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The Maximum Number of Triangles in a Graph of Given Maximum Degree

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Abstract: We prove that any graph on n vertices with max degree d has at most $q\binom{d+1}{3}+\binom{r}{3}$ triangles, where $n=q(d+1)+r$, $0\le r\le d$. This resolves a conjecture of Gan-Loh-Sudakov.

1 Introduction

Fix positive integers d and n with $d+1\le n\le 2d+1$. Galvin [7] conjectured that the maximum number of cliques in an n-vertex graph with maximum degree d comes from a disjoint union $K_{d+1}\sqcup K_r$ of a clique on $d+1$ vertices and a clique on $r:=n-d-1$ vertices. Cutler and Radcliffe [4] proved this conjecture. Engbers and Galvin [6] then conjectured that, for any fixed $t\ge 3$, the same graph $K_{d+1}\sqcup K_r$ maximizes the number of cliques of size t, over all $(d+1+r)$-vertex graphs with maximum degree d. Engbers and Galvin [6]; Alexander, Cutler, and Mink [1]; Law and McDiarmid [11]; and Alexander and Mink [2] all made progress on this conjecture before Gan, Loh, and Sudakov [9] resolved it in the affirmative. Gan, Loh, and Sudakov then extended the conjecture to arbitrary $n\ge 1$ (for any d).

Conjecture (Gan-Loh-Sudakov Conjecture)**.** *Any graph on n vertices with maximum degree d has at most $q\binom{d+1}{3}+\binom{r}{3}$ triangles, where $n=q(d+1)+r$, $0\le r\le d$.*

They showed their conjecture implies that, for any fixed $t\ge 4$, any max-degree d graph on $n=q(d+1)+r$ vertices has at most $q\binom{d+1}{t}+\binom{r}{t}$ cliques of size t. In other words, considering triangles is enough to resolve the general problem of cliques of any fixed size.

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The Gan-Loh-Sudakov conjecture (GLS conjecture) has attracted substantial attention. Cutler and Radcliffe [5] proved the conjecture for $d \leq 6$ and showed that a minimal counterexample, in terms of number of vertices, must have $q = O(d)$. Gan [8] proved the conjecture if $d+1-\frac{9}{4096}d \leq r \leq d$ (there are some errors in his proof, but they can be mended). Using fourier analysis, the author [3] proved the conjecture for Cayley graphs with $q \geq 7$. Kirsch and Radcliffe [10] investigated a variant of the GLS conjecture in which the number of edges is fixed instead of the number of vertices (with still a maximum degree condition).

In this paper, we fully resolve the Gan-Loh-Sudakov conjecture.

Theorem 1. *For any positive integers $n, d \geq 1$, any graph on n vertices with maximum degree d has at most $q\binom{d+1}{3} + \binom{r}{3}$ triangles, where $n = q(d+1)+r$, $0 \leq r \leq d$.*

Analyzing the proof shows that $qK_{d+1} \sqcup K_r$ is the unique extremal graph if $r \geq 3$, and that $qK_{d+1} \sqcup H$, for any H on r vertices, are the extremal graphs if $0 \leq r \leq 2$.

The heart of the proof is the following Lemma, of independent interest, which says that, in any graph, we can find a closed neighborhood whose removal from the graph removes few triangles. Theorem 1 will follow from its repeated application.

Lemma 1. *In any graph G, there is a vertex v whose closed neighborhood meets at most $\binom{d(v)+1}{3}$ triangles.*

As mentioned above, Theorem 1, together with the work of Gan, Loh, and Sudakov [9], yields the general result, for cliques of any fixed size.

Theorem 2. *Fix $t \geq 3$. For any positive integers $n, d \geq 1$, any graph on n vertices with maximum degree d has at most $q\binom{d+1}{t} + \binom{r}{t}$ cliques of size t, where $n = q(d+1)+r$, $0 \leq r \leq d$.*

Theorem 2 gives another proof of (the generalization of) Galvin's conjecture (to $n \geq 2d+2$) that a disjoint union of cliques maximizes the total number of cliques in a graph with prescribed number of vertices and maximum degree.

Finally, the author would like to point out a connection to a related problem, that of determining the minimum number of triangles that a graph of fixed number of vertices n and prescribed minimum degree δ can have. The connection stems from a relation, observed in [2] and [9], between the number of triangles in a graph and the number of triangles in its complement:

$$|T(G)| + |T(G^c)| = \binom{n}{3} - \frac{1}{2}\sum_v d(v)[n-1-d(v)].$$

Lo [12] resolved this “dual" problem when $\delta \leq \frac{4n}{5}$. His results resolve the GLS conjecture for regular graphs for $q = 2,3$, and the GLS conjecture implies his results, up to an additive factor of $O(\delta^2)$, for $q = 2,3$, and yields an extension of his results for $q \geq 4$ — these are the optimal results asymptotically, in the natural regime of $\frac{\delta}{n}$ fixed, and $n \to \infty$.

