PDE-CDT Core Course

Analysis of Partial Differential Equations - Part III

Lecture 3

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Hyperbolic Systems of Conservation Laws

\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \cdots, u_m) = 0, \\
\cdots \\
\frac{\partial}{\partial t} u_m + \frac{\partial}{\partial x} f_m(u_1, \cdots, u_m) = 0,
\end{array} \right.
\end{align*}

\[ u_t + f(u)_x = 0 \]

\[ u = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m \quad \text{conserved quantities} \]

\[ f(u) = (f_1(u), \cdots, f_m(u))^\top \quad \text{fluxes} \]
Euler equations of gas dynamics (1755)

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0 \quad \text{(conservation of mass)} \\
(\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(conservation of momentum)} \\
(\rho E)_t + (\rho Ev + pv)_x &= 0 \quad \text{(conservation of energy)}
\end{align*}
\]

\[\rho = \text{mass density} \quad v = \text{velocity}\]

\[E = e + \frac{v^2}{2} = \text{energy density per unit mass (internal + kinetic)}\]

\[p = p(\rho, e) \quad \text{constitutive relation}\]
Hyperbolic Systems

\[ u_t + f(u)_x = 0 \quad \quad \text{for} \quad u = u(t, x) \in \mathbb{R}^m \]

\[ u_t + A(u)u_x = 0 \quad \quad A(u) = \nabla f(u) \]

The system is **strictly hyperbolic** if each \( m \times m \) matrix \( A(u) \) has real distinct eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u) \]

Right eigenvectors \( r_1(u), \cdots, r_m(u) \) \quad (column vectors)

Left eigenvectors \( l_1(u), \cdots, l_m(u) \) \quad (row vectors)

\[ Ar_i = \lambda_i r_i \quad \quad l_i A = \lambda_i l_i \]

Choose the bases so that

\[ l_i \cdot r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
Invariance of Hyperbolicity under Change of Coordinates

**Theorem**

- Let \( u \) be a smooth solution of the strictly hyperbolic system

\[
 u_t + A(u)u_x = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}
\]

- Assume \( \Phi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a smooth diffeomorphism, with inverse \( \psi \)

Then \( w := \Phi(u) \) solves the strictly hyperbolic system

\[
 w_t + B(w)w_x = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}
\]

for

\[
 B(w) := \nabla \Phi(\psi(w)) A(\psi(w)) \nabla \psi(w) \quad w \in \mathbb{R}^m
\]
Dependence of Eigenvalues and Eigenvectors on $u$

Theorem

Assume that the matrix function $A(u)$ is smooth, strictly hyperbolic. Then

- The eigenvalues $\lambda_k(u)$ depend smoothly on $u \in \mathbb{R}^m$, $k = 1, \cdots, m$

- We can select the right eigenvectors $r_k(u)$ and left eigenvector $l_k(u)$ to depend smoothly on $u \in \mathbb{R}^m$ and satisfy the normalization

$$|r_k(u)|, |l_k(u)| = 1, \quad k = 1, \cdots, m.$$

*We are not only globally and smoothly defining the eigenvalues and eigenspaces of $A(u)$, but also globally providing the eigenspaces of $A(u)$ with an orientation.*
Linear Hyperbolic Systems

\[ u_t + Au_x = 0 \quad \quad u(0, x) = \phi(x) \]

\[ \lambda_1 < \cdots < \lambda_m \text{ eigenvalues} \quad \quad r_1, \cdots, r_m \text{ eigenvectors} \]

Explicit solutions: **Linear superposition of travelling waves**

\[ u(t, x) = \sum_i \phi_i(x - \lambda_i t) r_i \]

\[ \phi_i(s) = l_i \cdot \phi(s) \]
Nonlinear Effects

\[ u_t + A(u)u_x = 0 \]

eigenvalues depend on \( u \) \implies \text{waves change shape}
eigenvectors depend on $u \implies$ nontrivial wave interactions
Loss of Regularity

\[ u_t + (u^2/2)_x = 0 \quad u_t + uu_x = 0 \]

\[ f(u) = u^2/2 \quad \text{characteristic speed: } f'(u) = u \]

