$$

We are free to switch the roles of Φ and Ψ, so that we also have

$$2\int_{\mathbb{D}}\Big|\mathbb{E}\Phi'(rz)\Psi(rz)\Big|^2(1-|z|^2)\mathrm{d}A(z)\le\log\frac{1}{1-r^2}. \tag{5.2.5}$$

Since

$$\partial_z\mathbb{E}\Phi(rz)\Psi(rz)=r\mathbb{E}\Phi'(rz)\Psi(rz)+r\mathbb{E}\Phi(rz)\Psi'(rz),$$

it follows from (5.2.4) and (5.2.5) that

$$\begin{aligned}\int_{\mathbb{D}}&\Big|\partial_z\mathbb{E}\Phi(rz)\Psi(rz)\Big|^2(1-|z|^2)\mathrm{d}A(z)\\ &\le 2r^2\int_{\mathbb{D}}\Big(\Big|\mathbb{E}\Phi(rz)\Psi'(rz)\Big|^2+\Big|\mathbb{E}\Phi'(rz)\Psi(rz)\Big|^2\Big)(1-|z|^2)\mathrm{d}A(z)\le 2r^2\log\frac{1}{1-r^2}.\end{aligned} \tag{5.2.6}$$

Proof of Theorem 1.7.2. A variant of the Littlewood-Paley identity states that for an analytic function f in the Hardy space $H^2(\mathbb{D})$,

$$\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)\mathrm{d}A(z)=\int_{\mathbb{T}}|f(z)|^2\mathrm{d}s(z)-\int_{\mathbb{D}}|f(z)|^2\mathrm{d}A(z),$$

so that with $F(z) = \mathbb{E}\Phi(rz)\Psi(rz)$, (5.2.6) asserts that

$$\int_{\mathbb{T}} |F(rz)|^2 \mathrm{d}s(z) - \int_{\mathbb{D}} |F(rz)|^2 \mathrm{d}A(z) \le 2r^2 \log \frac{1}{1-r^2}. \tag{5.2.7}$$

In terms of the Taylor expansion of F,

$$F(z) = \sum_{j=2}^{+\infty} \hat{F}(j) z^j,$$

the estimate (5.2.7) amounts to

$$\sum_{j=2}^{+\infty} \frac{j\, r^{2j}}{j+1} |\hat{F}(j)|^2 \le 2r^2 \log \frac{1}{1-r^2}. \tag{5.2.8}$$

By integration, we see from (5.2.9) that

$$\begin{aligned} \int_{\mathbb{D}} |F(rz)|^2 \mathrm{d}A(z) = \sum_{j=2}^{+\infty} \frac{r^{2j}}{j+1} |\hat{F}(j)|^2 &\le 2\int_0^r \sum_{j=2}^{+\infty} \frac{j\, t^{2j-1}}{j+1} |\hat{F}(j)|^2 \mathrm{d}t \\ &\le 2\int_0^r t \log \frac{1}{1-t^2} \mathrm{d}t = (1-r^2)\log(1-r^2) + r^2 \le r^2. \end{aligned} \tag{5.2.9}$$

It now follows from (5.2.7) combined with the estimate (5.2.9) that

$$\int_{\mathbb{T}} |F(rz)|^2 \mathrm{d}s(z) \le 2r^2 \log \frac{1}{1-r^2} + r^2, \tag{5.2.10}$$

as claimed. □

6. Möbius invariance and the mock-Bloch space

6.1. **Möbius invariance of the Dirichlet symbol.** For a Möbius automorphism ϕ of the unit disk $\mathbb{D}$, let $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$ be the unitary transformations on $L^2(\mathbb{D})$ given by (1.10.1). If ϕ, ψ are two such Möbius automorphisms, we see that

$$\mathbf{U}_\psi \mathbf{U}_\phi f = \mathbf{U}_\psi(\phi'(f\circ\phi)) = \psi'(\phi'\circ\psi)(f\circ\phi\circ\psi) = (\phi\circ\psi)'(f\circ\phi\circ\psi) = \mathbf{U}_{\phi\circ\psi}(f),$$

which puts us in the context of representation theory. In particular, we find that $\mathbf{U}_\phi^* = \mathbf{U}_\phi^{-1} = \mathbf{U}_{\phi^{-1}}$.

