PDE-CDT Core Course

Analysis of Partial Differential Equations-Part III

Lecture 3

EPSRC Centre for Doctoral Training in
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Hyperbolic Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t}u_1 + \frac{\partial}{\partial x}f_1(u_1, \cdots, u_m) = 0, \\ \qquad\qquad \cdots\cdots \\ \frac{\partial}{\partial t}u_m + \frac{\partial}{\partial x}f_m(u_1, \cdots, u_m) = 0, \end{cases}$$

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$ conserved quantities

$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_m(\mathbf{u}))^\top$ fluxes

Euler equations of gas dynamics (1755)

$$\begin{cases} \rho_t + (\rho v)_x = 0 & \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x = 0 & \text{(conservation of momentum)} \\ (\rho E)_t + (\rho E v + p v)_x = 0 & \text{(conservation of energy)} \end{cases}$$

ρ = mass density $\qquad v$ = velocity

$E = e + v^2/2$ = energy density per unit mass (internal + kinetic)

$p = p(\rho, e)$ constitutive relation

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})$ (column vectors)

Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad \mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i$$

Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Invariance of Hyperbolicity under Change of Coordinates

Theorem

- *Let* $\mathbf{u}$ *be a smooth solution of the strictly hyperbolic system*

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \textit{in } \mathbb{R}_+ \times \mathbb{R}$$

- *Assume* $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ *is a smooth diffeomorphism, with inverse* Ψ

Then $\mathbf{w} := \Phi(\mathbf{u})$ *solves the strictly hyperbolic system*

$$\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = 0 \qquad \textit{in } \mathbb{R}_+ \times \mathbb{R}$$

for $\quad \mathbf{B}(w) := \nabla\Phi(\Psi(\mathbf{w}))\mathbf{A}(\Psi(\mathbf{w}))\nabla\Psi(\mathbf{w}) \qquad \mathbf{w} \in \mathbb{R}^m$

Dependence of Eigenvalues and Eigenvectors on $\mathbf{u}$

Theorem

Assume that the matrix function $\mathbf{A}(\mathbf{u})$ is smooth, strictly hyperbolic. Then

- *The eigenvalues $\lambda_k(\mathbf{u})$ depend smoothly on $\mathbf{u} \in \mathbb{R}^m, k = 1, \cdots, m$*
- *We can select the right eigenvectors $\mathbf{r}_k(\mathbf{u})$ and left eigenvector $\mathbf{l}_k(\mathbf{u})$ to depend smoothly on $\mathbf{u} \in \mathbb{R}^m$ and satisfy the normalization*

$$|\mathbf{r}_k(\mathbf{u})|, |\mathbf{l}_k(\mathbf{u})| = 1, \qquad k = 1, \cdots, m.$$

*We are not only globally and smoothly defining the eigenvalues and eigenspaces of $\mathbf{A}(\mathbf{u})$, but also globally providing the eigenspaces of $\mathbf{A}(\mathbf{u})$ with an orientation.

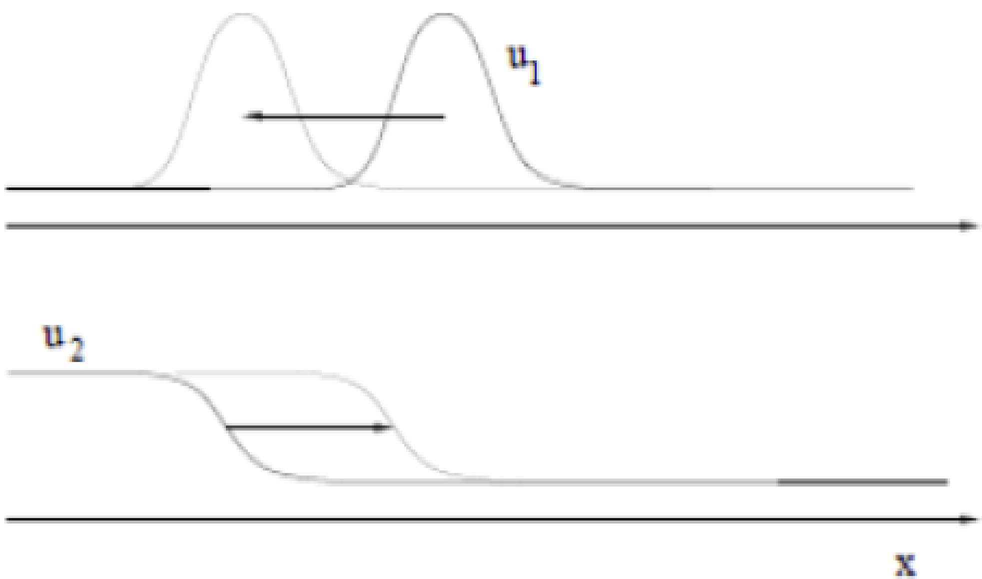
Linear Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \qquad \mathbf{u}(0,x) = \phi(x)$$

$\lambda_1 < \cdots < \lambda_m$ eignevalues $\qquad \mathbf{r}_1, \cdots, \mathbf{r}_m$ eigenvectors

Explicit solutions: **Linear superposition of travelling waves**

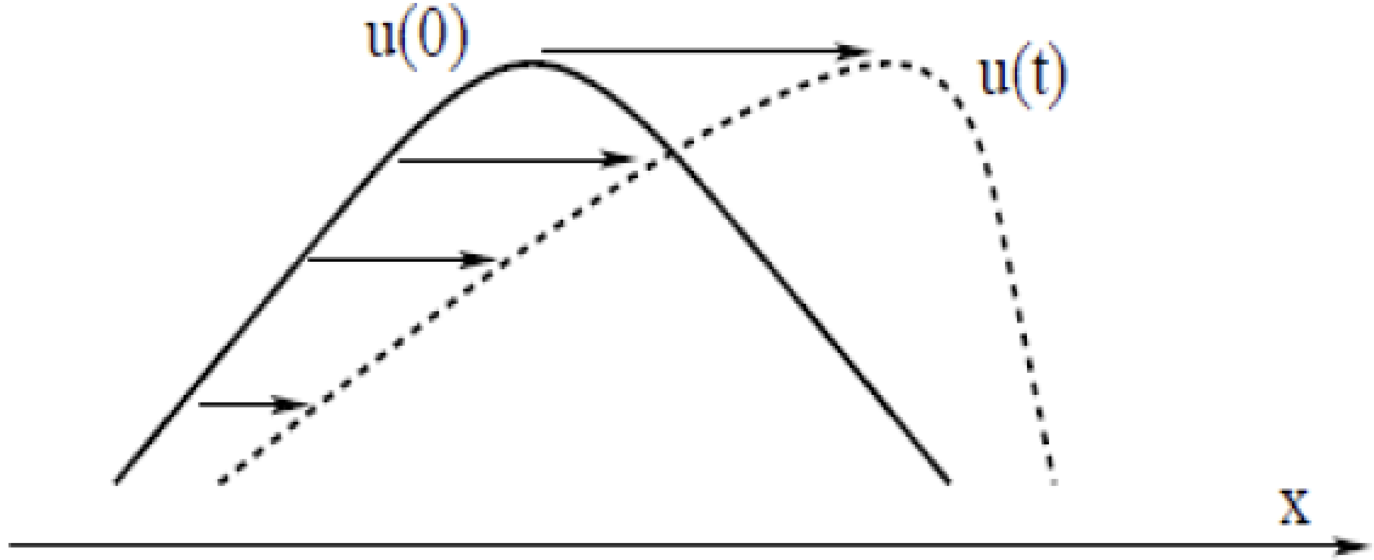
$$\mathbf{u}(t,x) = \sum_i \phi_i(x - \lambda_i t)\,\mathbf{r}_i \qquad \phi_i(s) = \mathbf{l}_i \cdot \phi(s)$$