2 Notation

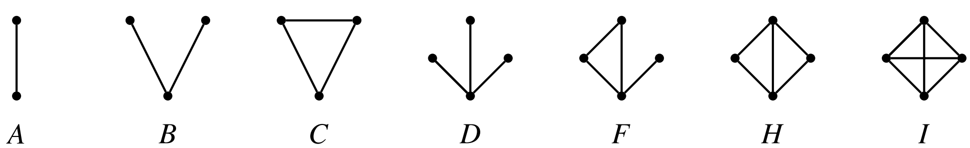
Denote by E the edge set of G; for two vertices u,v, we write “$uv \in E$" if there is an edge between u and v and “$uv \notin E$" otherwise — in particular, for any u, $uu \notin E$. For a vertex v, let $|T_{N[v]}|$ denote the number of triangles with at least one vertex in the closed neighborhood $N[v] := \{u : uv \in E\} \cup \{v\}$, and let $|T(G-N[v])|$ denote the number of triangles with all vertices in the graph $G-N[v]$ (the subgraph induced by the vertices not in $N[v]$). Finally, $d(v)$ denotes the degree of v.

3 Proof of Theorem 1

For a graph G, let $W(G) = \{(x,u,v,w) : ux, vx, wx \in E, uv, uw, vw \notin E\}$.

Lemma 2. *For any graph G, $6\sum_v |T_{N[v]}| + |W(G)| = \sum_v d(v)^3$.*

Proof. Let $\Omega = \{(z,u,v,w) : uv, uw, vw \in E$ and $[zu \in E$ or $zv \in E$ or $zw \in E]\}$, $\Sigma = \{(x,u,v,w) : ux, vx, wx \in E\}$, and $W = W(G)$. Note that repeated vertices in the 4-tuples are allowed. First observe that, since there are 6 ways to order the vertices of a triangle, $\sum_v 6|T_{N[v]}| = |\Omega|$. Any 4-tuple in Σ, W, or Ω gives rise to one of the induced subgraphs shown below, since one vertex must be adjacent to all the others.



Since $|\Sigma| = \sum_v d(v)^3$, it thus suffices to show that for each of the induced subgraphs above, the number of times it comes from a 4-tuple in Σ is the sum of the number of times it comes from 4-tuples in Ω and W. Any fixed copy of A, say on vertices u and v, comes 0 times from a 4-tuple in Ω (since it has no triangles), and 2 times from each of W and Σ $((u,v,v,v),(v,u,u,u))$. Any fixed copy of B, say on vertices u,v,w with $vu, vw \in E$, comes 0 times from Ω, and 6 times from each of W and Σ $((v,u,u,w),(v,u,w,u),(v,u,w,w),(v,w,u,u),(v,w,u,w),(v,w,w,u))$. Any fixed copy of C comes 18 times from each of Ω and Σ (3 choices for the first vertex and then 6 for the ordered triangle), and 0 times from W. Similarly, any fixed copy of D comes 6 times from each of W and Σ, and 0 times from Ω; finally, F,H,I come $6,12,24$ times, respectively, from each of Ω and Σ, and 0 times from W. $\square$

We now prove our key lemma, previously mentioned in the introduction.

Lemma 1. *In any graph G, there is a vertex v whose closed neighborhood meets at most $\binom{d(v)+1}{3}$ triangles, i.e. $|T_{N[v]}| \le \binom{d(v)+1}{3}$.*

Proof. By Lemma 2, since $|W(G)| \ge |\{(x,u,u,u) : ux \in E\}| = \sum_x d(x)$, we have $\sum_v |T_{N[v]}| \le \sum_v \frac{1}{6}[d(v)^3 - d(v)]$. By the pigeonhole principle, there is some v with

$$|T_{N[v]}| \le \frac{1}{6}[d(v)^3 - d(v)] = \binom{d(v)+1}{3}.$$

$\square$

Lemma 3. *For any positive integers $a \geq b \geq 1$, it holds that $\binom{a}{3}+\binom{b}{3} \leq \binom{a+1}{3}+\binom{b-1}{3}$. Consequently, for any positive integers a,b and any positive integer c with $\max(a,b) \leq c \leq a+b$, it holds that $\binom{a}{3}+\binom{b}{3} \leq \binom{c}{3}+\binom{a+b-c}{3}$.*

Proof. $\binom{a+1}{3}-\binom{a}{3}=\binom{a}{2}$, and $\binom{b}{3}-\binom{b-1}{3}=\binom{b-1}{2}$. Iterate to get the consequence. □

We now finish the proof of Theorem 1.

Proof of Theorem 1. With a fixed d, we induct on n. For $n=1$, the result is obvious. Take some $n \geq 2$, and suppose the theorem holds for all smaller values of n. Let G be a max-degree d graph on n vertices. By Lemma 1, we may take v with $|T_{N[v]}| \leq \binom{d(v)+1}{3}$. Write $n = q(d+1)+r$ for $0 \leq r \leq d$. Note $|T(G)| = |T(G-N[v])|+|T_{N[v]}|$. Since $G-N[v]$ has maximum degree (at most) d, if $d(v)+1 \leq r$, then induction and Lemma 3 give

$$|T(G)| \leq q\binom{d+1}{3}+\binom{r-(d(v)+1)}{3}+\binom{d(v)+1}{3} \leq q\binom{d+1}{3}+\binom{r}{3},$$

and if $d(v)+1 > r$, then induction and Lemma 3 give

$$|T(G)| \leq (q-1)\binom{d+1}{3}+\binom{d+1+r-(d(v)+1)}{3}+\binom{d(v)+1}{3} \leq q\binom{d+1}{3}+\binom{r}{3}.$$

The maximum degree condition ensured $d+1+r-(d(v)+1) \geq 0$ and $d(v)+1 \leq d+1$. □

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