Lemma 6.1.1. *We have that*

$$\bar{w}\, \mathbf{U}_\phi^* \mathrm{s}_w = \bar{\phi}(w)\, \mathrm{s}_{\phi(w)} - \bar{\phi}(0)\, \mathrm{s}_{\phi(0)}, \qquad w \in \mathbb{D}.$$

Proof. This is a direct computation. □

Proof of Theorem 1.10.3. In view of the definition of the operator $\mathbf{T}_\phi = \mathbf{U}_\phi \mathbf{T} \bar{\mathbf{U}}_\phi^*$, we see that

$$\oslash W[\mathbf{T}_\phi](z) = z^2 \langle \mathbf{U}_\phi \mathbf{T} \bar{\mathbf{U}}_\phi^* \bar{\mathrm{s}}_z, \mathrm{s}_z \rangle_{\mathbb{D}} = z^2 \langle \mathbf{T} \bar{\mathbf{U}}_\phi^* \bar{\mathrm{s}}_z, \mathbf{U}_\phi^* \mathrm{s}_z \rangle_{\mathbb{D}},$$

and by Lemma 6.1.1, it follows that

$$\begin{aligned} z^2 \langle \mathbf{T} \bar{\mathbf{U}}_\phi^* \bar{\mathrm{s}}_z, \mathbf{U}_\phi^* \mathrm{s}_z \rangle_{\mathbb{D}} = \phi(z)^2 \langle \mathbf{T} \bar{\mathrm{s}}_{\phi(z)}, \mathrm{s}_{\phi(z)} \rangle_{\mathbb{D}} &- \phi(0)\phi(z) \langle \mathbf{T} \bar{\mathrm{s}}_{\phi(z)}, \mathrm{s}_{\phi(0)} \rangle_{\mathbb{D}} \\ &- \phi(0)\phi(z) \langle \mathbf{T} \mathrm{s}_{\phi(0)}, \mathrm{s}_{\phi(z)} \rangle_{\mathbb{D}} + \phi(0)^2 \langle \mathbf{T} \mathrm{s}_{\phi(0)}, \mathrm{s}_{\phi(0)} \rangle_{\mathbb{D}}, \end{aligned}$$

which is the claimed invariance. □

6.2. **The mock-Bloch space is bigger than the Bloch space.** We show that the product of two Dirichlet space functions need not be in the Bloch space.

Proof of Theorem 1.10.2. Let $r_1, r_2, r_3, \ldots$ be a increasing sequence on $]0,1[$ tending rapidly to 1. We let f and g be the functions

$$f(z) := \sum_{j=1}^{+\infty} j^{-1}(1-r_j^2)\frac{z}{1-r_jz}, \quad g(z) := \sum_{j=1}^{+\infty} \frac{j^{-1}}{\sqrt{\log\frac{1}{1-r_j^2}}}\log\frac{1}{1-r_jz}.$$

Then

$$\|f\|_\nabla^2 = \int_{\mathbb{D}} |f'|^2 \mathrm{d}A = \int_{\mathbb{D}} \left|\sum_{j=1}^{+\infty} j^{-1}\tfrac{1-r_j^2}{(1-r_jz)^2}\right|^2 \mathrm{d}A(z) = \sum_{j,k=1}^{+\infty}(jk)^{-1}\frac{(1-r_j^2)(1-r_k^2)}{(1-r_jr_k)^2} < +\infty$$

if the sequence $\{r_j\}_j$ is sparse enough. In a similar manner,

$$\|g\|_\nabla^2 = \int_{\mathbb{D}} |g'|^2 \mathrm{d}A = \int_{\mathbb{D}} \left|\sum_{j=1}^{+\infty} \frac{j^{-1}}{\sqrt{\log\frac{1}{1-r_j^2}}}\frac{r_j}{1-r_jz}\right|^2 \mathrm{d}A(z) = \sum_{j,k=1}^{+\infty}(jk)^{-1}\frac{\log\frac{1}{1-r_jr_k}}{\sqrt{\log\frac{1}{1-r_j^2}}\sqrt{\log\frac{1}{1-r_k^2}}} < +\infty$$

if the sequence is sparse enough. We could require for instance that simultaneously the following conditions should hold:

$$\log\frac{1}{1-r_jr_k} \le 2^{-|j-k|}\sqrt{\log\frac{1}{1-r_j^2}}\sqrt{\log\frac{1}{1-r_k^2}}$$