Global solutions only in a space of discontinuous functions
1-D Example

\[ \begin{aligned}
    U_t + \left( \frac{U^2}{2} \right)_x &= 0 \\
    U_t \bigg|_{t=0} &= U_0(x) \\
    U_t + uu_x &= 0
\end{aligned} \]

\[ \begin{aligned}
    \frac{dx}{dt} &= u, \quad \rightarrow \quad X = X_j + U_0(x_j)t \\
    \frac{du}{dt} &= 0, \quad \rightarrow \quad U = U_0(x_j), \quad j = 1, 2
\end{aligned} \]

When \( t \rightarrow t_* = \frac{X_2 - X_1}{U_0(x_1) - U_0(x_2)} > 0 \), \( U(t, x) \) is Multi-Valued.
Smooth Solutions – Evolution of Wave Components

\[ u_t = -A(u)u_x \]

\[ \lambda_i(u) = i\text{-th eigenvalue, } l_i(u), r_i(u) = i\text{-th eigenvectors} \]

\[ u_x^i := l_i \cdot u_x = [i\text{-th component of } u_x] = [\text{density of } i\text{-waves in } u] \]

\[ u_x = \sum_{i=1}^{m} u_x^i r_i(u) \quad u_t = -\sum_{i=1}^{m} \lambda_i(u) u_x^i r_i(u) \]

Differentiate the 1st equation w.r.t. \( t \) and the 2nd w.r.t \( x \)

\[ \rightarrow \text{Evolution equation for scalar components } u_x^i: \]

\[ (u_x^i)_t + (\lambda_i u_x^i)_x = \sum_{j>k} (\lambda_j - \lambda_k) \left( l_i \cdot [r_j, r_k] \right) u_x^j u_x^k \]
Source Terms

\((\lambda_j - \lambda_k) \left( \mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k] \right) u^j_x u^k_x \)

= amount of \(i\)-waves produced by the interaction of \(j\)-waves with \(k\)-waves

\(\lambda_j - \lambda_k = \) [difference in speed]

= [rate at which \(j\)-waves and \(k\)-waves cross each other]

\(u^j_x u^k_x = \) [density of \(j\)-waves] \(\times\) [density of \(k\)-waves]

\([\mathbf{r}_j, \mathbf{r}_k] = (\nabla \mathbf{r}_k) \mathbf{r}_j - (\nabla \mathbf{r}_j) \mathbf{r}_k \) (Lie bracket)

= [directional derivative of \(\mathbf{r}_k\) in the direction of \(\mathbf{r}_j\)]

= [directional derivative of \(\mathbf{r}_j\) in the direction of \(\mathbf{r}_k\)]

\(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k] = i\)-th component of the Lie bracket \([\mathbf{r}_j, \mathbf{r}_k]\) along the basis of eigenvectors \(\{\mathbf{r}_1, \cdots, \mathbf{r}_m\}\)
Shock Solutions

\[ u_t + f(u)_x = 0 \]

\[ u(t, x) = \begin{cases} 
    u^- & \text{if } x < \lambda t \\
    u^+ & \text{if } x > \lambda t
\end{cases} \]

is a weak solution

if and only if the Rankine-Hugoniot Equations hold:

\[ \lambda [u^+ - u^-] = f(u^+) - f(u^-) \]

[Speed of the shock] × [Jump in the state] = [Jump in the flux]
Derivation of the Rankine-Hugoniot Equations

\[ 0 = \int \int \left\{ u \phi_t + f(u) \phi_x \right\} \, dx \, dt = \int \int_{\Omega^+ \cup \Omega^-} \text{div} \ (u \phi, f(u) \phi) \, dx \, dt \]
\[ = \int_{\partial \Omega^+} \mathbf{n}^+ \cdot \mathbf{v} \, ds + \int_{\partial \Omega^-} \mathbf{n}^- \cdot \mathbf{v} \, ds \]
\[ = \int \left[ \lambda (u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t) \, dt . \]

