GLOBAL SOLUTIONS OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE DISCONTINUOUS INITIAL DATA

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Abstract

We prove the global existence of weak solutions to the Navier-Stokes equations for compressible, heat-conducting flow in one space dimension with *large, discontinuous* initial data, and we obtain a-priori estimates for these solutions which are *independent of time*, sufficient to determine their asymptotic behavior. In particular, we show that, as time goes to infinity, the solution tends to a constant state determined by the initial mass and the initial energy, and that the magnitudes of singularities in the solution decay to zero.

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1. Introduction

We prove the global existence of weak solutions to the Navier-Stokes equations for compressible, heat-conducting flow in one space dimension with *large, discontinuous* initial data, and we obtain a-priori estimates for these solutions, which are *independent of time*, sufficient to determine their asymptotic behavior.

The equations under consideration express the conservation of mass and momentum and the balance of energy:

$$
\begin{aligned}
v_t - u_x &= 0,\\
u_t + p(v,e)_x &= \left(\tfrac{\epsilon u_x}{v}\right)_x,\\
\left(e + \tfrac{u^2}{2}\right)_t + (up(v,e))_x &= \left(\tfrac{\epsilon u u_x + \lambda e_x}{v}\right)_x .
\end{aligned}
\tag{1.1}
$$

Here v, u, e, and p represent respectively the specific volume, velocity, specific internal energy, and pressure; ϵ and λ are fixed positive viscosity parameters; and x is the Lagrangian coordinate, so that x =constant corresponds to a particle trajectory. We shall assume that e, v, and p are related by the equation of state of an ideal, polytropic fluid:

$$e = pv/(\gamma - 1) = c_v\theta,$$

where θ is the temperature, and $\gamma > 1$ and $c_v > 0$ are constants.

In the present paper we study the initial boundary-value problem for (1.1); thus $0 < x < 1$ without loss of generality, and the boundary conditions

$$u(i,t) = 0, \quad e_x(i,t) = 0, \qquad i = 0, 1, \tag{1.2}$$

are to hold for $t > 0$. On the other hand, our existence results can be extended with little difficulty to the Cauchy problem, as in [17] and [18].

We now give a precise formulation of our results. Let initial data

$$(v,u,e)|_{t=0} = (v_0(x), u_0(x), e_0(x)), \quad 0 \le x \le 1, \tag{1.3}$$

be given, satisfying

$$C_0^{-1} \le v_0(x) \le C_0, \quad e_0(x) \ge C_0^{-1}, \quad \|u_0\|_{L^4} + \|e_0\|_{L^2} + \mathrm{TV}(v_0) \le C_0, \tag{1.4}$$

for a constant $C_0 > 0$. Define the following functionals for weak solutions of (1.1):

$$
\begin{aligned}
\mathcal{E}(t) &= \sup_{0\le s\le t} \left(\sigma(s)\|u_x(\cdot,s)\|^2 + \sigma^2(s)\|e_x(\cdot,s)\|^2\right)\\
&\quad + \int_0^t (\|u_x(\cdot,s)\|^2 + \|e_x(\cdot,s)\|^2 + \sigma(s)\|u_t(\cdot,s)\|^2 + \sigma^2(s)\|e_t(\cdot,s)\|^2)ds,
\end{aligned}
\tag{1.5}
$$

$$
\begin{aligned}
\mathcal{F}(t) &= \sup_{0\le s\le t} \left(\sigma^2(s)\|u_t(\cdot,s)\|^2 + \sigma^3(s)\|e_t(\cdot,s)\|^2\right)\\
&\quad + \int_0^t \sigma^2(s)\|u_{xt}(\cdot,s)\|^2 ds + \int_0^t \sigma^3(s)\|e_{xt}(\cdot,s)\|^2 ds,
\end{aligned}
\tag{1.6}
$$

where $\sigma(t)$ =min$\{t,1\}$, and $\|\cdot\|$ denotes the norm in $L^2(0,1)$. The following theorem then gives the main result of this paper.

Theorem 1.1 (Well-Posedness and Large-Time Behavior). *Given initial data (v_0,u_0,e_0) satisfying (1.4), there is a global weak solution (v,u,e) such that $v,u\in C([0,\infty);L^2)$, $e\in C((0,\infty);L^2)$ with $e(\cdot,t)\rightharpoonup e_0$ weakly in L^2 as $t\to 0$. Furthermore, there is a constant M depending on C_0, but independent of $t>0$, such that the following hold:*

$$M^{-1}\le v(x,t)\le M,\qquad M^{-1}\le e(x,t)\le M\sigma^{-1}(t), \tag{1.7}$$

$$TV_{[0,1]}\left(v(\cdot,t)\right)\le M,\qquad \|v(\cdot,t')-v(\cdot,t)\|\le M|t'-t|^{1/2}, \tag{1.8}$$

$$\|u_x(\cdot,t)\|\le M\sigma^{-1/2}(t),\qquad \|e_x(\cdot,t)\|\le M\sigma^{-1}(t), \tag{1.9}$$

$$\|u(\cdot,t)\|_{L^\infty}\le M\sigma^{-1/4}(t),\qquad \mathcal{E}(t)+\mathcal{F}(t)\le M. \tag{1.10}$$

Finally, the solution tends to a constant as $t\to\infty$ in the sense that

$$\|(v-v_\infty)(\cdot,t)\|_{L^\infty(0,1)}+\|(u,e-e_\infty)(\cdot,t)\|_{H^1(0,1)}\to 0, \tag{1.11}$$

where $v_\infty=\int_0^1 v_0(x)dx$ and $e_\infty=\int_0^1\left(e_0(x)+\frac{u_0^2(x)}{2}\right)dx$.

The construction of solutions proceeds in two steps. In the first, we assume that v_0 is piecewise H^1, having a finite number of points of discontinuity. Solutions for this case are obtained as limits of approximate solutions derived from a suitable semidiscrete difference approximation to (1.1). Then in the second step we exploit the uniform total variation estimate (1.8) to complete the solution operator for the more general case that $v_0\in BV$. This entire construction parallels the analysis of Hoff [10] and [11], in which global existence was obtained for small initial data. Our results thus improve upon [10] and [11] by allowing for *large, discontinuous* initial data, and by achieving estimates (1.7)-(1.10) which are independent of time, sufficient to obtain the large-time behavior (1.11) of solutions. The crucial step here is to obtain pointwise upper and lower bounds for the specific volume v. Our derivation of these bounds starts from the idea of Kazhikov and Shelukhin [19], but elaborates upon it in a new and significant way. We therefore give the complete details of this argument (Lemma 2.2 below).

The pointwise bounds in (1.7) for the specific volume (equivalently, the density) show that neither vacuum states nor concentration states can occur, no matter how large the initial data is. This is one of several important differences between the Navier-Stokes equations and the inviscid Euler equations, for which vacuum states may in fact occur for large initial data and for certain equations of state (cf. [2, 4]). It is also relevant in this regard that solutions of the Navier-Stokes equations show certain instabilities when vacuum states are allowed (cf. Hoff and Serre [14]).

Once these pointwise bounds (1.7) for v have been established, the higher-order regularity assertions of Theorem 1.1 follow very much as in [10] and [11]; we therefore omit most of the details of their proofs. The asymptotic behavior (1.11) can then be derived as a consequence of the time-independent bounds (1.7)-(1.10), together with the weak form of the equations (1.1). This argument is given in Section 3.