and

$$\frac{1}{(1-r_jr_k)^2} \le 2^{-|j-k|}\frac{1}{(1-r_j^2)(1-r_k^2)}.$$

By construction, we have

$$f'(z)g(z) = \sum_{j,k=1}^{+\infty}(jk)^{-1}\frac{1-r_j^2}{(1-r_jz)^2}\frac{\log\frac{1}{1-r_kz}}{\sqrt{\log\frac{1}{1-r_k^2}}},$$

so that

$$(1-r_l^2)f'(r_l)g(r_l) = \sum_{j,k=1}^{+\infty}(jk)^{-1}\frac{1-r_j^2}{(1-r_jr_l)^2}\frac{\log\frac{1}{1-r_kr_l}}{\sqrt{\log\frac{1}{1-r_k^2}}} \ge l^{-2}\sqrt{\log\frac{1}{1-r_l^2}}$$

which with a sufficiently sparse sequence $\{r_j\}_j$ can be made to tend to infinity. Since both f and g have nonnegative Taylor coefficients,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \ge f'(x)g(x), \qquad 0 \le x < 1,$$

so it would follow that

$$\|fg\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|(fg)'(z)| \ge \sup_l (1-r_l^2)f'(r_l)g(r_l) = +\infty.$$

On the other hand, there is a rank 1 operator $\mathbf{T}$ such that $f(z)g(z) = \oslash\mathcal{P}[\mathbf{T}](z)$, so fg definitely belongs to the mock-Bloch space $\mathcal{B}^{\mathrm{mock}}(\mathbb{D})$. □

7. Characterization of Dirichlet symbols of Grunsky operators

7.1. **Grunsky operators.** Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a univalent function. In other words, φ is a conformal mapping onto a simply connected domain. The associated *Grunsky operator* $\mathbf{\Gamma}_\varphi$ is given by (1.11.2), and it is well-known that $\mathbf{\Gamma}_\varphi$ is a norm contraction on $L^2(\mathbb{D})$, and that it maps into the Bergman space $A^2(\mathbb{D})$. This contractiveness is usually referred to as the *Grunsky inequalities*, and in this form it was studied in, e.g., [3]. Without loss of generality, we assume that $\varphi(0) = 0$ and $\varphi'(0) = 1$. We recall that the Dirichlet symbol associated with $\mathbf{\Gamma}_\varphi$ is given by (1.11.2).

Proof of Theorem 1.11.1. We first show that any symbol $Q(z,w)=\boldsymbol{Q}[\boldsymbol{\Gamma}_\varphi](z,w)$ for a normalized univalent function φ has the properties (a) and (b). Since $\boldsymbol{Q}[\boldsymbol{\Gamma}_\varphi](z,w)=zw\mathcal{P}[\boldsymbol{\Gamma}_\varphi](z,w)$ it follows that (a) holds. We note that if $\psi(z):=1/\varphi(1/z)$ and if $\xi:=1/z$, $\eta:=1/w$, then

$$Q(z,w) \;=\; \boldsymbol{Q}[\boldsymbol{\Gamma}_\varphi](z,w) \;=\; \log\frac{zw(\varphi(z)-\varphi(w))}{(z-w)\varphi(z)\varphi(w)} \;=\; \log\frac{\xi^{-1}\eta^{-1}(\varphi(\xi^{-1})-\varphi(\eta^{-1}))}{(\xi^{-1}-\eta^{-1})\varphi(\xi^{-1})\varphi(\eta^{-1})} \;=\; \log\frac{\psi(\xi)-\psi(\eta)}{\xi-\eta}.$$

In other words,

$$\psi(\xi)-\psi(\eta)=(\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})},$$

so that

$$\begin{aligned}(7.1.1)\quad 0=\partial_\xi\partial_\eta(\psi(\xi)-\psi(\eta))&=\partial_\xi\partial_\eta\Big\{(\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}\Big\}\\ &=\Big\{\xi^{-2}\partial_z Q(\xi^{-1},\eta^{-1})-\eta^{-2}\partial_w Q(\xi^{-1},\eta^{-1})+(\xi-\eta)\xi^{-2}\eta^{-2}\Big(\partial_z\partial_w Q(\xi^{-1},\eta^{-1})\\ &\qquad+(\partial_z Q(\xi^{-1},\eta^{-1}))(\partial_z Q(\xi^{-1},\eta^{-1}))\Big)\Big\}\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}.\end{aligned}$$