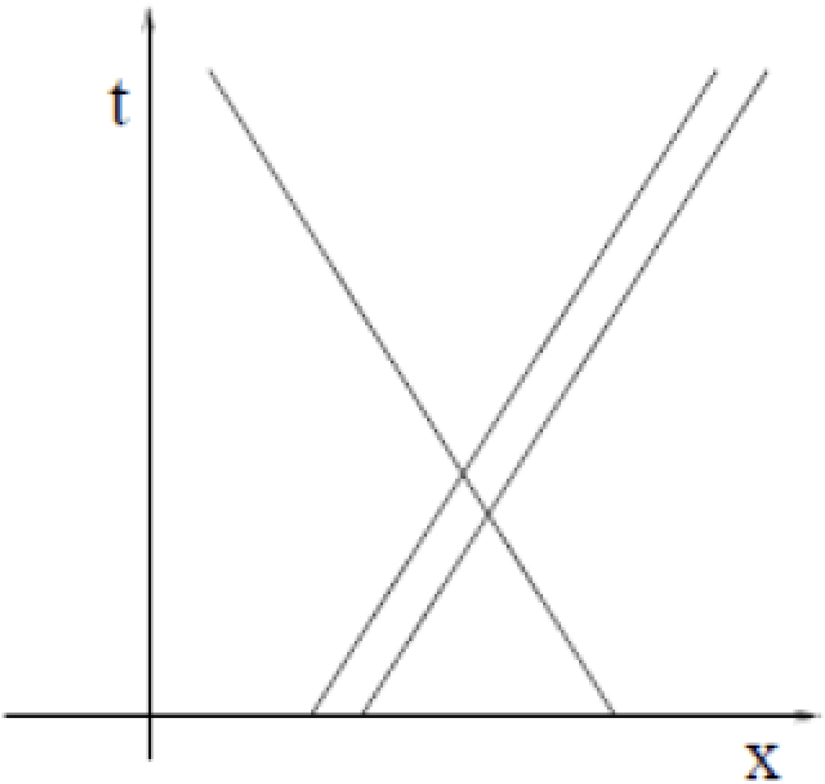
Nonlinear Effects

$$u_t + A(u)u_x = 0$$

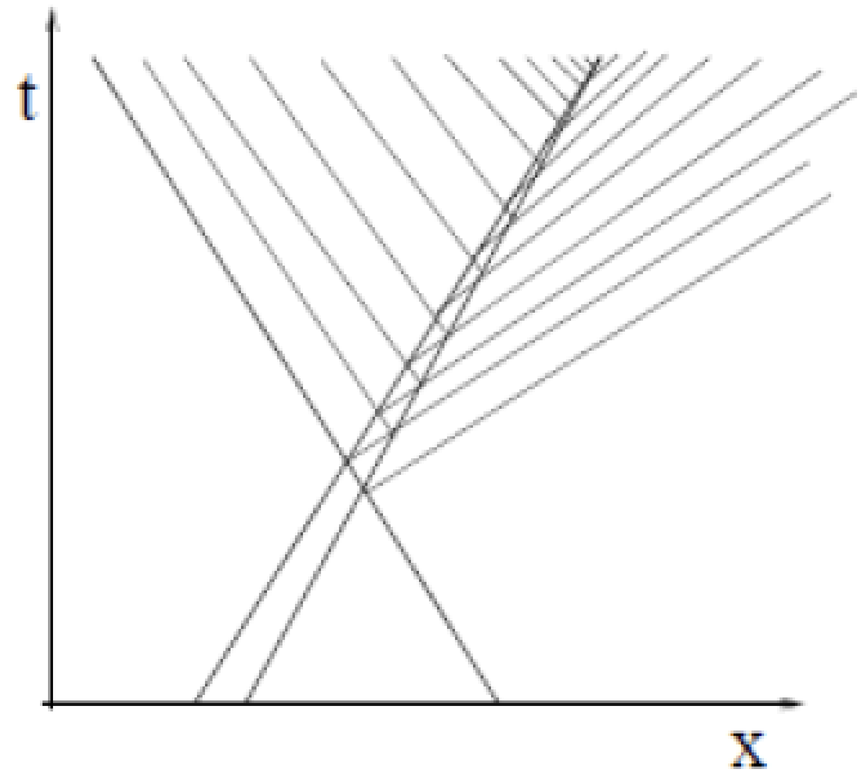
eigenvalues depend on u $\implies$ waves change shape