The construction of solutions for the intermediate case that v_0 is piecewise H^1 plays a largely auxiliary role in the present paper. Still, there are interesting and important observations that can be made concerning the propagation of singularities for this case. These observations are made in [10] and [11] as well, but their range or validity is now extended to the case of large initial data. Thus let v_0 be piecewise smooth, having jump discontinuities at isolated points $y_1 < \cdots < y_N$. Then, by applying the Rankine-Hugoniot conditions to (1.1) (together with the hypothesis that u and e are continuous in positive time) as in [10] and [11], we find at the heuristic level that discontinuities in v, p, u_x, and e_x occur only at $x = y_i$, and satisfy the following jump conditions:

$$\left[p(v,e) - \frac{\epsilon u_x}{v}\right] = 0, \qquad \left[\frac{e_x}{v}\right] = 0, \tag{1.12}$$

where $[w]$ denotes a jump of the function w across $x = y_i : [w(y_i,t)] = w(y_i + 0,t) - w(y_i - 0,t)$. Now, combining (1.12) with the first equation in (1.1), we find that

$$[\log v]_t = \epsilon^{-1}[p] = \frac{\gamma - 1}{\epsilon} e \left[\frac{1}{v}\right] .$$

Since e is positive and $1/v$ is a decreasing function of $\log v$, we therefore anticipate *exponential decay* of the magnitudes of these discontinuities as $t \to \infty$. This is indeed the case, and its proof depends crucially on the fact that the pointwise bounds for v and e of Theorem 1.1 are independent of time. The precise statement is as follows.

Theorem 1.2 (Discontinuities and Large-Time Behavior)**.** *Assume, in addition to the hypotheses of Theorem 1.1, that v_0 is piecewise H^1, having isolated jump discontinuities at points $y_1 < \cdots < y_N$. Then, for $t > 0$, each of the quantities $v(\cdot,t), p(\cdot,t), u_x(\cdot,t)$, and $e_x(\cdot,t)$ has one-sided limits at each y_i, and the jump conditions* (1.12) *are satisfied in a strict pointwise sense. Moreover,*

$$[\log v(y_i,t)] = [\log v(y_i,0)] \exp\left(-\epsilon^{-1}\int_0^t \alpha_i(s)\, ds\right), \tag{1.13}$$

where

$$\alpha_i(t) = -(\gamma - 1)e(y_i,t)\frac{[(1/v)(y_i,t)]}{[\log v(y_i,t)]}. \tag{1.14}$$

Finally, there is a constant M depending on C_0, but independent of t and N, such that

$$\begin{aligned} |[(v,p,u_x,e_x)(y_d,t)]| &\leq M \min\left(\exp\{-M^{-1}t^{1/2}\},\ \sigma(t)^{-3/2}\exp\{-M^{-1}t\}\right) \\ &\to 0, \quad \text{as } t \to \infty. \end{aligned}$$

Uniqueness is a rather delicate issue for solutions which are as general as those of Theorem 1.1, owing to the absence of uniform regularity in the initial layer near $t = 0$. By imposing slightly stronger conditions on u_0 and e_0, however, we can improve the smoothing rates implicit in the definitions of $\mathcal{E}(t)$ and $\mathcal{F}(t)$ sufficiently to prove that solutions are in fact unique and depend continuously on their initial values. The following theorem is established in Hoff [10] and [13] for small solutions and can be extended to the present case with little difficulty.

Theorem 1.3 (Regularity and Stability). *Assume, in addition to the hypotheses of Theorem 1.1, that*

$$TV(u_0) + TV(e_0) \leq C_0. \tag{1.15}$$

Then there exists $M > 0$ independent of t such that the solution of Theorem 1.1 satisfies the additional estimates:

$$\|u_x(\cdot,t)\| \leq M\sigma^{-1/4}(t), \qquad \|e_x(\cdot,t)\| \leq M\sigma^{-1/4}(t), \tag{1.16}$$

$$TV_{[0,1]}(u(\cdot,t)) \leq M\sigma^{-1/4}(t), \qquad TV_{[0,1]}(e(\cdot,t)) \leq M\sigma^{-1/4}(t). \tag{1.17}$$

Moreover, solutions satisfying (1.7)-(1.10) *and* (1.16)-(1.17) *are unique and depend continuously on their initial data in the sense that, if (v_1, u_1, e_1) and (v_2, u_2, e_2) are any two such solutions, and if $S(t)$ is defined by*

$$S(t) = \|(v_2 - v_1)(\cdot,t)\| + \|(u_2 - u_1)(\cdot,t)\|_{-\alpha} + \|(e_2 - e_1)(\cdot,t)\|_{-\beta},$$

where α and β are small and positive ($\|\cdot\|_{-r}$ denotes the norm in the negative Sobolev space $H^{-r}(0,1)$), then given $T > 0$, there is a constant $C(T)$ such that, for $0 \leq t \leq T$,

$$S(t) \leq C(T)S(0).$$

We remark that the results of Theorems 1.1 and 1.2 can be converted to equivalent statements for the Navier-Stokes equations in Eulerian coordinates (cf. [3]):

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= (\epsilon u_x)_x, \\ (\rho E)_t + (u(\rho E + p))_x &= (\epsilon u u_x)_x + (\lambda e_x)_x, \end{aligned} \tag{1.18}$$

where $E = e + u^2/2$ is the total specific energy. Corresponding statements concerning uniqueness and continuous dependence are more subtle, however, owing to the fact that the change of variables from Lagrangian to Eulerian coordinates is solution–dependent, and our solutions are only minimally regular.

The Navier-Stokes system (1.1) has been studied by a great many authors in a large variety of contexts. Besides the results of Hoff [10, 11] and Kazhikov and Shelukhin [19] referred to earlier, the reader may consult Amosov and Zlotnick [1], Jiang [16], Matsumura and Yanagi [21], and Fujita-Yashima, Padula and Novotny [7], and the references contained therein. As far as we are aware, the present paper is the first to give time-independent estimates for large, discontinuous initial data, for the full nonbarotropic system (1.1). See also

Hoff [12] for a global existence result for the corresponding multidimensional system, but with small, discontinuous initial data, and Lions [20] for the global existence of weak solutions for the isentropic Navier-Stokes equations in the presence of vacuum.

The approach developed here has also been applied to establishing the global well-posedness and large-time behavior of discontinuous solutions with large initial data to the Navier-Stokes equations for a compressible *reacting* flow, describing dynamic combustion (see [3, 6, 8, 9, 22]). These results will be presented in our forthcoming paper [5].

2. Difference Approximations and A-Priori Estimates

In this section we construct semidiscrete difference approximations of solutions to (1.1)-(1.4), and we derive various a-priori bounds for these approximations required for the subsequent analysis.

Let h be an increment in x such that $Kh = 1$ for $K \in \mathbb{Z}_+$, $x_k = kh$ for $k \in \{0, 1, \cdots, K\}$, and $x_j = jh$ for $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots K - \frac{1}{2}\}$. Approximations $v_j(t), u_k(t), e_j(t)$ to $v(x_j, t), u(x_k, t)$, $e(x_j, t)$ are then constructed as follows:

$$\dot{v}_j = \delta u_j, \tag{2.1}$$

$$\dot{u}_k + \delta p_k = \epsilon \delta \left(\frac{\delta u}{v} \right)_k, \tag{2.2}$$

$$\dot{e}_j + p_j \delta u_j = \epsilon \frac{(\delta u_j)^2}{v_j} + \lambda \delta \left(\frac{\delta e}{v} \right)_j. \tag{2.3}$$

Here $p_j = p(v_j, e_j)$, $e_j = e(v_j, e_j)$, v_k is taken to be the average $v_k = \frac{v_{k+\frac{1}{2}} + v_{k-\frac{1}{2}}}{2}$ with $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots, K - \frac{1}{2}\}$, $k \in \{0, 1, \cdots, K\}$, and δ is the operator defined by

$$\delta w_l = \frac{w_{l+\frac{1}{2}} - w_{l-\frac{1}{2}}}{h}, \qquad l = k, \text{ or } j.$$

For the time being we assume only that initial data $(v_j(0), u_k(0), e_j(0))$ for the above ordinary differential equations has been specified and satisfies

$$u_0 = u_K = 0, \qquad \delta e_0 = \delta e_K = 0, \tag{2.4}$$

and

$$C_0^{-1} \le v_j(0) \le C_0, \quad e_j(0) \ge C_0^{-1}, \quad \sum_k u_k^4(0) h + \sum_j e_j^2(0) h \le C_0. \tag{2.5}$$

We also assume that there are distinguished points $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1, N = N(h), N^4 h \le 1$, such that

$$\sum_{k=k_i} |[v_{k_i}(0)]| + \sum_{k \ne k_i} |\delta v_k(0)|^2 h \le C_0. \tag{2.6}$$

Now, standard theory of ordinary differential equations applies to show that the initial-value problem (2.1)-(2.5) has a unique solution $(v_j(t), u_k(t), e_j(t))$, defined at least for small time. The a-priori bounds to be derived in this section will show that these solutions exist *globally* in time, and will provide sufficient compactness both to extract limiting solutions as $h \to 0$ as well as to determine their asymptotic behavior.