\[ \mathbf{v} \doteq (u \phi, f(u) \phi) \]
Alternative Formulation

\[
\lambda (u^+ - u^-) = f(u^+) - f(u^-)
\]

\[
= \int_0^1 \nabla f(\theta u^+ + (1 - \theta)u^-) \cdot (u^+ - u^-) d\theta
\]

\[
= A(u^+, u^-) \cdot (u^+ - u^-)
\]

\[
A(u, v) := \int_0^1 \nabla f(\theta u + (1 - \theta)v) \cdot (u - v) d\theta
\]

\[
= \text{[averaged Jacobian matrix]}
\]

The Rankine-Hugoniot conditions hold if and only if

\[
\lambda (u^+ - u^-) = A(u^+, u^-)(u^+ - u^-)
\]

- The jump \( u^+ - u^- \) is an eigenvector of the averaged matrix \( A(u^+, u^-) \)
- The speed \( \lambda \) coincides with the corresponding eigenvalue
The Rankine-Hugoniot condition for the scalar conservation law \( u_t + f(u)_x = 0 \)

\[
\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s) \, ds
\]

[speed of the shock] = [slope of secant line through \( u^- \), \( u^+ \) on the graph of \( f \)]

= [average of the characteristic speeds between \( u^- \) and \( u^+ \)]
Points of Approximate Jump

The function $u = u(t, x)$ has an approximate jump at a point $(\tau, \xi)$ if there exists states $u^- = u^+$ and a speed $\lambda$ such that, setting

$$U(t, x) := \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

there holds: $\lim_{\rho \to 0^+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \left| u(t, x) - U(t - \tau, x - \xi) \right| dx dt = 0$

Theorem

If $u$ is a weak solution to the system $u_t + f(u)_x = 0$, then the Rankine-Hugoniot equations hold at each point of approximate jump.
**Problem:** Given $u^- \in \mathbb{R}^m$, find the states $u^+ \in \mathbb{R}^m$ which, for some speed $\lambda$, satisfy the Rankine-Hugoniot equations:

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-) = A(u^-, u^+)(u^+ - u^-)$$

**Alternative Formulation:** Fix $i \in \{1, \cdots, m\}$. The jump $u^+ - u^-$ is a (right) $i$-eigenvector of the averaged matrix $A(u^-, u^+)$ if and only if it is orthogonal to all (left) eigenvectors $l_j(u^+, u^-)$ of $A(u^-, u^+)$:

$$l_j(u^-, u^+) \cdot (u^+ - u^-) = 0 \quad \text{for all } j \neq i$$

Implicit Function Theorem $\implies$ For each $i$, there exists a curve $s \to S_i(s)(u^-)$ of points that satisfy $(RH)_i$. 

![Diagram](image-url)
Non-uniqueness of Weak solutions

Example: a Cauchy problem for Burgers’ equation

\[ u_t + \left(\frac{u^2}{2}\right)_x = 0 \]

\[ u(0, x) = \begin{cases} 
1 & \text{if } \ x \geq 0 \\
0 & \text{if } \ x < 0 
\end{cases} \]

Each \( \alpha \in [0, 1] \) yields a weak solution

\[ u_\alpha(t, x) = \begin{cases} 
0 & \text{if } \ x < \alpha t/2 \\
\alpha & \text{if } \ \alpha t/2 < x < (1 + \alpha)t/2 \\
1 & \text{if } \ x \geq (1 + \alpha)t/2 
\end{cases} \]
Admissibility Conditions on Shocks

\[ u_t + f(u)_x = 0 \]

- Solutions should be stable w.r.t. small initial perturbations
- Solutions should be limits of suitable approximations and/or physical regularisations (\textit{Vanishing viscosity, relaxation, \ldots})
- Any convex entropy should not increase
Stability conditions: the scalar case

Perturb the shock with left and right states $u^-, u^+$ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\iff$

$speed \ of \ jump \ behind \ \leq \ speed \ of \ jump \ ahead$

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$
speed of a shock = slope of a secant line to the graph of $f$