eigenvectors depend on u $\Longrightarrow$ nontrivial wave interactions



linear

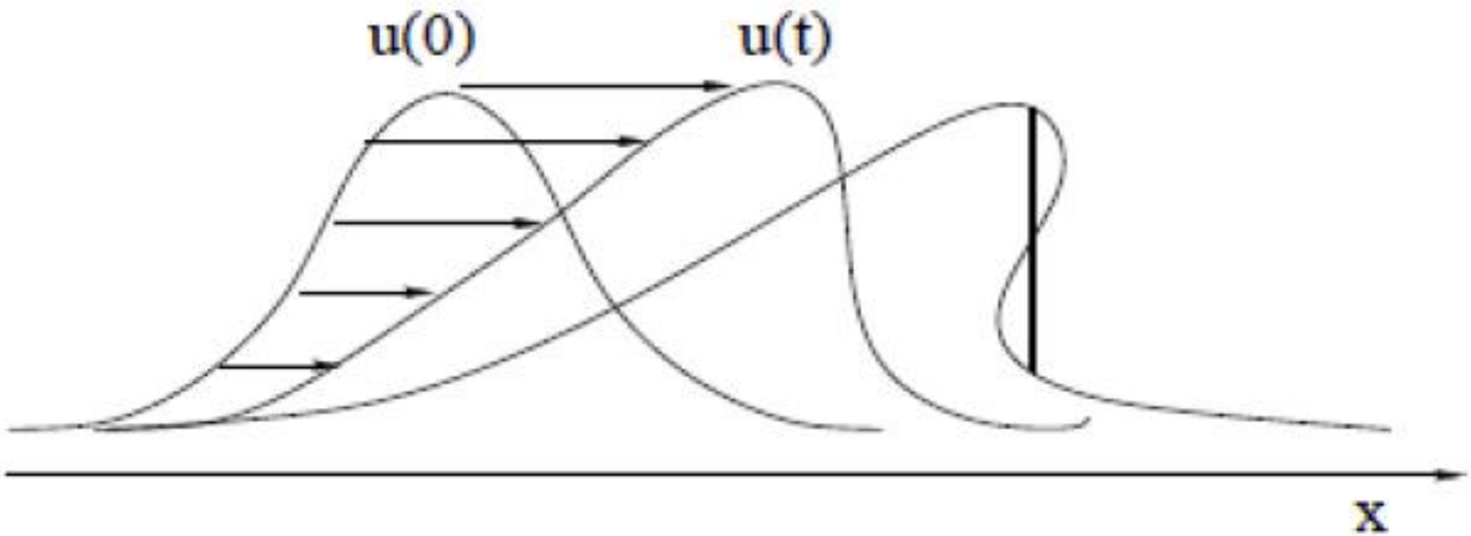


nonlinear

Loss of Regularity

$$u_t + (u^2/2)_x = 0 \qquad u_t + uu_x = 0$$

$$f(u) = u^2/2 \qquad \text{characteristic speed: } f'(u) = u$$



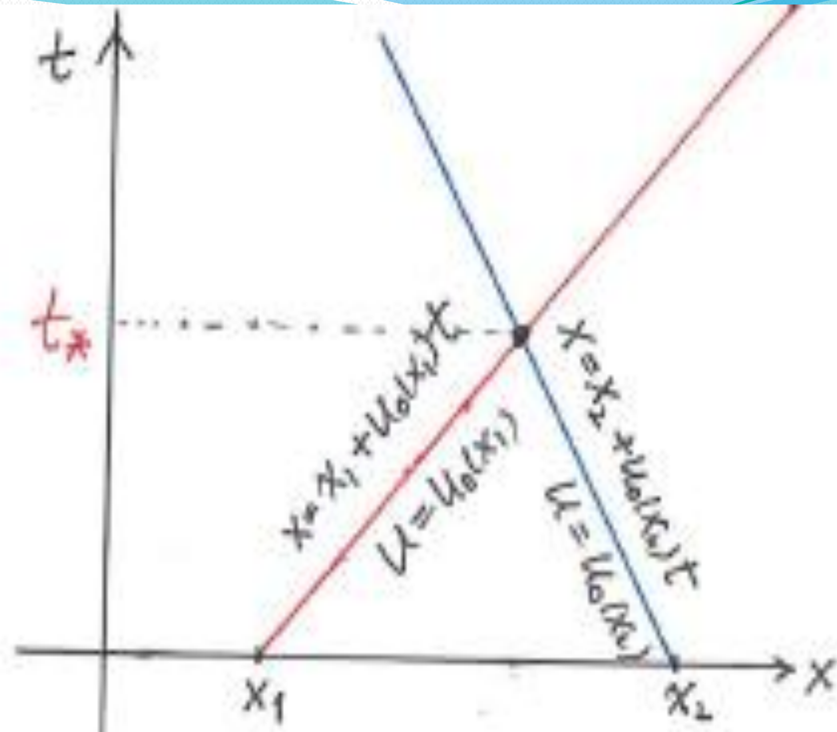
Global solutions only in a space of discontinuous functions

1-D Example.

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$$u_t + u\,u_x = 0$$

$$\begin{cases} \frac{dx}{dt} = u, \longrightarrow x = x_j + u_0(x_j)t \\ \frac{du}{dt} = 0 \longrightarrow u = u_0(x_j),\ j=1,2 \end{cases}$$



$u_0(x_1) > u_0(x_2)$

When $t \rightarrow t_* = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} > 0$, $u(t,x)$ is Multi-Valued

Smooth Solutions – Evolution of Wave Components

$$\mathbf{u}_t = -\mathbf{A}(\mathbf{u})\mathbf{u}_x$$

$\lambda_i(\mathbf{u}) = i$-th eigenvalue, $\quad \mathbf{l}_i(\mathbf{u}), \mathbf{r}_i(\mathbf{u}) = i$-th eigenvectors

$u_x^i := \mathbf{l}_i \cdot \mathbf{u}_x = [i\text{-th component of } \mathbf{u}_x] = [\text{density of } i\text{-waves in } \mathbf{u}]$

$$\mathbf{u}_x = \sum_{i=1}^{m} u_x^i \, \mathbf{r}_i(\mathbf{u}) \qquad \mathbf{u}_t = -\sum_{i=1}^{m} \lambda_i(\mathbf{u}) u_x^i \, \mathbf{r}_i(\mathbf{u})$$

Differentiate the 1st equation w.r.t. t and the 2nd w.r.t x

$\Longrightarrow$ Evolution equation for scalar components u_x^i:

$$\left(u_x^i\right)_t + \left(\lambda_i u_x^i\right)_x = \sum_{j>k} (\lambda_j - \lambda_k)\Big(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]\Big) u_x^j u_x^k$$

Source Terms

$$(\lambda_j - \lambda_k)\Big(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]\Big) u_x^j u_x^k$$

= amount of i-waves produced by the interaction of j-waves with k-waves

$\lambda_j - \lambda_k$ = [difference in speed]
= [rate at which j-waves and k-waves cross each other]

$u_x^j u_x^k$ = [density of j-waves] $\times$ [density of k-waves]

$[\mathbf{r}_j, \mathbf{r}_k] = (\nabla \mathbf{r}_k)\mathbf{r}_j - (\nabla \mathbf{r}_j)\mathbf{r}_k$ (Lie bracket)
= [directional derivative of $\mathbf{r}_k$ in the direction of $\mathbf{r}_j$]
−[directional derivative of $\mathbf{r}_j$ in the direction of $\mathbf{r}_k$]

$\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]$ = i-th component of the Lie bracket $[\mathbf{r}_j, \mathbf{r}_k]$ along the basis of eigenvectors $\{\mathbf{r}_1, \cdots, \mathbf{r}_m\}$

Shock Solutions

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$$\mathbf{u}(t,x) = \begin{cases} \mathbf{u}^- & \text{if} \quad x < \lambda t \\ \mathbf{u}^+ & \text{if} \quad x > \lambda t \end{cases}$$ is a weak solution

if and only if the **Rankine-Hugoniot Equations** hold:

$$\lambda \left[\mathbf{u}^+ - \mathbf{u}^-\right] = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)$$

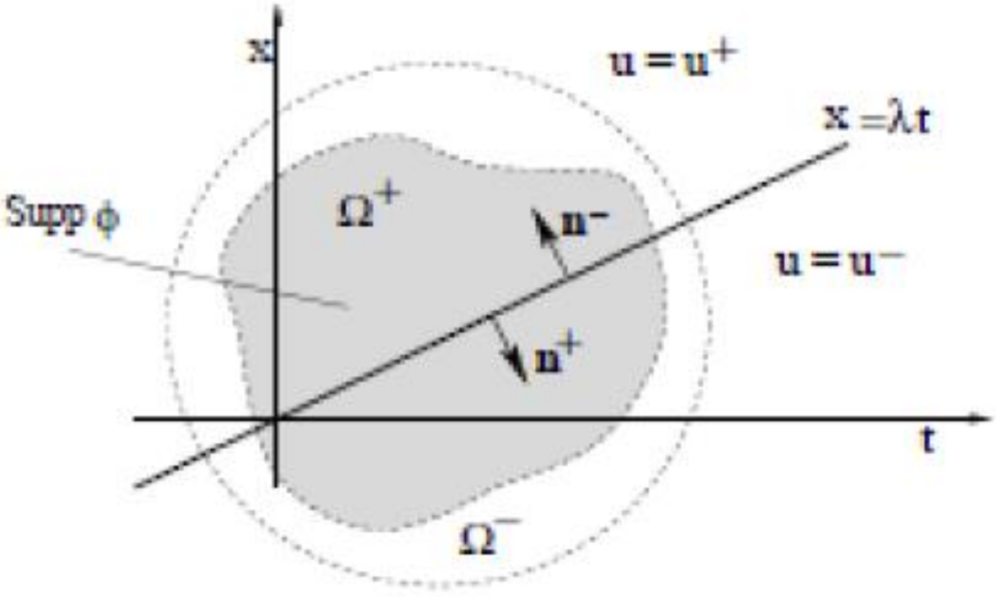
[Speed of the shock] $\times$ **[Jump in the state]** $=$ **[Jump in the flux]**

Derivation of the Rankine - Hugoniot Equations

$$0 = \iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt = \iint_{\Omega^+ \cup \Omega^-} \text{div } (u\phi, f(u)\phi)\, dxdt$$

$$= \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds$$

$$= \int \left[\lambda(u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t)\, dt\, .$$



$$\mathbf{v} \doteq \left(u\phi\, ,\ f(u)\phi \right)$$

Alternative Formulation

$$\begin{aligned}\lambda(\mathbf{u}^+ - \mathbf{u}^-) &= \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-) \\ &= \int_0^1 \nabla \mathbf{f}(\theta \mathbf{u}^+ + (1-\theta)\mathbf{u}^-) \cdot (\mathbf{u}^+ - \mathbf{u}^-) d\theta \\ &= \mathbf{A}(\mathbf{u}^+, \mathbf{u}^-) \cdot (\mathbf{u}^+ - \mathbf{u}^-)\end{aligned}$$