In the following, M will denote a generic positive constant independent of t and h. We begin with certain basic energy estimates for the difference approximations.

Lemma 2.1. *The solutions $(v_j(t), u_k(t), e_j(t))$ of (2.1)-(2.5) satisfy the following relations:*

$$\sum_j v_j(t)h = 1, \tag{2.7}$$

$$\sum_j e_j(t)h + \frac{1}{2}\sum_k u_k^2(t)h = \sum_j e_j(0)h + \frac{1}{2}\sum_k u_k^2(0)h, \tag{2.8}$$

$$E(t) + \int_0^t V(s)ds = E(0) < \infty, \tag{2.9}$$

where

$$\begin{cases} E(t) = \sum_j \left((e_j - 1 - \log e_j) + (\gamma - 1)(v_j - 1 - \log v_j)\right)(t)h + \frac{1}{2}\sum_k u_k^2(t)h, \\ V(t) = \sum_j \left(\frac{\epsilon(\delta u_j)^2}{v_j e_j}\right)(t)h + \sum_k \left(\frac{\lambda(\delta e_k)^2}{v_k e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}}\right)(t)h. \end{cases}$$

Proof: *Step 1.* We obtain from (2.1) that

$$\frac{d}{dt}\sum_j v_j(t)h = \sum_j \delta u_j(t)h = u_K - u_0 = 0. \tag{2.10}$$

Step 2. Summing over j in (2.3) and integrating with respect to t, we obtain

$$\sum_j e_j(t)h + \int_0^t \sum_j (p_j \delta u_j)hds = \sum_j e_j(0)h + \int_0^t \sum_j \frac{\epsilon(\delta u_j)^2}{v_j}hds. \tag{2.11}$$

We obtain in a similar way from (2.2) that

$$\frac{1}{2}\sum_k u_k^2(t)h - \int_0^t \sum_j (p_j \delta u_j)hds = \frac{1}{2}\sum_k u_k^2(0)h - \int_0^t \sum_j \frac{\epsilon(\delta u_j)^2}{v_j}hds. \tag{2.12}$$

Then (2.8) follows from (2.11) and (2.12).

Step 3. Differentiating the energy $E = E(t)$, we obtain

$$\frac{d}{dt}E(t) = \sum_j \left((\dot e_j - \frac{1}{e_j}\dot e_j) + (\gamma - 1)(\dot v_j - \frac{1}{v_j}\dot v_j)\right)h + \sum_k u_k \dot u_k h.$$

Then applying (2.3) yields

$$\frac{d}{dt}E(t) = \sum_j (\dot{e}_j + (\gamma-1)\dot{v}_j)h - \left\{ \sum_j \left(\frac{\epsilon(\delta u_j)^2}{v_j e_j} \right) h + \sum_k \left(\frac{\lambda(\delta e_k)^2}{v_k e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}} \right) h \right\} + \sum_k u_k \dot{u}_k h.$$

Integrating over $[0,t]$ and using (2.10), we then get

$$E(t) - E(0) = \sum_j e_j h + \frac{1}{2}\sum_k u_k^2 h - \left(\sum_j e_j(0)h + \frac{1}{2}\sum_k u_k^2(0)h \right) - \int_0^t \left\{ \sum_j \left(\frac{\epsilon(\delta u_j)^2}{v_j e_j} \right) h + \sum_k \left(\frac{\lambda(\delta e_k)^2}{v_k e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}} \right) h \right\} ds.$$

The result (2.9) then follows from this and (2.8). ■

Next we derive pointwise upper and lower bounds for the specific volume $v = v_j(t)$ as well as a pointwise lower bound for the internal energy $e = e_j(t)$. These bounds constitute the most important part of the entire analysis: the fact that they give *time-independent* estimates insures that subsequent estimates for higher-order terms are time-independent as well, and it is this time-independence that allows for the determination of the asymptotic behavior of solutions.

Lemma 2.2. *There exists a constant $M > 0$, independent of t and h, such that*

$$M^{-1} \le v_j(t) \le M < \infty. \tag{2.13}$$

Proof: *Step 1.* We claim that, for any $t > 0$, there exist $j_1(t)$ and positive constants α, β with $\alpha < \beta$ such that

$$\alpha \le v_{j_1},\ \ e_{j_1} \le \beta.$$

To see this, observe that (2.9) implies

$$\sum_j (f(v_j) + f(e_j))hds \le M,$$

where $f(s) = s - 1 - \log s \ge 0$. Therefore, there exists $j_1 = j_1(t)$ such that

$$f(v_{j_1}), f(e_{j_1}) \le M. \tag{2.14}$$

The result now follows from the properties of function f.

Step 2. We claim that, for any $t \ge 0$,

$$v_j(t) = \frac{1 + \frac{\gamma-1}{\epsilon}\int_0^t P(s)Q_j(s)e_j(s)ds}{P(t)Q_j(t)}, \tag{2.15}$$

where

$$
(2.16)\qquad \begin{cases} P(t) = v_{j_1}(0)\exp\{\frac{1}{\epsilon}\int_0^t p_{j_1}(s)ds\}, \\ Q_j(t) = \frac{1}{v_j(0)v_{j_1}(t)}\exp\{\frac{1}{\epsilon}\sum_{(j_1,j)}(u_k(0)-u_k(t))h\}, \end{cases}
$$

and

$$
j_1 = j_1(t), \qquad \sum_{(j_1,j)} = \begin{cases} \sum_{j_1<k<j}, & j_1 < j, \\ 0, & j_1 = j, \\ -\sum_{j<k<j_1}, & j_1 > j. \end{cases}
$$

To prove this, we let $L = \log v$ and rewrite (2.2) in the form

$$
(2.17)\qquad \epsilon\,\delta\dot{L}_k = \dot{u}_k + \delta p_k.
$$

Integrating with respect to t, we obtain

$$
\epsilon\,\delta L_k(t) = \epsilon\,\delta L_k(0) + u_k(t) - u_k(0) + \int_0^t \delta p_k(s)ds,
$$

and summing over j,

$$
(2.18)\qquad \begin{aligned} &\epsilon(L_j(t) - L_{j_1}(t)) \\ &= \epsilon(L_j(0) - L_{j_1}(0)) + \sum_{(j_1,j)}(u_k(t)-u_k(0))h + \int_0^t (p_j(s) - p_{j_1}(s))ds. \end{aligned}
$$

We divide both sides of (2.18) by ϵ and take the exponential to obtain

$$
\frac{v_j(t)}{v_{j_1}(t)} = \frac{v_j(0)}{v_{j_1}(0)}\exp\left\{\frac{1}{\epsilon}\sum_{(j_1,j)}(u_k(t)-u_k(0))h\right\}\exp\left\{\frac{1}{\epsilon}\int_0^t (p_j(s)-p_{j_1}(s))ds\right\},
$$

which implies

$$
(2.19)\qquad v_j(t) = \frac{\exp\{\frac{1}{\epsilon}\int_0^t p_j(s)ds\}}{P(t)Q_j(t)}.
$$

Multiplying (2.19) by $\left\{\frac{(\gamma-1)}{\epsilon}e_j(t)\right\}$ and rearranging then give

$$
(2.20)\qquad \frac{d}{dt}\exp\left\{\frac{\gamma-1}{\epsilon}\int_0^t \frac{e_j(s)}{v_j(s)}ds\right\} = \frac{(\gamma-1)}{\epsilon}e_j(t)P(t)Q_j(t).
$$

Finally, integrating (2.20) with respect to t and taking into account that

$$
p_j = (\gamma-1)\frac{e_j}{v_j},
$$

we obtain

$$
\exp\left\{\frac{1}{\epsilon}\int_0^t p_j(s)ds\right\} = 1 + \frac{\gamma-1}{\epsilon}\int_0^t P(s)Q_j(s)e_j(s)ds,
$$

which implies (2.15).