Stability conditions:

- when $u^- < u^+$ the graph of $f$ should remain above the secant line
- when $u^- > u^+$, the graph of $f$ should remain below the secant line
General stability conditions

Scalar case: stability holds if and only if

\[
\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}
\]

for every intermediate state \( u^* \in [u^-, u^+] \)
Vector Valued Case: $\mathbf{u}^+ = S_i(\sigma)(\mathbf{u}^-)$ for some $\sigma \in \mathbb{R}$

Admissibility Condition (T.-P. Liu)

The speed $\lambda(\sigma)$ of the shock joining $\mathbf{u}^-$ with $\mathbf{u}^+$ must be less or equal to the speed of every smaller shock, joining $\mathbf{u}^-$ with an intermediate state $\mathbf{u}^* = S_i(s)(\mathbf{u}^-)$, $s \in [0, \sigma]$:

$$\lambda(\mathbf{u}^-, \mathbf{u}^+) \leq \lambda(\mathbf{u}^-, \mathbf{u}^*)$$

- The Liu condition singles out precisely the solutions which are limits of vanishing viscosity approximations

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad u^\varepsilon \to u \quad \text{as} \ \varepsilon \to 0$$
Admissibility Condition (P. Lax)

A shock connecting the states $u^-, u^+$, travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \geq \lambda_i(u^-, u^+) \geq \lambda_i(u^+)$$

- Geometric meaning: characteristics flow toward the shock from both sides
- The Liu condition implies the Lax condition
Mathematical Entropy – Entropy Flux

\[ u_t + f(u)_x = 0 \]

**Definition:** A function \( \eta : \mathbb{R}^m \rightarrow \mathbb{R} \) is called an **Entropy**, with **Entropy Flux** \( q : \mathbb{R}^m \rightarrow \mathbb{R} \) if

\[ \nabla \eta(u) \nabla f(u) = \nabla q(u) \]

For **smooth** solutions \( u = u(t, x) \), this implies

\[
\begin{align*}
\eta(u)_t + q(u)_x &= \nabla \eta(u) u_t + \nabla q(u) u_x \\
&= -(\nabla \eta(u) \nabla f(u)) u_x + \nabla q(u) u_x = 0
\end{align*}
\]

\[ \implies \eta(u) \text{ is an additional conserved quantity, with flux } q(u) \]
Existence of Entropy – Entropy Flux Pairs

\[ \nabla \eta(u) \nabla f(u) = \nabla q(u). \]

\[
\begin{pmatrix}
\frac{\partial \eta}{\partial u_1} & \cdots & \frac{\partial \eta}{\partial u_m}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial q}{\partial u_1} & \cdots & \frac{\partial q}{\partial u_m}
\end{pmatrix}
\]

- A systems of \( m \) equations for 2 unknown functions: \( \eta(u) \) and \( q(u) \)
- Over-determined if \( m > 2 \)
- However, most of physical systems (described by several conservation laws) are endowed with natural entropies
A weak solution $u$ of the hyperbolic system $u_t + f(u)_x = 0$ is **Entropy Admissible** if

$$\eta(u)_t + q(u)_x \leq 0$$

in the sense of distributions, for every entropy-entropy flux pair $(\eta, q)$ with $\nabla^2 \eta(u) \geq 0$, i.e. convex.

$$\iint \left\{ \eta(u) \varphi_t + q(u) \varphi_x \right\} \, dx \, dt \geq 0 \quad \varphi \in C_c^\infty, \quad \varphi \geq 0$$

- Smooth solutions conserve all entropies
- Solutions with shocks are admissible if they dissipate all convex entropies
\[ u_\varepsilon^t + f(u_\varepsilon^\varepsilon)_x = \varepsilon u_\varepsilon^{xx} \quad u_\varepsilon^\varepsilon \to u \quad \text{as } \varepsilon \to 0 \]