Changing back to (z,w)-coordinates, we obtain that

$$0=z^2\partial_z Q(z,w)-w^2\partial_w Q(z,w)+(w-z)zw\Big(\partial_z\partial_w Q(z,w)+(\partial_z Q(z,w))(\partial_z Q(z,w))\Big),$$

which is the same as

$$\frac{w^2\partial_w Q(z,w)-z^2\partial_z Q(z,w)}{(w-z)zw}=\partial_z\partial_w Q(z,w)+(\partial_z Q(z,w))(\partial_z Q(z,w)),$$

that is, property (b).

We turn to the reverse implication, to show that a holomorphic function Q in $\mathbb{D}^2$ with the properties (a) and (b) is necessarily of the form $\boldsymbol{Q}[\boldsymbol{\Gamma}_\varphi]$ for some normalized conformal mapping φ. In view of the above calculation (7.1.1), condition (b) asserts that

$$\partial_\xi\partial_\eta\Big\{(\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}\Big\}=0$$

which means that locally in $\mathbb{D}_\mathrm{e}^2$,

$$(\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}=G_1(\xi)+G_2(\eta),$$

where G_1,G_2 are holomorphic but with possible logarithmic branching at infinity. Letting $\eta\to\xi$, we find that $G_1(\xi)+G_2(\xi)=0$, so that $G_2(\eta)=-G_1(\eta)$. So the above identity becomes

$$(7.1.2)\qquad (\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}=G_1(\xi)-G_1(\eta).$$

We still need to know that G_1 is a globally well-defined function in $\mathbb{D}_\mathrm{e}$ (without logarithmic branching). We differentiate both sides with respect to ξ:

$$G_1'(\xi)=\partial_\xi\Big((\xi-\eta)\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}\Big)=\Big\{1-\xi^{-2}(\xi-\eta)\partial_z Q(\xi^{-1},\eta^{-1})\Big\}\,\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}=\mathrm{e}^{Q(\xi^{-1},\xi^{-1})},$$

where in the last step we plugged in $\eta=\xi$, which is allowed since the expression is independent of η. As $|\xi|\to+\infty$, we have $Q(\xi^{-1},\xi^{-1})=\mathrm{O}(|\xi|^{-2})$, so that $\mathrm{e}^{Q(\xi^{-1},\xi^{-1})}=1+\mathrm{O}(|\xi|^{-2})$, which rules out a ξ^{-1} term, and hence there is no logarithmic branching. In addition, we see that $G_1'(\infty)=1$. If we put, for some constant c, $\psi:=G_1+c$, then by (7.1.2),

$$\mathrm{e}^{Q(\xi^{-1},\eta^{-1})}=\frac{\psi(\xi)-\psi(\eta)}{\xi-\eta}.$$

Since the left-hand side is holomorphic and does not vanish in $\mathbb{D}_\mathrm{e}^2$, it follows that ψ is univalent on $\mathbb{D}_\mathrm{e}$. But then there must exist a point in the complex plane $\mathbb{C}$ which is not in the image $\psi(\mathbb{D}_\mathrm{e})$, and by adjusting c we can make sure that $0\notin\psi(\mathbb{D}_\mathrm{e})$. Then winding things backwards we get φ from ψ in the above fashion, and $Q(z,w)$ is seen to be of the form (1.11.2), as claimed. □

8. Zachary Chase's construction of a permutation

8.1. **Permutation of bases.** We consider a permutation $\pi : \mathbb{Z}_+ \to \mathbb{Z}_+$. We use the permutation to define that $\beta_j := \bar{\alpha}_{\pi(j)}$, which in turn defines the second Gaussian process $\Psi(z)$. In this case, the formula (1.7.5) reduces to

$$\int_{\mathbb{T}} |\mathbb{E}\Phi(r\zeta)\Psi(r\zeta)|^2 \mathrm{d}s(\zeta) = \sum_{l=2}^{+\infty} r^{2l}\Big(\sum_{j,k:j+k=l} (jk)^{-\frac{1}{2}}\delta_{j,\pi(k)}\Big)^2, \tag{8.1.1}$$

where $\delta_{j,k}$ denotes the Kronecker delta, which equals 1 if $j = k$ and 0 otherwise. Since the sum of Kronecker deltas is squared, it makes sense to try to concentrate the times they equal 1 to certain values of l.