Step 3. We claim that, for every $t > 0$, there exists $j_2 = j_2(t)$ such that

$$\sum_{k<j_2(t)} u_k(0)h + \int_0^t \sigma_{j_2(t)}(s)ds = \sum_j \left(v_j(0) \sum_{k<j} u_k(0)h \right) h$$
$$- \int_0^t \sum_j \left(\frac{u^2_{j+\frac{1}{2}} + u^2_{j-\frac{1}{2}}}{2} + (\gamma-1)e_j \right) hds + O(h^{1/2}), \tag{2.21}$$

where $\sigma_j = \left(\frac{\epsilon\delta u}{v}\right)_j - p_j$, and $O(h^{1/2})$ denotes terms bounded by $Mh^{1/2}$.

To prove this, we let

$$\Psi_j(t) = \sum_{k<j} u_k(0)h + \int_0^t \sigma_j(s)ds,$$

so that $\dot{\Psi}_j = \sigma_j$. Now,

$$\delta\Psi_k = u_k(0) + \int_0^t \left(\delta \left(\frac{\epsilon\delta u}{v} \right)_k - \delta p_k \right) ds = u_k(0) + \int_0^t \dot{u}_k(s)ds = u_k(t),$$

and therefore

$$\begin{aligned} \delta^2\Psi_j &= \delta u_j = \frac{v_j}{\epsilon}\sigma_j + \frac{v_j}{\epsilon}p_j = \frac{v_j}{\epsilon}\dot{\Psi}_j + \frac{\gamma-1}{\epsilon}e_j \\ &= \frac{1}{\epsilon}(v_j\Psi_j)_t - \frac{1}{\epsilon}\Psi_j\delta u_j + \frac{\gamma-1}{\epsilon}e_j, \end{aligned}$$

that is,

$$(v_j\Psi_j)_t = \epsilon\delta^2\Psi_j + \Psi_j\delta u_j - (\gamma-1)e_j.$$

Letting

$$\Psi_k = \frac{\Psi_{k+\frac{1}{2}} + \Psi_{k-\frac{1}{2}}}{2},$$

we obtain

$$\begin{aligned} \Psi_j\delta u_j &= \Psi_j \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{h} \\ &= \frac{u_{j+\frac{1}{2}}\Psi_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\Psi_{j-\frac{1}{2}}}{h} + u_{j+\frac{1}{2}}\frac{\Psi_j - \Psi_{j+1}}{2h} + u_{j-\frac{1}{2}}\frac{\Psi_{j-1} - \Psi_j}{2h} \\ &= \delta(u\Psi)_j - \frac{u_{j+\frac{1}{2}}\delta\Psi_{j+\frac{1}{2}}}{2} - \frac{u_{j-\frac{1}{2}}\delta\Psi_{j-\frac{1}{2}}}{2}, \end{aligned}$$

which implies

$$(v_j\Psi_j)_t = \epsilon\delta^2\Psi_j - \left(\frac{u^2_{j+\frac{1}{2}} + u^2_{j-\frac{1}{2}}}{2} + (\gamma-1)e_j \right).$$

Applying the boundary conditions

$$\delta\Psi_0 = u_0 = 0, \qquad \delta\Psi_K = u_K = 0,$$

we then obtain

$$\sum_{j=\frac{1}{2}}^{K-\frac{1}{2}}(v_j\Psi_j)(t)h = \sum_{j=\frac{1}{2}}^{K-\frac{1}{2}}(v_j\Psi_j)(0)h - \int_0^t \sum_j \left(\frac{u^2_{j+\frac{1}{2}} + u^2_{j-\frac{1}{2}}}{2} + (\gamma-1)e_j \right) hds.$$

Let

$$\Psi(t) = \sum_j (v_j\Psi_j)(t)h.$$

Then, since $\sum_j v_j(t)h = 1$, there exist points $j^{'} = j^{'}(t), j^{''} = j^{''}(t)$ such that

$$\Psi_{j'}(t) \le \Psi(t) \le \Psi_{j''}(t), \qquad \text{for fixed } t.$$

In fact, there exists $j_2 = j_2(t)$ such that, either

$$\Psi_{j_2}(t) \le \Psi(t) \le \Psi_{j_2+1}(t),$$

or

$$\Psi_{j_2+1}(t) \le \Psi(t) \le \Psi_{j_2}(t).$$

On the other hand, for any $j = j(t)$,

$$|\Psi_{j+1} - \Psi_j| = h|\delta\Psi_{j+\frac{1}{2}}| \le h^{1/2}\left(\sum_k u_k^2 h\right)^{1/2} \le Mh^{1/2},$$

and so

$$|\Psi_{j_2(t)} - \Psi(t)| \le Mh^{1/2}.$$

We conclude that, for each $t > 0$, there exists $j_2(t)$ such that

$$\begin{aligned} \Psi_{j_2}(t) \;\; &= \;\; \sum_j \left(v_j(0) \sum_{k<j} u_k(0)h \right) h \\ &\quad - \int_0^t \sum_j \left(\frac{u^2_{j+\frac{1}{2}} + u^2_{j-\frac{1}{2}}}{2} + (\gamma-1)e_j \right) hds + O(h^{1/2}). \end{aligned} \tag{2.22}$$

This proves (2.21).

Step 4. We now give a second representation for v. Specifically, we shall show that, for any $t \ge 0$,

$$v_j(t) = (1+O(h^{1/2}))D_j(t)\mathrm{exp}\left\{ -\frac{1}{\epsilon}\int_0^t \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + (\gamma-1)e_l \right) hds \right\} \times$$

$$\left\{ 1 + \frac{1+O(h^{1/2})}{\epsilon}\int_0^t \frac{(\gamma-1)e_j(s)}{D_j(s)} \mathrm{exp}\left\{ \frac{1}{\epsilon}\int_0^s \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + (\gamma-1)e_l \right) hd\tau \right\} ds \right\},$$

where

$$D_j(t) = v_j(0)\mathrm{exp}\left\{ \frac{1}{\epsilon}\left(\sum_{(j_2(t),j)} u_k(t)h - \sum_{k<j} u_k(0)h + \sum_l v_l(0)\left(\sum_{k<l} u_k(0)h \right) h \right) \right\},$$

and $j_2(t)$ is as in (2.21).