For any entropy-entropy flux pair

\[ (\eta(u), q(u)) \quad \nabla^2 \eta(u) \geq 0, \]

multiply \( \nabla \eta(u_\varepsilon^\varepsilon) \) both sides of the system yields

\[ \eta(u_\varepsilon^\varepsilon)_t + q(u_\varepsilon^\varepsilon)_x = \varepsilon \eta(u_\varepsilon^\varepsilon)_xx - \varepsilon (u_x)^\top \nabla^2 \eta(u_\varepsilon^\varepsilon) u_x \]

\[ \leq \varepsilon \eta(u_\varepsilon^\varepsilon)_xx \to 0 \]

in the sense of distributions.
Pressureless Euler Equations

\[ \partial_t \rho + \partial_x (\rho v) = 0, \quad \partial_t (\rho v) + \partial_x (\rho v^2) = 0 \]

\[ \alpha = \frac{1}{\sqrt{1 + \sigma^2}} (\sigma \rho - [\rho v]) > 0, \quad \sigma = \frac{\sqrt{\rho_+ v_+} + \sqrt{\rho_- v_-}}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-) \]
Isentropic Euler Equations

Pressure Function \( p(\rho) = \kappa \rho^\gamma \)

\[
\frac{\partial t}{\partial t} \rho + \frac{\partial x}{\partial x} (\rho v) = 0, \quad \frac{\partial t}{\partial t} (\rho v) + \frac{\partial x}{\partial x} (\rho v^2 + p(\rho)) = 0
\]
Isentropic Euler Equations  

\[ \partial_t \rho + \partial_x (\rho v) = 0, \quad \partial_t (\rho v) + \partial_x (\rho v^2 + p(\rho)) = 0 \]

\((t, x) \rightarrow (t, y) : y_t = \rho(t, x), y_x = -(\rho v)(t, x); \quad \tau(t, y) = 1/\rho(t, x)\)
Global in Time Solutions to the Cauchy Problem

\[ u_t + f(u)_x = 0, \quad u(0, x) = u(x) \]

- Construct a sequence of approximate solutions \( \{u^\nu\}_{\nu \geq 1} \)
- Show that (a subsequence) converges: \( u^\nu \to u \) in \( L^1_{loc} \)
- Show that the limit \( u \) is an entropy solution.

Need: a-priori bound on the total variation (J. Glimm, 1965)
Building Block: The Riemann Problem

\[ u_t + f(u)_x = 0, \quad u(0, x) = \begin{cases} 
    u^- & x < 0 \\
    u^+ & x > 0
\end{cases} \]

- B. Riemann 1860: $2 \times 2$ Isentropic Euler equations
- P. Lax 1957: $m \times m$ systems (+ special assumptions)
- T.-P. Liu 1975: $m \times m$ systems (generic case)

*The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative*
Solution to the Riemann problem

- is invariant w.r.t. rescaling symmetry: $u^\theta(t, x) = u(\theta t, \theta x)$ for $\theta > 0$
- describes local behavior of BV solutions near each point $(t_0, x_0)$
- describes large-time asymptotics as $t \to +\infty$ (for small total variation)
Riemann Problem for Linear Systems

\[ u_t + A u_x = 0 \]

\[ u(0, x) = \begin{cases} 
  u^- & \text{if } x < 0 \\
  u^+ & \text{if } x > 0 
\end{cases} \]

\[ x/t = \lambda_1 \quad \omega_0 = u^- \]

\[ x/t = \lambda_2 \quad \omega_1 \]

\[ x/t = \lambda_3 \quad \omega_2 = u^+ \]

\[ u^+ - u^- = \sum_{j=1}^{n} c_j r_j \quad \text{(sum of eigenvectors of } A) \]

Intermediate states: \[ \omega_i = u^- + \sum_{j \leq i} c_j r_j \]

\[ i\text{-th jump: } \omega_i - \omega_{i-1} = c_i r_i \text{ travels with speed } \lambda_i \]
Scalar Conservation Law

\[ u_t + f(u)_x = 0 \quad u \in \mathbb{R} \]

CASE 1: Linear flux: \[ f(u) = \lambda u. \]