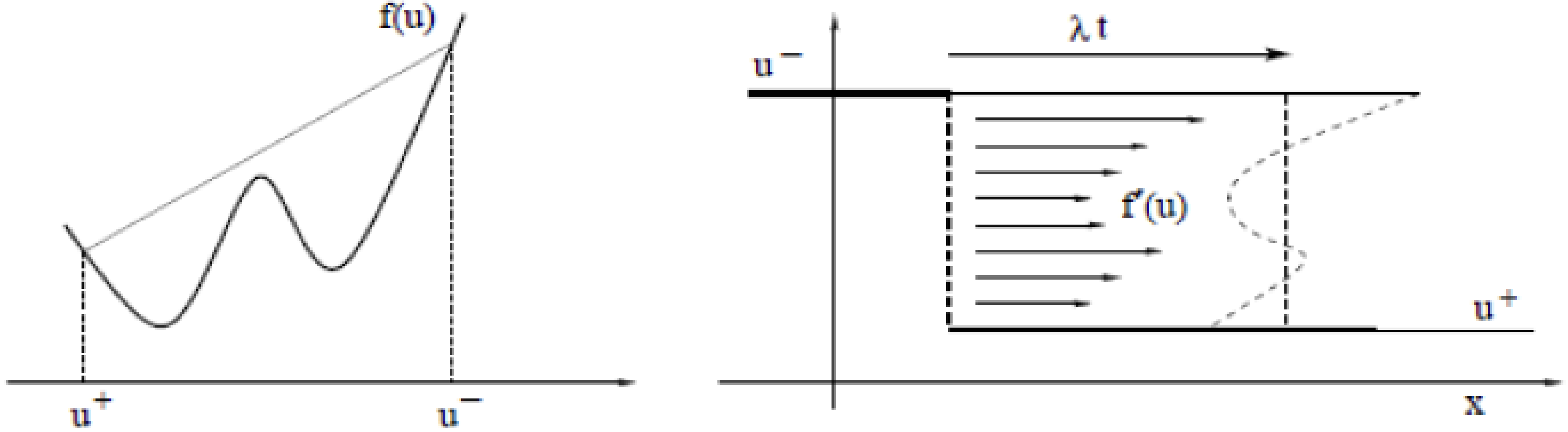
$\mathbf{A}(\mathbf{u}, \mathbf{v}) := \int_0^1 \nabla \mathbf{f}(\theta \mathbf{u} + (1-\theta)\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) d\theta$
=[averaged Jacobian matrix]

The Rankine-Hugoniot conditions hold if and only if

$$\lambda(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{A}(\mathbf{u}^+, \mathbf{u}^-)(\mathbf{u}^+ - \mathbf{u}^-)$$

- The jump $\mathbf{u}^+ - \mathbf{u}^-$ is an eigenvector of the averaged matrix $\mathbf{A}(\mathbf{u}^+, \mathbf{u}^-)$
- The speed λ coincides with the corresponding eigenvalue

The Rankine-Hugoniot condition for the scalar conservation law $u_t + f(u)_x = 0$



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s)\, ds$$

[speed of the shock] = [slope of secant line through u^-, u^+ on the graph of f]

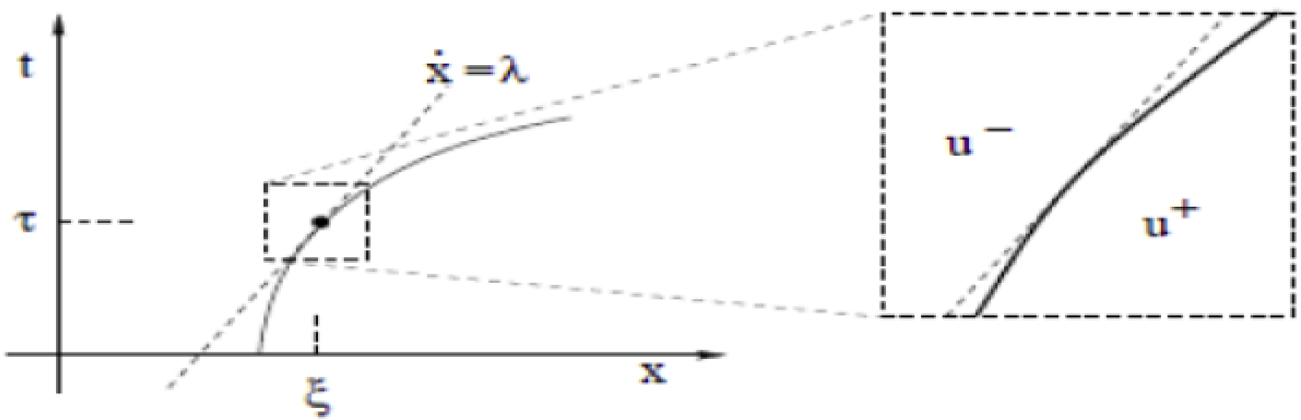
= [average of the characteristic speeds between u^- and u^+]

Points of Approximate Jump

The function $\mathbf{u} = \mathbf{u}(t,x)$ has an approximate jump at a point (τ,ξ) if there exists states $\mathbf{u}^- = \mathbf{u}^+$ and a speed λ such that, setting

$$U(t,x) := \begin{cases} \mathbf{u}^- & \text{if} \quad x < \lambda t \\ \mathbf{u}^+ & \text{if} \quad x > \lambda t \end{cases}$$

there holds: $\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big| \mathbf{u}(t,x) - U(t-\tau, x-\xi) \Big| dxdt = 0$



Theorem

If $\mathbf{u}$ is a weak solution to the system $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$, then the Rankine-Hugoniot equations hold at each point of approximate jump.

Construction of Shock Waves

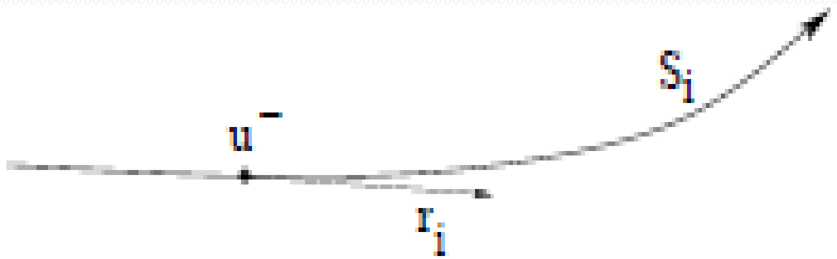
Problem: Given $\mathbf{u}^- \in \mathbb{R}^m$, find the states $\mathbf{u}^+ \in \mathbb{R}^m$ which, for some speed λ, satisfy the Rankine-Hugoniot equations:

$$\lambda(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-) = \mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)(\mathbf{u}^+ - \mathbf{u}^-)$$

Alternative Formulation: Fix $i \in \{1, \cdots, m\}$. The jump $\mathbf{u}^+ - \mathbf{u}^-$ is a (right) i-eigenvector of the avergaed matrix $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$ if and only if it is orthogonal to all (left) eigenvectors $\mathbf{l}_j(\mathbf{u}^+, \mathbf{u}^-)$ of $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$:

$$\mathbf{l}_j(\mathbf{u}^-, \mathbf{u}^+) \cdot (\mathbf{u}^+ - \mathbf{u}^-) = 0 \qquad \text{for all } j \neq i$$

Implicit Function Theorem $\Longrightarrow$ For each i, there exists a curve $s \to S_i(s)(\mathbf{u}^-)$ of pints that satisfy $(RH)_i$.