To prove this, we first write (2.2) in the form

$$\dot{u}_k(t) + \delta p_k(t) = \epsilon(\delta \dot{L})_k(t),$$

where again $L = \log v$. Summing and integrating yield

$$\begin{aligned}\sum_{(j_2,j)}(u_k(t) - u_k(0))h \ &+ \int_0^t (p_j(s) - p_{j_2}(s))ds \\ &= \epsilon(L_j(t) - L_j(0) - L_{j_2}(t) + L_{j_2}(0)),\end{aligned}$$

where $j_2 = j_2(t)$. Rearranging and applying the relation

$$\frac{d}{ds}L_{j_2(t)}(s) = \left(\frac{\delta u}{v}\right)_{j_2(t)}(s),$$

we then obtain

$$\begin{aligned}&\epsilon L_j(t) - \int_0^t p_j(s)ds \\ &= \int_0^t \left\{\left(\frac{\epsilon\delta u}{v}\right)_{j_2(t)}(s) - p_{j_2(t)}(s)\right\} ds + \sum_{(j_2(t),j)}(u_k(t) - u_k(0))h + \epsilon L_j(0) \\ &= \int_0^t \sigma_{j_2(t)}(s)ds + \sum_{(j_2,j)}(u_k(t) - u_k(0))h + \epsilon L_j(0),\end{aligned}$$

which by (2.21) implies

$$\begin{aligned}&\epsilon L_j(t) - \int_0^t p_j(s)ds \\ &= \sum_{l=\frac{1}{2}}^{K-\frac{1}{2}} v_l(0)\left(\sum_{k<l} u_k(0)h\right) h - \int_0^t \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + (\gamma - 1)e_l\right) hds \\ &\quad + \sum_{(j_2(t),j)} u_k(t)h - \sum_{k<j} u_k(0)h + \epsilon L_j(0) + O(h^{1/2}).\end{aligned}$$

Dividing by ϵ and taking the exponential, we get

$$\begin{aligned}&v_j(t)\exp\left\{-\frac{1}{\epsilon}\int_0^t p_j(s)ds\right\} \\ &= \exp\left\{-\frac{1}{\epsilon}\int_0^t \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + (\gamma - 1)e_l\right) hds\right\} D_j(t)(1 + O(h^{1/2})),\end{aligned}$$

so that

$$\frac{1}{v_j(t)}\exp\left\{\frac{1}{\epsilon}\int_0^t p_j(s)ds\right\}$$

$$(2.23)\qquad = \frac{\exp\left\{\frac{1}{\epsilon}\int_0^t\sum_l\left(\frac{u^2_{l+\frac12}+u^2_{l-\frac12}}{2}+(\gamma-1)e_l\right)hds\right\}}{D_j(t)}(1+O(h^{1/2})).$$

Multiplying (2.23) by $\left\{\frac{\gamma-1}{\epsilon}e_j(t)\right\}$ and rearranging, we obtain

$$\exp\left\{\tfrac{1}{\epsilon}\textstyle\int_0^t p_j(s)ds\right\} = 1+\ \frac{\gamma-1}{\epsilon}\left(1+O(h^{1/2})\right)\int_0^t\frac{e_j(s)}{D_j(s)}$$

$$\times\exp\left\{\frac{1}{\epsilon}\int_0^s\sum_l\left(\frac{u^2_{l+\frac12}+u^2_{l-\frac12}}{2}+(\gamma-1)e_l\right)hd\tau\right\}ds.$$

Substituting this back into (2.23), we then obtain the second representation (2.23) for $v_j(\cdot)$.

Step 5. We complete the proof of the lemma by obtaining the required pointwise bounds for $v_j(\cdot)$. Thus define

$$m_v(t)=\min_{x\in[0,1]}v(x,t),\quad m_e(t)=\min_{x\in[0,1]}e(x,t),$$
$$M_v(t)=\max_{x\in[0,1]}v(x,t),\quad M_e(t)=\max_{x\in[0,1]}e(x,t).$$

Now, by (2.9), there exists $M_0>0$, independent of t and h, such that

$$M_0^{-1}\le D_j,Q_j\le M_0.$$

Observe that $f(s)=s-1-\log s$ is a convex function of s. Thus, by (2.9),

$$f\left(\sum_j e_jh\right)\le\sum_j f(e_j)h\le M,$$

which implies

$$(2.24)\qquad \alpha\le\sum_j e_j(t)h\le\beta.$$

This inequality, together with (2.9), gives that

$$0\le a_1\le\frac{1}{\epsilon}\sum_j\left((\gamma-1)e_j+\frac{u^2_{j+\frac12}-u^2_{j-\frac12}}{2}\right)h\le a_2,$$

where

$$a_1=\frac{(\gamma-1)\alpha}{\epsilon},\qquad a_2=\frac{1}{\epsilon}((\gamma-1)\beta+2E(0)).$$

The second representation (2.23) for $v_j(\cdot)$ implies

$$v_j(t) \leq M \exp\{-a_1 t\} \left(1 + \int_0^t M_e(s) \exp\{a_1 s\} ds\right),$$

that is,

$$M_v(t) \leq M \exp\{-a_1 t\} \left(1 + \int_0^t M_e(s) \exp\{a_1 s\} ds\right). \tag{2.25}$$

At this point we need to obtain bounds for $M_e(\cdot)$ and $m_e(\cdot)$. Note that

$$e_j^{1/2} = e_{j_2}^{1/2} + \sum_{(j_2,j)} \left(e_{k+\frac{1}{2}}^{1/2} - e_{k-\frac{1}{2}}^{1/2}\right) = e_{j_2}^{1/2} + \sum_{(j_2,j)} \left(\frac{\delta e_k}{e_{k+\frac{1}{2}}^{1/2} + e_{k-\frac{1}{2}}^{1/2}}\right) h.$$

Using the fact that $a + b \geq 2\sqrt{ab}$, we obtain

$$\begin{aligned}
\frac{|\delta e_k|}{e_{k+\frac{1}{2}}^{1/2} + e_{k-\frac{1}{2}}^{1/2}} &\leq \frac{|\delta e_k|}{2(e_{k+\frac{1}{2}} e_{k-\frac{1}{2}})^{1/2}} \left(e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}\right)^{1/4} \\
&\leq \frac{(\delta e_k)^2}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}} + M \left(e_{k+\frac{1}{2}} + e_{k-\frac{1}{2}}\right),
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{(j_2,j)} \left(\frac{|\delta e_k|}{e_{k+\frac{1}{2}}^{1/2} + e_{k-\frac{1}{2}}^{1/2}}\right) h \\
&\leq M_v(t) \sum_k \frac{(\delta e_k)^2}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}} v_k} h + M \left(\sum_k e_{k+\frac{1}{2}} h + \sum_k e_{k-\frac{1}{2}} h\right) \\
&\leq M + M_v(t) \sum_k \frac{(\delta e_k)^2}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}} v_k} h.
\end{aligned}$$

Thus

$$M_e(t) \leq M(1 + M_v(t) B(t)), \tag{2.26}$$

$$m_e(t) \geq M^{-1} \left(1 - M_v(t) B(t)\right), \tag{2.27}$$

where

$$B(t) = \sum_k \left(\frac{(\delta e_k)^2}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}} v_k}\right)(t) h.$$

Substituting (2.26) into (2.25), we then obtain

$$M_v(t) \leq M \exp\{-a_1 t\} \left\{1 + \int_0^t \exp\{a_1 s\} \left(1 + M_v(s) B(s)\right) ds\right\}.$$

Now define

$$A(t) = M_v(t) \exp\{a_1 t\}.$$

Then

$$\begin{aligned} A(t) &\leq M\left\{1+\int_0^t (\exp\{a_1 s\}+A(s)B(s))ds\right\} \\ &\leq M\left\{\exp\{a_1 t\}+\int_0^t A(s)B(s)ds\right\}. \end{aligned}$$

Therefore,

$$A(t) \leq M\exp\{a_1 t\} + M\int_0^t B(s)\exp\left\{M\int_s^t B(\tau)d\tau + a_1 s\right\} ds.$$