Jump travels with speed \( \lambda \) (contact discontinuity)
CASE 2: the flux $f$ is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \implies$ centered rarefaction wave

$u^+ < u^- \implies$ stable shock

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$
A class of nonlinear hyperbolic systems

\[ u_t + f(u)_x = 0 \]

\[ A(u) = Df(u) \quad A(u)r_i(u) = \lambda_i(u)r_i(u) \]

Assumption (H) (P. Lax, 1957): Each \( i \)-th characteristic field is

- either genuinely nonlinear, so that \( \nabla \lambda_i \cdot r_i > 0 \) for all \( u \)
- or linearly degenerate, so that \( \nabla \lambda_i \cdot r_i = 0 \) for all \( u \)
genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors $r_i$

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors $r_i$
Shock and Rarefaction curves

\[ u_t + f(u)_x = 0 \quad \text{A}(u) = Df(u) \]

**i-rarefaction** curve through \( u_0 \): \( \sigma \mapsto R_i(\sigma)(u_0) \)

= integral curve of the field of eigenvectors \( r_i \) through \( u_0 \)

\[ \frac{du}{d\sigma} = r_i(u), \quad u(0) = u_0 \]

**i-shock** curve through \( u_0 \): \( \sigma \mapsto S_i(\sigma)(u_0) \)

= set of points \( u \) connected to \( u_0 \) by an i-shock, so that \( u - u_0 \) is an i-eigenvector of the averaged matrix \( A(u, u_0) \)
Elementary waves

\[ u_t + f(u)_x = 0 \]

\[ u(0,x) = \begin{cases} 
  u^-, & \text{if } x < 0 \\
  u^+, & \text{if } x > 0
\end{cases} \]

CASE 1 (Centered rarefaction wave). Let the \( i \)-th field be genuinely nonlinear.

If \( u^+ = R_i(\sigma)(u^-) \) for some \( \sigma > 0 \), then

\[ u(t,x) = \begin{cases} 
  u^-, & \text{if } x < t\lambda_i(u^-), \\
  R_i(s)(u^-), & \text{if } x = t\lambda_i(s) \quad s \in [0, \sigma] \\
  u^+, & \text{if } x > t\lambda_i(u^+)
\end{cases} \]

is a weak solution of the Riemann problem.
A centered rarefaction wave

\[ u = u^- \quad \text{and} \quad u = u^+ \]

\[ \frac{x}{t} = \lambda_i(u^-) \quad \text{and} \quad \frac{x}{t} = \lambda_i(u^+) \]
CASE 2 (Shock or contact discontinuity). Assume that
\[ u^+ = S_i(\sigma)(u^-) \] for some \( i, \sigma \). Let \( \lambda = \lambda_i(u^-, u^+) \) be the shock speed.

Then the function
\[
u(t, x) = \begin{cases} 
u^- & \text{if } x < \lambda t, \\ \nu^+ & \text{if } x > \lambda t, \end{cases}
\]
is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff \( \sigma < 0 \).
Solution to a 2 x 2 Riemann problem

1 - rarefaction

2 - shock

\[ u = u^- \]

\[ u = u^+ \]
Solution of the general Riemann problem (P. Lax, 1957)

\[ u_t + f(u)_x = 0 \]
\[ u(0,x) = \begin{cases} 
  u^- & \text{if } x < 0 \\
  u^+ & \text{if } x > 0
\end{cases} \]

Problem: Find states \( \omega_0, \omega_1, \ldots, \omega_m \) such that

\[ \omega_0 = u^- \quad \omega_m = u^+ \]

and every couple \( \omega_{i-1}, \omega_i \) are connected by an elementary wave (shock or rarefaction)

\[ \begin{cases} 
  \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\
  \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0
\end{cases} \]
define: \[ \Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases} \]

\[
(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)
\]

Jacobian matrix at the origin: \[ J = \begin{pmatrix} r_1(u^-) & r_2(u^-) & \cdots & r_n(u^-) \end{pmatrix} \]
always has full rank