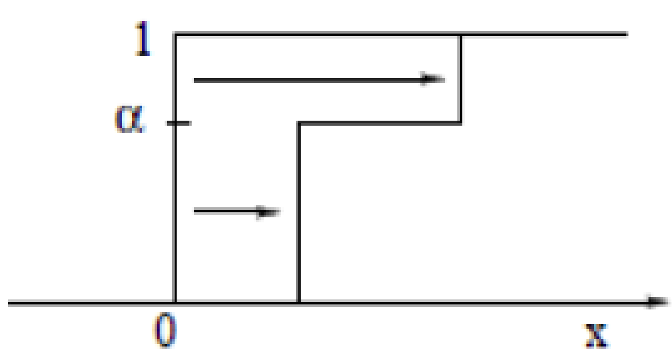
Non-uniqueness of Weak solutions

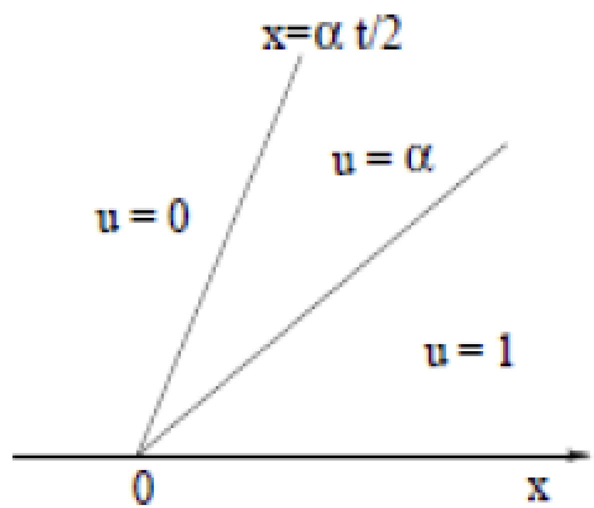
Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \qquad u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Each $\alpha \in [0,1]$ yields a weak solution

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2 \\ \alpha & \text{if} \quad \alpha t/2 < x < (1+\alpha)t/2 \\ 1 & \text{if} \quad x \geq (1+\alpha)t/2 \end{cases}$$





Admissibility Conditions on Shocks

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

- **Solutions should be stable w.r.t. small initial perturbations**
- **Solutions should be limits of suitable approximations and/or physical regularisations (Vanishing viscosity, relaxation, $\cdots$)**
- **Any convex entropy should not increase**
- $\cdots\cdots$

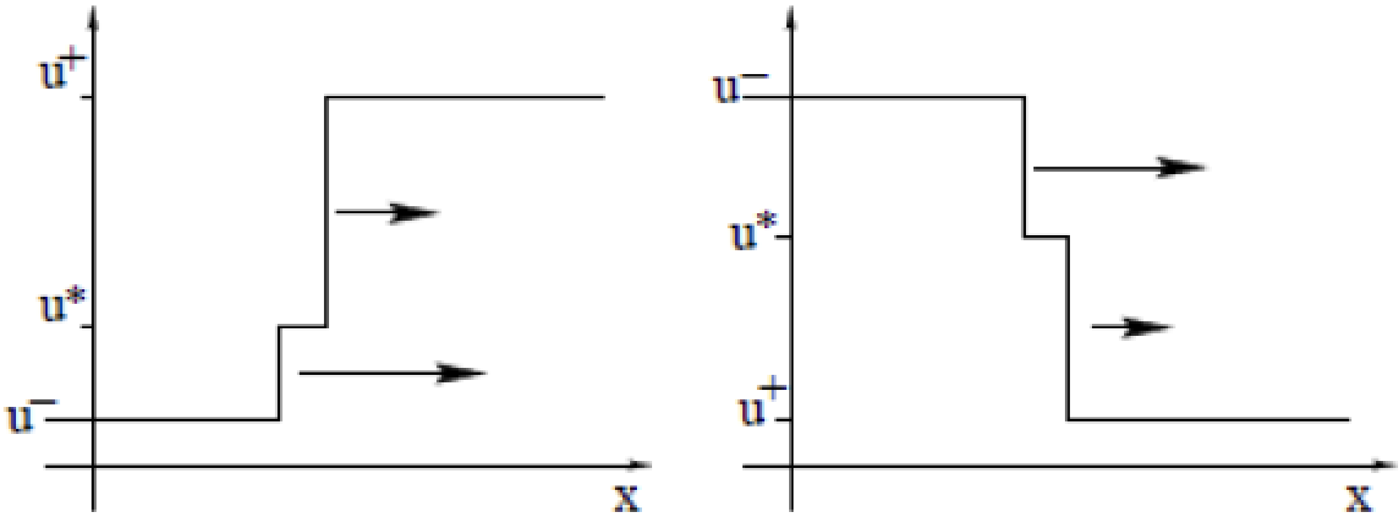
Stability conditions: the scalar case

Perturb the shock with left and right states u^-, u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

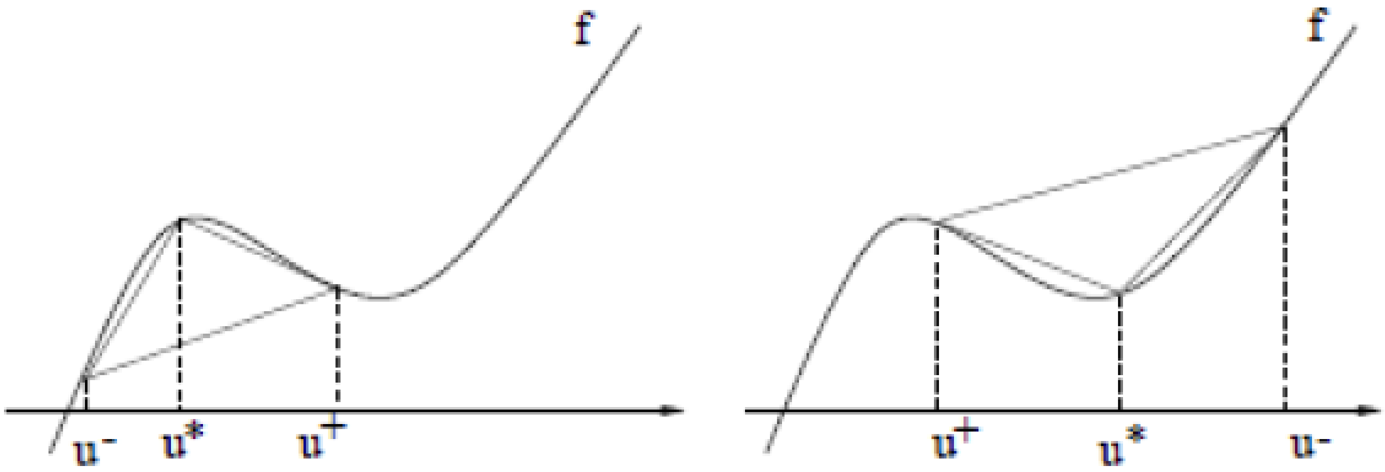
Initial shock is stable $\Longleftrightarrow$

[speed of jump behind] $\leq$ [speed of jump ahead]

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$



speed of a shock = slope of a secant line to the graph of f



Stability conditions:

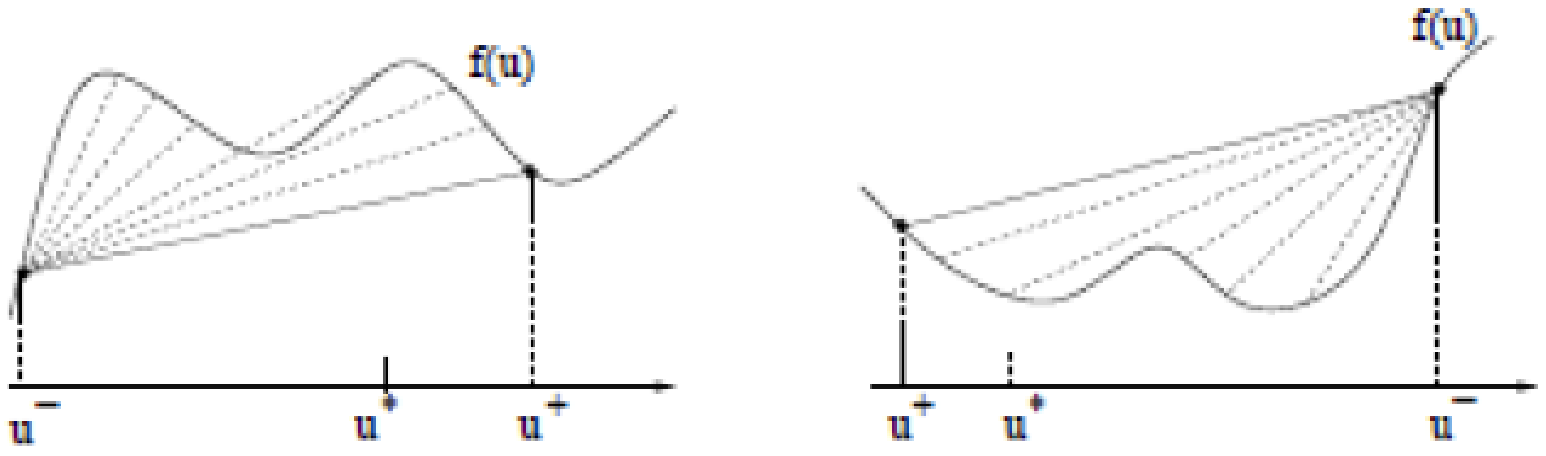
- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

General stability conditions

Scalar case: stability holds if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

for every intermediate state $u^* \in [u^-, u^+]$



Vector Valued Case: $\mathbf{u}^+ = S_i(\sigma)(\mathbf{u}^-)$ for some $\sigma \in \mathbb{R}$

Admissibility Condition (T.-P. Liu)

The speed $\lambda(\sigma)$ of the shock joining $\mathbf{u}^-$ with $\mathbf{u}^+$ must be less or equal to the speed of every smaller shock, joining $\mathbf{u}^-$ with an intermediate state $\mathbf{u}^* = S_i(s)(\mathbf{u}^-), s \in [0, \sigma]$:

$$\lambda(\mathbf{u}^-, \mathbf{u}^+) \leq \lambda(\mathbf{u}^-, \mathbf{u}^*)$$

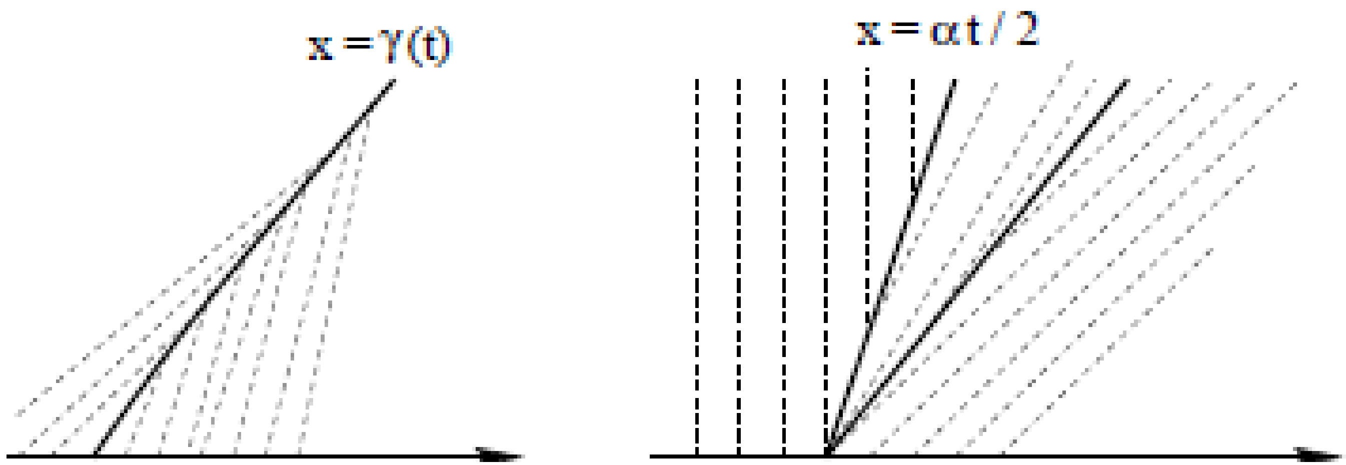
- The Liu condition singles out precisely the solutions which are limits of vanishing viscosity approximations

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}_{xx}^\varepsilon \qquad \mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{as } \varepsilon \to 0$$

Admissibility Condition (P. Lax)

A shock connecting the states u^-, u^+, travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \;\geq\; \lambda_i(u^-, u^+) \;\geq\; \lambda_i(u^+)$$



- Geometric meaning: characteristics flow toward the shock from both sides
- The Liu condition implies the Lax condition

Mathematical Entropy – Entropy Flux

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

Definition: A function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called an **Entropy**, with **Entropy Flux** $q : \mathbb{R}^m \to \mathbb{R}$ if

$$\nabla\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u})$$

For **smooth** solutions $\mathbf{u} = \mathbf{u}(t, x)$, this implies

$$\begin{aligned} \eta(\mathbf{u})_t + q(\mathbf{u})_x &= \nabla\eta(\mathbf{u})\,\mathbf{u}_t + \nabla q(\mathbf{u})\,\mathbf{u}_x \\ &= -(\nabla\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u}))\mathbf{u}_x + \nabla q(\mathbf{u})\,\mathbf{u}_x = 0 \end{aligned}$$

$\Longrightarrow$ $\eta(\mathbf{u})$ is an additional conserved quantity, with flux $q(\mathbf{u})$

Existence of Entropy – Entropy Flux Pairs

$$\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u}).$$

$$\left(\frac{\partial \eta}{\partial u_1} \cdots \frac{\partial \eta}{\partial u_m}\right) \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ & \cdots & \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix} = \left(\frac{\partial q}{\partial u_1} \cdots \frac{\partial q}{\partial u_m}\right)$$

- A systems of m equations for 2 unknown functions: $\eta(\mathbf{u})$ and $q(\mathbf{u})$
- Over-determined if $m > 2$
- However, most of physical systems (described by several conservation laws) are endowed with natural entropies

Entropy Admissibility Condition

A weak solution $\mathbf{u}$ of the hyperbolic system $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$ is **Entropy Admissible** if

$$\eta(\mathbf{u})_t + q(\mathbf{u})_x \leq 0$$

in the sense of distributions, for every entropy-entropy flux pair (η, q) with $\nabla^2\eta(\mathbf{u}) \geq$, i.e. convex.

$$\iint \{\eta(\mathbf{u})\varphi_t + q(\mathbf{u})\varphi_x\}\, dxdt \geq 0 \qquad \varphi \in C_c^\infty,\ \varphi \geq 0$$

- Smooth solutions conserve all entropies
- Solutions with shocks are admissible if they dissipate all convex entropies

Consistency with Vanishing Viscosity Approximations

$$\mathbf{u}^\varepsilon_t + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}^\varepsilon_{xx} \qquad \mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{as } \varepsilon \to 0$$