Applying the fact that

$$\int_0^t B(\tau)d\tau \leq M,$$

we obtain

$$A(t) \leq M(1+\exp\{a_1 t\}),$$

which implies

$$M_v(t) = \exp\{-a_1 t\}A(t) \leq M. \tag{2.28}$$

Next, (2.27)-(2.28), together with the second representation (2.23) for $v_j(\cdot)$, imply that

$$\begin{aligned} m_v(t) &\geq M^{-1}\left(\exp\{-a_2 t\}+\int_0^t m_e(s)\exp\{-a_2(t-s)\}ds\right) \\ &\geq M^{-1}\left(\exp\{-a_2 t\}+\int_0^t \exp\{-a_2(t-s)\}\{1-MB(s)\}ds\right) \\ &\geq M^{-1}\left(1-\exp\{-a_2 t\}-M\int_0^t \exp\{-a_2(t-s)\}B(s)ds\right). \end{aligned}$$

Notice that

$$\begin{aligned} &\lim_{t\to\infty}\int_0^t \exp\{-a_2(t-s)\}B(s)ds \\ &= \lim_{t\to\infty}\left(\int_0^{\frac{t}{2}} \exp\{-a_2(t-s)\}B(s)ds + \int_{\frac{t}{2}}^t \exp\{-a_2(t-s)\}B(s)ds\right) \\ &\leq \lim_{t\to\infty}\left(\exp\{-\frac{a_2}{2}t\}\int_0^\infty B(s)ds + \int_{\frac{t}{2}}^t B(s)ds\right) = 0. \end{aligned}$$

Therefore, there exists $t_0 > 0$ such that

$$m_v(t) \geq \frac{M^{-1}}{2}, \qquad t \geq t_0 > 0. \tag{2.29}$$

On the other hand,

$$
\begin{aligned}
P(t) &= \sum_j P(t)v_j(t)h = \sum_j \left(\frac{1 + \frac{\gamma-1}{\epsilon}\int_0^t e_j(s)P(s)Q_j(s)ds}{Q_j(t)} \right) h \\
&\leq M\left(1 + \int_0^t P(s)(\sum_j e_j(s)h)ds\right) \leq M\left(1 + \int_0^t P(s)ds\right),
\end{aligned}
$$

which gives

$$
P(t) \leq M\exp\{Mt\}.
$$

Therefore, from (2.15),

$$
v_j(t) \geq \frac{M^{-1}}{P(t)} \geq M^{-1}\exp\{-Mt\} \geq M^{-1}\exp\{-Mt_0\}, \quad \text{when} \quad 0 \leq t \leq t_0.
$$

Combining this last relation with (2.29), we then obtain the required bounds for v_j. This completes the proof of Lemma 2.2.

■

In the following lemma, we derive a lower bound for the internal energy $e_j(\cdot)$.

Lemma 2.3. *There is a constant M, independent of t and h, such that*

$$
e_j(t) \geq \frac{1}{M(t+1)}. \tag{2.30}
$$

Proof: Setting $\omega_j = \frac{1}{e_j}$ and multiplying (2.3) by $\{-e_j^{-2}\}$, we obtain

$$
\dot{\omega}_j - \frac{p_j}{e_j^2}\delta u_j = -\frac{\epsilon\omega_j^2(\delta u_j)^2}{v_j} - \frac{\lambda}{e_j^2}\delta\left(\frac{\delta e}{v}\right)_j,
$$

which implies

$$
\begin{aligned}
\dot{\omega}_j &\leq \frac{(\gamma-1)\omega_j\delta u_j}{v_j} - \epsilon\frac{\omega_j^2(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\omega}{v}\right)_j \\
&\leq \left(\epsilon\frac{\omega_j^2(\delta u_j)^2}{v_j} + \frac{1}{\epsilon}\frac{(\gamma-1)^2}{v_j}\right) - \epsilon\frac{\omega_j^2(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\omega}{v}\right)_j.
\end{aligned}
$$

Therefore,

$$
\dot{\omega}_j \leq \frac{M}{v_j} + \lambda\delta\left(\frac{\delta\omega}{v}\right)_j,
$$

so that

$$
\frac{d\omega_j^{2q}}{dt} \leq \frac{qM\omega_j^{2q-1}}{v_j} + 2q\lambda\omega_j^{2q-1}\delta\left(\frac{\delta\omega}{v}\right)_j.
$$

Summing by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\sum_j \omega_j^{2q} h\right) &\leq qM \sum_j \frac{\omega_j^{2q-1}}{v_j} h - 2q\lambda \sum_k \left((2q-1)\omega_k^{2q-2} \frac{\delta\omega_k^2}{v_k}\right) h \\ &\leq qM \sum_j \omega_j^{2q-1} \frac{1}{v_j} h \leq qM \left\{\sum_j \omega_j^{2q}\right\}^{\frac{2q-1}{2q}} \cdot \left\{\sum_j v_j^{-2q} h\right\}^{\frac{1}{2q}}.\end{aligned}$$

Define

$$D(t) = \left(\sum_j \omega_j^{2q}(t) h\right)^{\frac{1}{2q}}.$$

Then we have

$$D^{2q-1}(t) D'(t) \leq \frac{M}{2} D^{2q-1}(t) \left(\sum_j v_j^{-2q}(t) h\right)^{\frac{1}{2q}},$$

which implies

$$D'(t) \leq \frac{M}{2} \left(\sum_j v_j^{-2q}(t) h\right)^{\frac{1}{2q}} \leq \frac{M}{2m_v(t)} \leq M.$$

Therefore

$$D(t) \leq D_0 + Mt.$$

Now let $q \to \infty$ to obtain

$$\|\omega(t)\|_{L^\infty} \leq \|\omega(0)\|_{L^\infty} + Mt,$$

which implies

$$\omega(t) \leq M(1+t).$$

This completes the proof. ∎

Remark: Lemmas 2.1-2.3, together with the bounds

$$e_i(t) \leq \left(\sum_j e_j(t) h\right) h^{-1} \leq Mh^{-1}, \quad u_i^2(t) \leq \left(\sum_j u_j^2(t) h\right) h^{-1} \leq Mh^{-1},$$

show that the initial-value problem (2.1)-(2.5) is solvable for all $t > 0$ with fixed $h > 0$.

In the following lemma we derive three auxiliary estimates relating to the discrete piecewise-H^1 norm of v and to the evolution in time of the jumps in v. In particular, we shall show that the magnitude of the jump discontinuity

$$[v_{k_i}] = v_{k_i+\frac{1}{2}} - v_{k_i-\frac{1}{2}}$$

at time t is bounded by the magnitude of the corresponding jump at time $t = 0$.

Lemma 2.4. *The following estimates hold for all $t > 0$:*
(a). *For $\hat{u}_j$ defined by $u^3_{j+\frac{1}{2}} - u^3_{j-\frac{1}{2}} = 3\hat{u}^2_j(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}})$,*

$$\sum_j \left((e_j - 1)^2 + u^4_{j+\frac{1}{2}} \right) h + \int_0^t \left(\sum_k (\delta e_k)^2 + \sum_j \left(\hat{u}^2_j e^2_j + \hat{u}^2_j (\delta u_j)^2 \right) \right) hds \leq M.$$

(b). *For the distinguished discontinuity points*

$$0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1,$$

$$|[v_{k_i}(t)]| \leq M \left(\exp\{-M^{-1} t^{1/2}\} |[v_{k_i}(0)]| + h^{1/2} \right). \tag{2.31}$$

(c). *Away from the distinguished discontinuities,*

$$\sup_t \sum_{k \neq k_i} (\delta v_k)^2 h + \int_0^t \left(\sum_{k \neq k_i} (1 + e_k)(\delta v_k)^2 + \sum_j (\delta u_j)^2 \right) hds \leq M(1 + N h^{1/2} t).$$

Proof: The first result follows from routine energy estimates together with the L^2 and pointwise estimates of Lemmas 2.1–2.3; and the third is derived exactly as in [10], pp. 29–30. We shall therefore give the details only for the second estimate, (2.31).