If \(|u^+ - u^-|\) is small, then the implicit function theorem yields existence and uniqueness of the intermediate states \(\omega_0, \omega_1, \ldots, \omega_n\)
General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)
Global solution to the Cauchy problem

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]

**Theorem** (Glimm, 1965).

Assume:

- system is strictly hyperbolic
- each characteristic field is either linearly degenerate or genuinely nonlinear

Then there exists a constant \( \delta > 0 \) such that, for every initial condition \( \bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n) \) with \( \text{Tot.Var.}(\bar{u}) \leq \delta \), the Cauchy problem has an entropy admissible weak solution \( u = u(t, x) \) defined for all \( t \geq 0 \).
Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in $t$-$x$ plane (Glimm scheme)

- at points where fronts interact (front tracking)
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans
Approximate the initial data $\bar{u}$ with a piecewise constant function.

Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$.

At each time $t_j$ where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem . . .

Need to check: \[ \begin{align*} &- \text{total variation remains small} \\ &- \text{number of wave fronts remains finite} \end{align*} \]
Interaction estimates

**GOAL:** estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves \( \sigma', \sigma'' \)

In incoming: a \( j \)-wave of strength \( \sigma' \) and an \( i \)-wave of strength \( \sigma'' \)

In outgoing: waves of strengths \( \sigma_1, \ldots, \sigma_m \). Then

\[
|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i, j} |\sigma_k| = O(1) \cdot |\sigma'\sigma''|
\]
Incoming: two \( i \)-waves of strengths \( \sigma' \) and \( \sigma'' \)

Outgoing: waves of strengths \( \sigma_1, \ldots, \sigma_m \). Then

\[
|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| = O(1) \cdot |\sigma'\sigma''| \left( |\sigma'| + |\sigma''| \right)
\]
Glimm functionals

Total strength of waves: \[ V(t) \doteq \sum_{\alpha} |\sigma_{\alpha}| \]

Wave interaction potential: \[ Q(t) \doteq \sum_{(\alpha, \beta) \in \mathcal{A}} |\sigma_{\alpha} \sigma_{\beta}| \]

\[ \mathcal{A} \doteq \text{couples of approaching wave fronts} \]
Changes in $V, Q$ at time $\tau$ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) = \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) = -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau^-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant $C_0$ large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

is nonincreasing, as long as $V$ remains small.

Total variation initially small $\implies$ global BV bounds

$$\text{Tot.Var.}\{u(t, \cdot)\} \leq V(t) \leq V(0) + C_0 Q(0)$$

Front tracking approximations can be constructed for all $t \geq 0$.
Keeping finite the number of wave fronts

At each interaction point, the Accurate Riemann Solver yields a solution, possibly introducing several new fronts.

The total number of fronts can become infinite in finite time.

Need: a Simplified Riemann Solver, producing only one "non-physical" front.
A sequence of approximate solutions

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

\((u_\nu)_{\nu \geq 1}\) sequence of approximate front tracking solutions

- initial data satisfy \(\|u_\nu(0, \cdot) - \bar{u}\|_{L^1} \leq \varepsilon_\nu \to 0\)
- all shock fronts in \(u_\nu\) are entropy-admissible
- each rarefaction front in \(u_\nu\) has strength \(\leq \varepsilon_\nu\)
- at each time \(t \geq 0\), the total strength of all non-physical fronts in \(u_\nu(t, \cdot)\) is \(\leq \varepsilon_\nu\)
Existence of a convergent subsequence

\[ \text{Tot.Var.}\left\{ u_\nu(t, \cdot) \right\} \leq C \]

\[ \left\| u_\nu(t) - u_\nu(s) \right\|_{L^1} \leq (t - s) \cdot [\text{total strength of all wave fronts}] \cdot [\text{maximum speed}] \]

\[ \leq L \cdot (t - s) \]