For any entropy-entropy flux pair

$$(\eta(\mathbf{u}), q(\mathbf{u})) \qquad \nabla^2 \eta(\mathbf{u}) \geq 0,$$

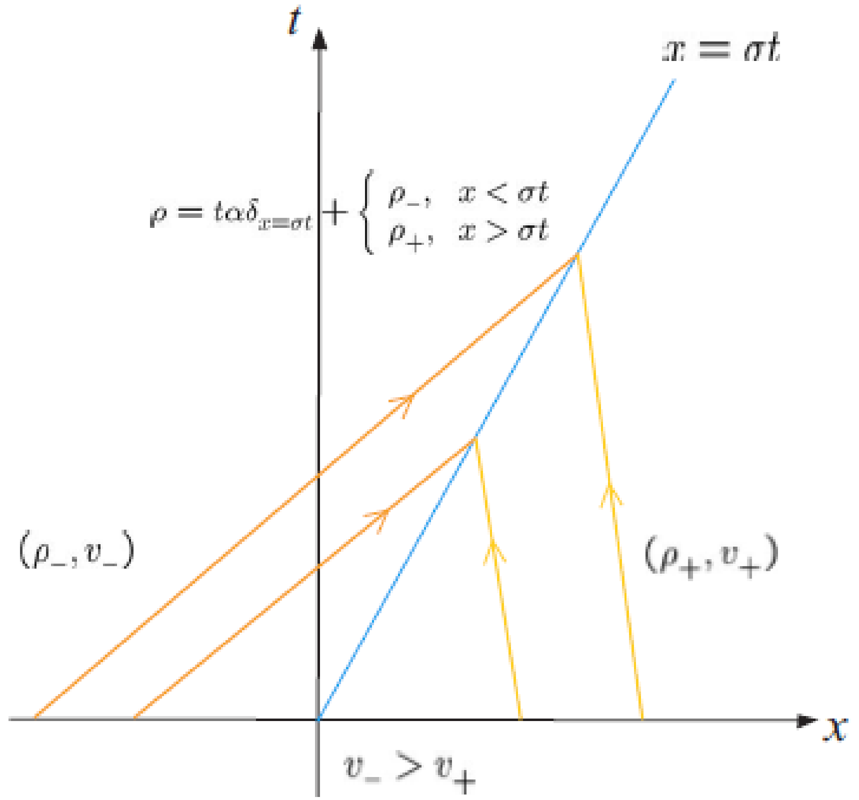
multiply $\nabla \eta(\mathbf{u}^\varepsilon)$ both sides of the system yields

$$\begin{aligned} \eta(\mathbf{u}^\varepsilon)_t + q(\mathbf{u}^\varepsilon)_x &= \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} - \varepsilon (\mathbf{u}_x)^\top \nabla^2 \eta(\mathbf{u}^\varepsilon) \mathbf{u}_x \\ &\leq \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} \to 0 \end{aligned}$$

in the sense of distributions.

Pressureless Euler Equations

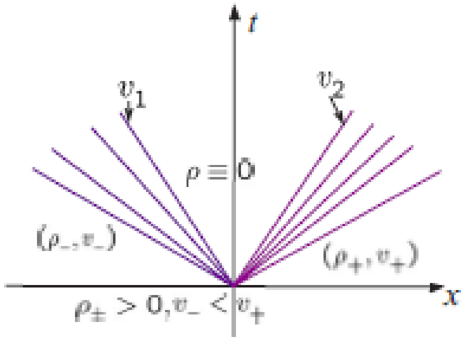
$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$



$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

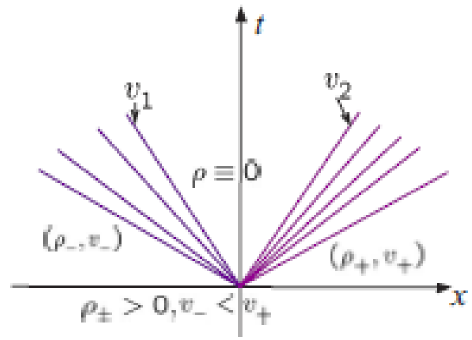
Isentropic Euler Equations Pressure Function $p(\rho) = \kappa\rho^\gamma$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

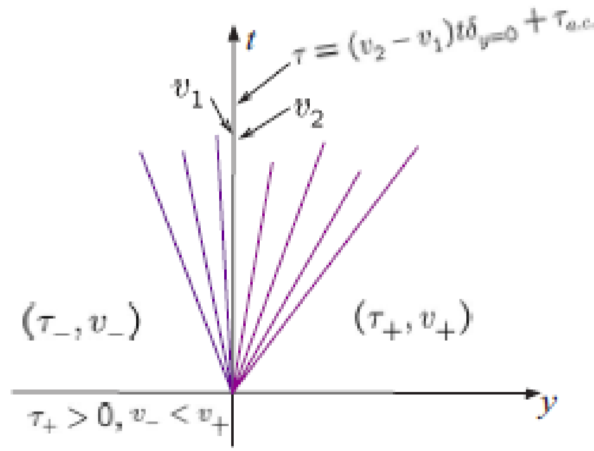


Isentropic Euler Equations Pressure Function $p(\rho) = \kappa\rho^\gamma$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



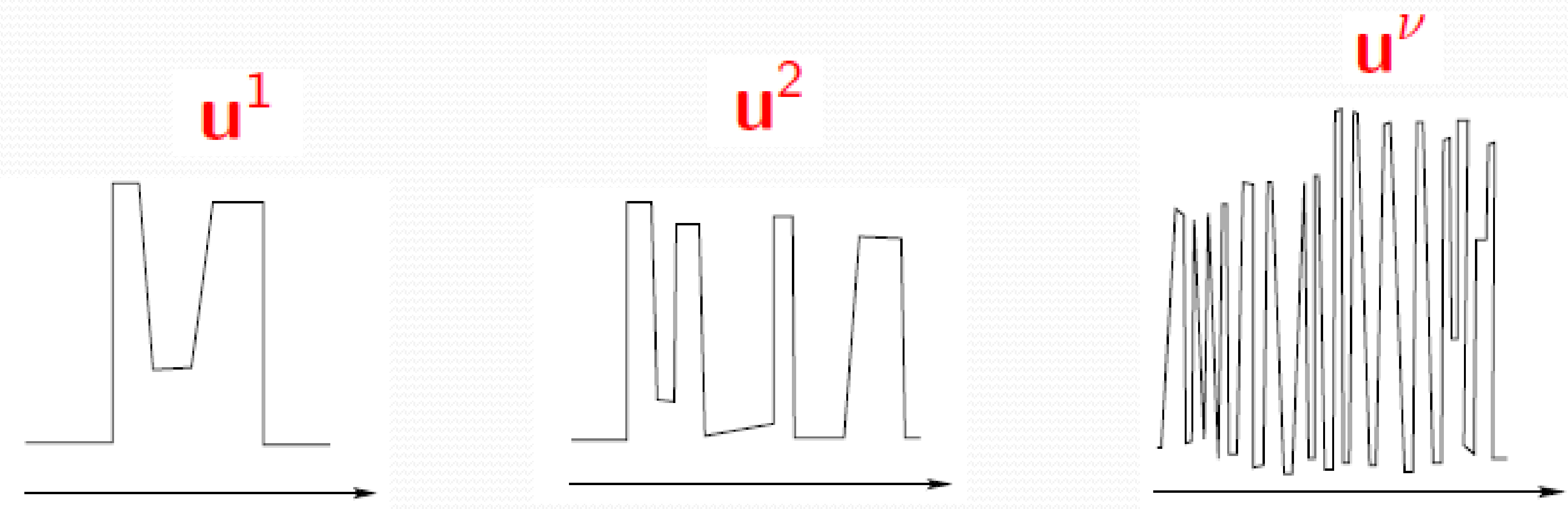
$$(t,x) \to (t,y): \; y_t = \rho(t,x), y_x = -(\rho v)(t,x); \qquad \tau(t,y) = 1/\rho(t,x)$$



Global in Time Solutions to the Cauchy Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \mathbf{u}(x)$$