Suppressing the subscript k_i, we have

$$[\epsilon L]_t = \left[\frac{\epsilon \dot{v}}{v} \right] = \left[\frac{\epsilon \delta u}{v} \right] = [p] + h\dot{u}. \tag{2.32}$$

Taking α_i is as in (1.14), we can solve this first-order differential equation to obtain

$$\begin{aligned} |[v_{k_i}(t)]| \leq\ & M \left\{ \exp \left\{ \frac{1}{\epsilon} \int_0^t \alpha_i e_{k_i} ds \right\} |[v_{k_i}(0)]| \right. \\ & \left. + h^{1/2} \left(\int_0^t \exp \left\{ \frac{2}{\epsilon} \int_s^t \alpha_i e_{k_i} d\tau \right\} ds \right)^{1/2} + h^{3/4} \right\}. \end{aligned} \tag{2.33}$$

We need to derive a lower bound for the term $\int_0^t e_k ds$ appearing here. Observe that

$$t = \int_0^t ds \leq \left(\int_0^t e_k ds \right)^{1/2} \left(\int_0^t {e_k}^{-1} ds \right)^{1/2},$$

so that

$$\int_0^t e_k ds \geq \frac{t^2}{\int_0^t {e_k}^{-1} ds}.$$

However,

$$
\begin{aligned}
\int_0^t \frac{1}{e_k} ds &\le \int_0^t \left(\frac{1}{\sum_i e_i h} + \sum_k \left| \delta \left(\frac{1}{e} \right)_k \right| h \right) ds \\
&\le Mt + \int_0^t \sum_k \frac{|\delta e_k|}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}} h ds. \qquad (2.34)
\end{aligned}
$$

Also, from Lemma 2.3, we have

$$
\frac{1}{e_{k+\frac{1}{2}} e_{k-\frac{1}{2}}} = \frac{1}{2e_k} \left(\frac{1}{e_{k-\frac{1}{2}}} + \frac{1}{e_{k+\frac{1}{2}}} \right) \le M \frac{(1+t)}{e_k}. \qquad (2.35)
$$

Combining (2.34) with (2.35), we then obtain

$$
\begin{aligned}
\int_0^t \frac{1}{e_k} ds &\le Mt + M \int_0^t \sum_k \frac{|\delta e_k|}{e_k} (1+s) h ds \\
&\le Mt + M \left(\int_0^t \sum_k \frac{(\delta e_k)^2}{e_k^2} h ds \right)^{1/2} \left(\int_0^t (1+s)^2 ds \right)^{1/2} \\
&\le Mt + M(1+t)^{3/2} \le M(1+t)^{3/2}.
\end{aligned}
$$

Hence, for $t \ge 0$,

$$
\int_0^t e_k ds \ge \frac{t^2}{M(1+t)^{3/2}} \ge M^{-1} t^{1/2},
$$

which implies

$$
\exp \left\{ \int_0^t \alpha_i e_k ds \right\} \le M \exp \left\{ -M^{-1} t^{1/2} \right\}.
$$

This completes the proof. ∎

In the following lemma we derive certain discrete higher-order estimates for u and e. These will be crucial both for the passage to the limit as h goes to zero, as well as for the determination of the large-time behavior.

Lemma 2.5. *Define* $\sigma(t)=\min\{t,1\}$. *The solutions* $(v_j(t),u_k(t),e_j(t))$ *of* (2.1)-(2.5) *then satisfy the following estimates:*

$$(\mathbf{a}).\ \sup_t\left(\sigma(t)\sum_j(\delta u_j)^2(t)h\right)+\int_0^t\sigma(s)\sum_k\dot u_k^2(s)h\ ds\le M(1+Nh^{1/2}t),$$

$$(\mathbf{b}).\ \sup_t\left(\sigma^2(t)\sum_k(\delta e_k)^2(t)h\right)+\int_0^t\sigma^2(s)\sum_j\dot e_j^2(s)h\ ds\le M(1+Nh^{1/2}t),$$

$$(\mathbf{c}).\ \sup_t\left(\sigma^2(t)\left(\sum_k\dot u_k^2(t)h+\sum_j(\delta u_j)^4(t)h\right)\right)$$
$$+\int_0^t\sigma^2(s)\sum_j(\delta\dot u_j)^2(s)h\ ds\le M(1+Nh^{1/2}t),$$

$$(\mathbf{d}).\ \sup_t\left(\sigma^3(t)\sum_j\dot e_j^2(t)h\right)+\int_0^t\sigma^3(s)\sum_j\left(\frac{(\delta\dot e_j)^2}{v_j}\right)(s)h\ ds\le M(1+Nh^{1/2}t).$$

Proof: The estimates $(\mathbf{a})-(\mathbf{c})$ follow from fairly tedious, but routine energy estimates, and are similar to those given in [10], pp. 30–34 (except that the estimates given here are time-independent, owing to the fact that the pointwise bounds of Lemmas 2.2–2.3 are time-independent). We therefore omit their proofs and give only the details of the proof of $(\mathbf{d})$.

We differentiate (2.3) with respect to t, multiply the result by $\{\sigma^3\dot e_j\}$, and sum and integrate to obtain

$$\int_0^t\sigma^3\sum_j\left(\dot e_j\ddot e_j\right)h\ ds+\int_0^t\sigma^3\sum_j\left(\dot e_j\dot p_j(\delta u_j)\right)h\ ds+\int_0^t\sigma^3\sum_j\left(\dot e_jp_j(\delta\dot u_j)\right)hds$$
$$=2\epsilon\int_0^t\sigma^3\sum_j\left(\dot e_j\frac{(\delta u_j)(\delta\dot u_j)}{v_j}\right)hds-\epsilon\int_0^t\sigma^3\sum_j\left(\dot e_j\frac{(\delta u_j)^3}{v_j^2}\right)h\ ds$$
$$+\lambda\int_0^t\sigma^2\sum_j\left(\dot e_j\delta\left(\frac{\delta\dot e}{v}\right)_j\right)h\ ds+3\int_0^t\sigma^2\dot\sigma\sum_j\dot e_j^2hds.$$

Applying the boundary conditions (1.2), we then obtain

$$\sup_t\left(\sigma^3\sum_j\dot e_j^2h\right)+\int_0^t\sigma^3\sum_j\left(\frac{(\delta\dot e_j)^2}{v_j}\right)h\ ds$$
$$\le M\int_0^t\sigma^3\sum_j\left(\dot e_j^2(\delta u_j)+e_j\dot e_j(\delta u_j)^2+e_j\dot e_j(\delta\dot u_j)+(\delta u_j)(\delta\dot u_j)\dot e_j+\dot e_j(\delta u_j)^3\right)h\ ds$$
$$+M\int_0^t\sigma^3\sum_j\left(\frac{(\delta\dot e_j)(\delta e_j)(\delta u_j)}{v_j^2}\right)h\ ds.$$

It follows that

$$\sup_t\left(\sigma^3\sum_j \dot e_j^2 h\right)+\int_0^t\sigma^3\sum_j\left(\frac{(\delta\dot e_j)^2}{v_j}\right)h\,ds$$

$$\le M\|\sigma(\delta u_j)\|_{L^\infty}\int_0^t\sigma^2\sum_j\left(\dot e_j^2+\dot e_j(\delta u_j)^2+e_j\dot e_j(\delta u_j)+\frac{(\delta\dot e_j)}{v_j}(\delta e_j)+(\delta\dot e_j)\dot e_j\right)h\,ds$$

$$+M\int_0^t\sigma^3\sum_j\left(e_j\dot e_j(\delta\dot u_j)\right)hds.$$

Next, from the momentum equation (2.2),

$$\begin{aligned}|\delta u_j|^2 &\le M\left\{\|\frac{\epsilon\delta u_j}{v_j}-p_j\|_\infty+\|p_j\|_\infty\right\}\\
&\le M\left\{\sum_k\left(\delta\left(\frac{\epsilon\delta u}{v}-p\right)_k\right)^2h+\sum_j\left((\delta u_j)^2+p_j^2\right)h+\|e\|_\infty^2\right\}\\
&\le M\left\{1+\sum_k(\delta e_k)^2h+\sum_k\dot u_k^2h+\sum_j(\delta u_j)^2h\right\}.\end{aligned}$$

Therefore, by Lemma 2.5 **(a)** – **(c)**,

$$\begin{aligned}\|\sigma\delta u_j\|_{L^\infty} &\le M\left\{\sigma^2+\sigma^2\sum_k(\delta e_k)^2h+\sigma^2\sum_k\dot u_k^2h+\sigma^2\sum_j(\delta u_j)^2h\right\}^{1/2}\\
&\le M(1+Nh^{1/2}t)^{1/2}.\end{aligned}$$

The result **(d)** now follows.