Proof of Theorem 1.7.3. Let $d \geq 3$ be an integer. We define the permutation $\pi = \pi_d$ in terms of a disjoint partition into intervals $\mathbb{Z}_+ = I_1 \cup I_2 \cup I_3 \cup \ldots$, where I_m is an interval on $\mathbb{Z}_+$ which moves toward the right as m increases. On each interval I_m we let π_d permute the interval in question. The first interval is $I_1 := \{1, \ldots, d-1\}$, and we put $\pi_d(j) := d - j$ for $j \in I_1$. The second interval is $I_2 := \{d, \ldots, d^2 - d\}$, and we put $\pi_d(j) := d^2 - j$ for $j \in I_2$. The third interval is $I_3 := \{d^2 - d + 1, \ldots, d^3 - d^2 + d - 1\}$ and on it we put $\pi_d(j) := d^3 - j$. The fourth interval is $I_4 := \{d^3 - d^2 + d, \ldots, d^4 - d^3 + d^2 - d\}$, and on it we put $\pi_d(j) := d^4 - j$. The general formula is $\pi_d(j) := d^m - j$ on I_m, but the endpoints of interval I_m depend on whether m is even or odd. If m is odd, then $m = 2n - 1$ for some $n = 1, 2, 3, \ldots$, and

$$I_m = I_{2n-1} := \left\{\frac{d^{2n-1}+1}{d+1}, \ldots, \frac{d^{2n}-1}{d+1}\right\}$$

while if m is even, then $m = 2n$ for some $n = 1, 2, 3, \ldots$, and

$$I_m = I_{2n} := \left\{\frac{d^{2n}+d}{d+1}, \ldots, \frac{d^{2n+1}-d}{d+1}\right\}.$$

The permutation π_d is now well-defined, and we see that for $k \in I_m$, $\delta_{j,\pi_d(k)} = \delta_{j,d^m-k} = 0$ unless $j + k = d^m$. This means that only the parameter values l that are powers of d contribute to the sum (8.1.1). When $l = d^m$, we find that

$$\sum_{j,k:j+k=d^m} (jk)^{-\frac{1}{2}}\delta_{j,\pi_d(k)} = \sum_{j\in I_m} j^{-\frac{1}{2}}(d^m - j)^{-\frac{1}{2}} = \frac{1}{d^m}\sum_{j\in I_m}\Big(\frac{j}{d^m}\Big)^{-\frac{1}{2}}\Big(1 - \frac{j}{d^m}\Big)^{-\frac{1}{2}} = \int_{\frac{1}{d+1}}^{1-\frac{1}{d+1}} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\mathrm{d}t + \mathrm{O}(d^{-m+1}),$$

by thinking of the sum as the Riemann sum of the integral with step length d^{-m}. The integral is the incomplete Beta function, since by symmetry

$$\int_{\frac{1}{d+1}}^{1-\frac{1}{d+1}} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\mathrm{d}t = \pi - 2\int_0^{\frac{1}{d+1}} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\mathrm{d}t = \pi - 4(d+1)^{-\frac{1}{2}}\, {}_2F_1\Big(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{d+1}\Big),$$

where the last equality relates it to the standard hypergeometric function. As it is well-known that

$$\lim_{r\to 1^-} \frac{1}{\log\frac{1}{1-r^2}} \sum_{m=1}^{+\infty} r^{2d^m} = \frac{1}{\log d},$$

it follows from the obtained asymptotics that

$$\lim_{r\to 1^-} \frac{1}{\log\frac{1}{1-r^2}} \sum_{m=1}^{+\infty} r^{2d^m}\Big(\sum_{j,k:j+k=d^m}(jk)^{-\frac{1}{2}}\delta_{j,\pi_d(k)}\Big)^2 = \frac{1}{\log d}\left\{\pi - 4(d+1)^{-\frac{1}{2}}{}_2F_1\Big(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{d+1}\Big)\right\}^2.$$

Finally, choosing $d = 29$ gives us the value ≈ 1.7208. This is the asymptotic variance of the correlation $f(z) = \mathbb{E}\Phi(z)\Psi(z)$ based on using the permutation to define that $\beta_j = \bar{\alpha}_{\pi_d(j)}$. □

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