Helly's compactness theorem \( \implies \) a subsequence converges

\[ u_\nu \to u \quad \text{in} \quad L^1_{\text{loc}} \]
Claim: \( u = \lim_{\nu \to \infty} u_\nu \) is a weak solution

\[
\iint \left\{ \phi_t u + \phi_x f(u) \right\} \, dx \, dt = 0 \quad \text{for} \quad \phi \in C_c^1 \left( ]0, \infty[ \times \mathbb{R} \right)
\]

Need to show:

\[
\lim_{\nu \to \infty} \iint \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} \, dx \, dt = 0
\]
\[
\int_0^\infty \int_{-\infty}^\infty \left\{ \phi_t(t, x)u_\nu(t, x) + \phi_x(t, x)f(u_\nu(t, x)) \right\} \, dx \, dt \\
= \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot n \, d\sigma
\]

\[
\limsup_{\nu \to \infty} \left| \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot n \, d\sigma \right| \\
\leq \limsup_{\nu \to \infty} \left| \sum_{\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{N} \cup \mathcal{P}} \left[ \dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha)) \right] \phi(t, x_\alpha(t)) \right| \\
\leq \left( \max_{t, x} |\phi(t, x)| \right) \cdot \limsup_{\nu \to \infty} \left\{ \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{R}} \varepsilon_\nu |\sigma_\alpha| + \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{N} \cup \mathcal{P}} |\sigma_\alpha| \right\} \\
= 0
\]
The Glimm scheme

\[ u_t + f(u)_x = 0 \quad \text{for} \quad u(0, x) = \bar{u}(x) \]

Assume: all characteristic speeds satisfy \( \lambda_i(u) \in [0, 1] \)

This is not restrictive. If \( \lambda_i(u) \in [-M, M] \), simply change coordinates:

\[ y = x + Mt, \quad \tau = 2Mt \]

Choose:

- a grid in the \( t-x \) plane with step size \( \Delta t = \Delta x \)
- a sequence of numbers \( \theta_1, \theta_2, \theta_3, \ldots \) uniformly distributed over \([0, 1]\)

\[
\lim_{N \to \infty} \frac{\# \{ j ; 1 \leq j \leq N, \theta_j \in [0, \lambda] \} }{N} = \lambda \quad \text{for each} \quad \lambda \in [0, 1].
\]
Glimm approximations

Grid points: \( x_j = j \cdot \Delta x \), \( t_k = k \cdot \Delta t \)

- for each \( k \geq 0 \), \( u(t_k, \cdot) \) is piecewise constant, with jumps at the points \( x_j \). The Riemann problems are solved exactly, for \( t_k \leq t < t_{k+1} \)
- at time \( t_{k+1} \) the solution is again approximated by a piecewise constant function, by a sampling technique

\[
\theta_2 = \frac{1}{3} \\
\theta_1 = \frac{1}{2}
\]
Example: Glimm's scheme applied to a solution containing a single shock

\[ U(t, x) = \begin{cases} 
 u^+ & \text{if } x > \lambda t \\
 u^- & \text{if } x < \lambda t
\end{cases} \]

Fix \( T > 0 \), take \( \Delta x = \Delta t = T/N \)

\[ x(T) = \#\{j; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda]\} \cdot \Delta t \]

\[ = \frac{\#\{j; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda]\}}{N} \cdot T \rightarrow \lambda T \quad \text{as } N \rightarrow \infty \]
Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence \((\theta_k)_{k \geq 1}\)

\[
\lim_{N \to \infty} \frac{\#\{j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \}}{N} = \lambda \quad \text{for each} \ \lambda \in [0, 1].
\]

Need fast convergence to uniform distribution. Achieved by choosing:

\[
\theta_1 = 0.1, \quad \ldots, \quad \theta_{759} = 0.957, \quad \ldots, \quad \theta_{39022} = 0.22093, \quad \ldots
\]

Convergence rate:

\[
\lim_{\Delta x \to 0} \frac{\|u^{\text{Glimm}}(T, \cdot) - u^{\text{exact}}(T, \cdot)\|_{L^1}}{\sqrt{\Delta x} \cdot |\ln \Delta x|} = 0
\]

(A.Bressan & A.Marson, 1998)