∎

3. Proof of Theorems 1.1 and 1.2

The existence and regularity statements in Theorems 1.1 and 1.2 can now be derived by the technique of Hoff [10]–[11], with the aid of Lemmas 2.1-2.5. Briefly, we begin with initial data as in Theorem 1.2, that is, with v_0 piecewise H^1. Difference approximations $(v_j(t),u_k(t),e_j(t))$ are constructed as in Section 2, and these mesh functions are used to construct approximate solutions $(v^h(x,t),u^h(x,t),e^h(x,t))$ by a suitable interpolation procedure. The estimates of Lemmas 2.1–2.5, which are uniform in h, then apply to show that these approximate solutions are appropriately compact, that their limits are indeed weak solutions, and that these weak solutions inherit all the properties (1.7)–(1.15) asserted in Theorems 1.1 and 1.2. The uniform total variation estimate (1.8) for v, obtained for the case that v_0 is piecewise H^1, can then be applied to complete the solution operator to the more general data of Theorem 1.1, that is, data for which v_0 is of bounded variation. This entire construction requires a fairly lengthy, but straightforward analysis. The details for the present case are nearly identical to those of [10]–[11], the important differences being that all of the estimates given here are independent of time and are valid for large initial data. These details are therefore omitted.

The large-time behavior result of Theorem 1.1 can now be derived as an a posteriori consequence of the weak form of the equations (1.1) and the estimates (1.7)–(1.10) of Theorem 1.1. A similar result is given in Hoff [12] for the multidimensional case under the assumption that the initial data are *small*; this restriction is required only for the derivation of time-independent bounds, however. Once these bounds have been established, as in Theorem 1.1, the derivation of the large-time behavior is the same. We therefore give here just a brief sketch.

The energy estimates (1.10) show that the function $a(t) = \|u_x(\cdot,t)\|_{L^2}^2$ is integrable and of bounded variation on $[1,\infty)$. It follows that

$$\lim_{t\to\infty} a(t) = 0,$$

that is, that $u(\cdot,t) \to 0$ in H_0^1. A similar argument shows that

$$\|e_x(\cdot,t)\|_{L^2} \to 0,$$

and routine bounds based on these results and on (1.10) then show that

$$\|(e+u^2/2)_x(\cdot,t)\|_{L^2} \to 0.$$

This fact, together with the conservation law

$$\int_0^1 (e+u^2/2)(x,t)\,dx = e_\infty,$$

and the convergence of $u(\cdot,t)$, then shows that $e(\cdot,t) \to e_\infty$ in H^1 as $t\to\infty$.

The argument for v is somewhat more involved and depends upon the same dissipative structure responsible for the damping of singularities. We will give a heuristic sketch of the proof; a somewhat delicate smoothing and renormalization analysis is required to make the argument precise (see [12] for details). We define $v_\infty = \displaystyle\int_0^1 v_0(x)dx$, $e_\infty = \displaystyle\int_0^1 (e_0(x) + \frac{u_0^2(x)}{2})dx$, $p_\infty = p(v_\infty, e_\infty)$, and the so-called effective viscous flux

$$\begin{aligned} F &= \frac{\epsilon u_x}{v} - p(v,e) + p_\infty \\ &= \frac{\epsilon u_x}{v} - (\gamma-1)\left(\frac{e-e_\infty}{v} + e_\infty\left(\frac{1}{v} - \frac{1}{v_\infty}\right)\right). \end{aligned}$$

Observe from (1.1) that $F_x = u_t$. Now, by applying the energy estimates (1.10), especially the bound $\displaystyle\int_1^\infty\int_0^1 u_{xt}^2\,dxdt < \infty$, we can show that

$$\|u_t(\cdot,t)\|_{L^2} \to 0, \quad \text{as } t\to\infty.$$

It follows that

$$\|F(\cdot,t) - \bar{F}(t)\|_{L^\infty} \to 0, \quad t\to\infty,$$

where $\bar{F}$ is the space average of F. This in turn, together with the previous results that $u_x, e - e_\infty \to 0$ in L^2, substituted into the definition of F, then shows that

$$\|\frac{1}{v(\cdot,t)} - \int_0^1 \frac{dx}{v(x,t)}\|_{L^2} \to 0,$$

which implies

$$\|v(\cdot,t) - (\int_0^1 \frac{dx}{v(x,t)})^{-1}\|_{L^2} \to 0.$$

Notice that

$$\begin{aligned} v_\infty &= \lim_{t\to\infty} \left(\int_0^1 \left(v(x,t) - (\int_0^1 \frac{dx}{v(x,t)})^{-1} \right) dx + (\int_0^1 \frac{dx}{v(x,t)})^{-1} \right) \\ &= \left(\lim_{t\to\infty} \int_0^1 \frac{dx}{v(x,t)} \right)^{-1}, \end{aligned}$$

so that $v(\cdot,t) \to v_\infty$ in L^2 as $t \to \infty$. Then, by taking space averages in the definition of F, we find that $\bar{F}(t) \to 0$, and therefore that

$$\|F(\cdot,t)\|_{L^\infty} \to 0.$$

Finally, to show that v converges in L^∞, we define $w(x,t) = \epsilon(\log v(x,t) - \log v_\infty)$ and compute formally from (1.1),

$$\begin{aligned} w_t &= \frac{\epsilon u_x}{v} = F + p - p_\infty \\ &= F + (\gamma - 1)\left(\frac{e - e_\infty}{v} + e_\infty \left(\frac{1}{v} - \frac{1}{v_\infty} \right) \right). \end{aligned}$$

Define

$$\beta(x,t) = -\frac{(\gamma-1)e_\infty \,(1/v(x,t) - 1/v_\infty)}{\epsilon(\log v(x,t) - \log v_\infty)},$$

so that

$$w_t + \beta w = F + \frac{\gamma - 1}{v}(e - e_\infty).$$

Evidently $C^{-1} \le \beta(x,t) \le C$, so that this ordinary differential equation is strictly dissipative. Thus

$$\begin{aligned} |w(x,t)| &\le \left| e^{-\int_1^t \beta(x,s)ds} w(x,1) \right| \\ &\quad + \left| \left(\int_1^{t/2} + \int_{t/2}^t \right) e^{-\int_s^t \beta(x,\tau)d\tau} \left(F(x,s) + \frac{\gamma-1}{v(x,s)}(e(x,s) - e_\infty) \right) ds \right| \\ &\le Ce^{-t/C} + Cte^{-t/C} + C \sup_{s \ge t/2} (\|F(\cdot,s)\|_{L^\infty} + \|e(\cdot,s) - e_\infty\|_{L^\infty}) \\ &\quad \to 0, \quad \text{as } t \to \infty. \end{aligned}$$

This completes the proof of Theorems 1.1–1.2.

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