- **Construct a sequence of approximate solutions $\{\mathbf{u}^\nu\}_{\nu\geq 1}$**
- **Show that (a subsequence) converges: $\mathbf{u}^\nu \to \mathbf{u}$ in L^1_{loc}**
- **Show that the limit $\mathbf{u}$ is an entropy solution.**



Need: a-priori bound on the total variation (J. Glimm, 1965)

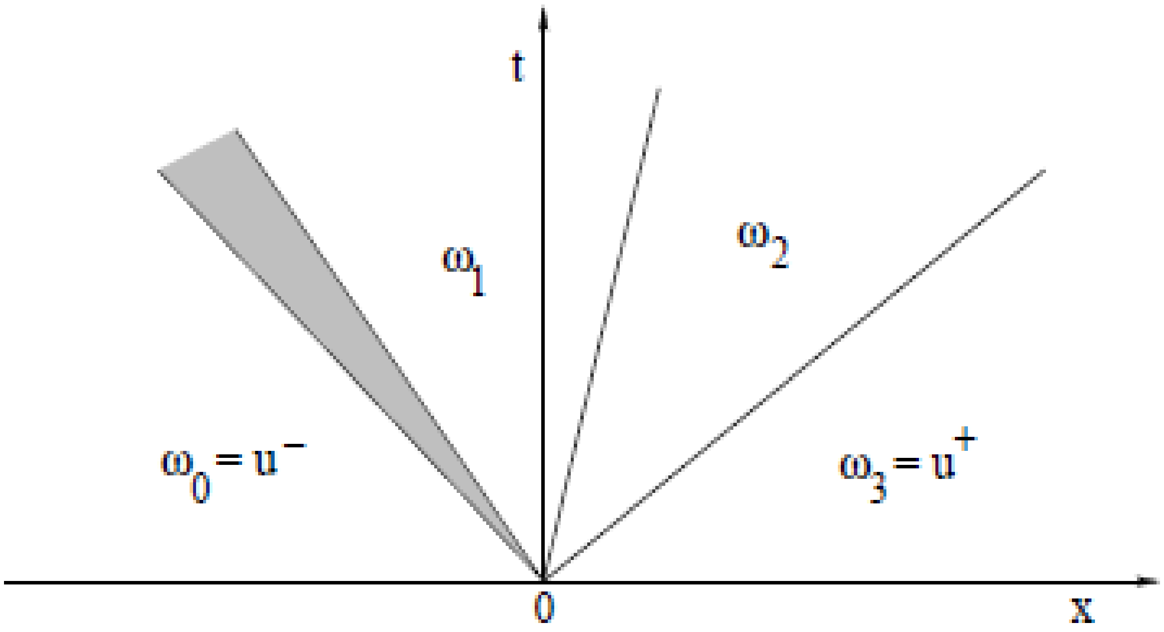
Building Block: The Riemann Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \begin{cases} \mathbf{u}^- & x < 0 \\ \mathbf{u}^+ & x > 0 \end{cases}$$

- **B. Riemann 1860:** 2×2 **Isentropic Euler equations**
- **P. Lax 1957:** $m \times m$ **systems (+ special assumptions)**
- **T.-P. Liu 1975:** $m \times m$ **systems (generic case)**

***The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative**

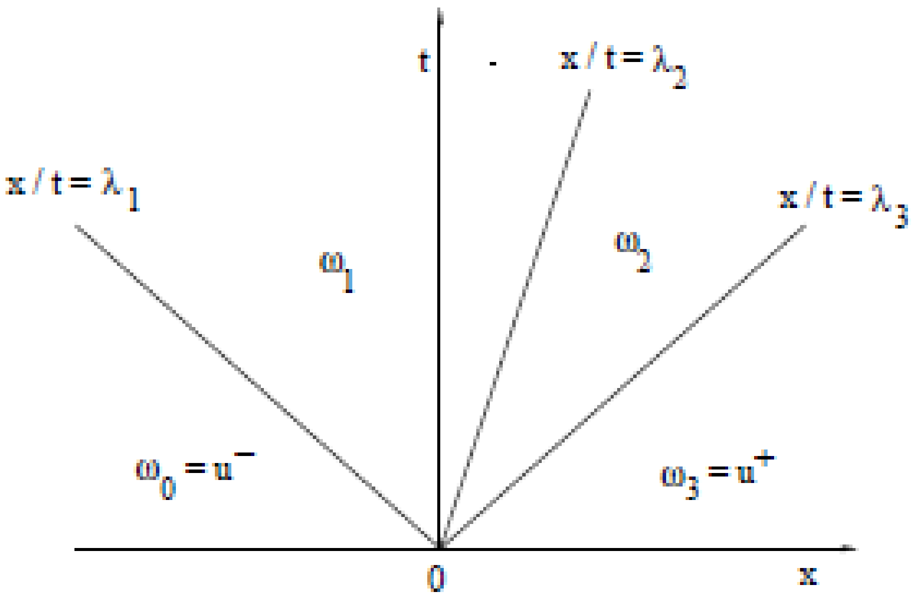
Solution to the Riemann problem



- is invariant w.r.t. rescaling symmetry: $u^\theta(t,x) \doteq u(\theta t, \theta x) \qquad \theta > 0$
- describes local behavior of BV solutions near each point (t_0, x_0)
- describes large-time asymptotics as $t \to +\infty$ (for small total variation)

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$



$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

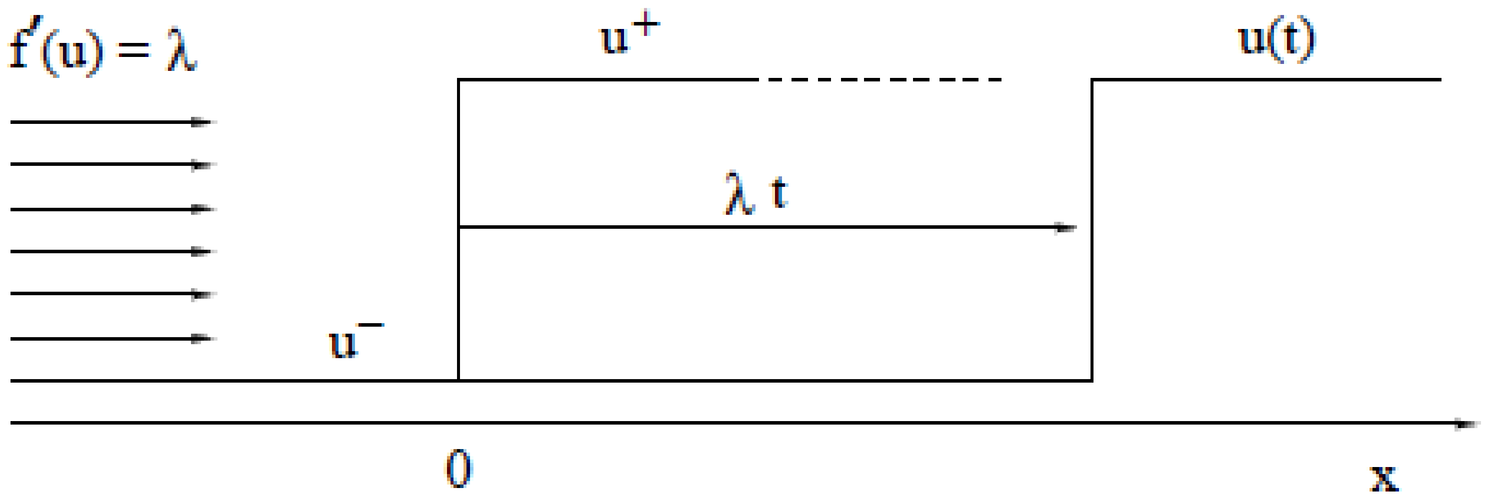
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \qquad u \in \mathbb{R}$$

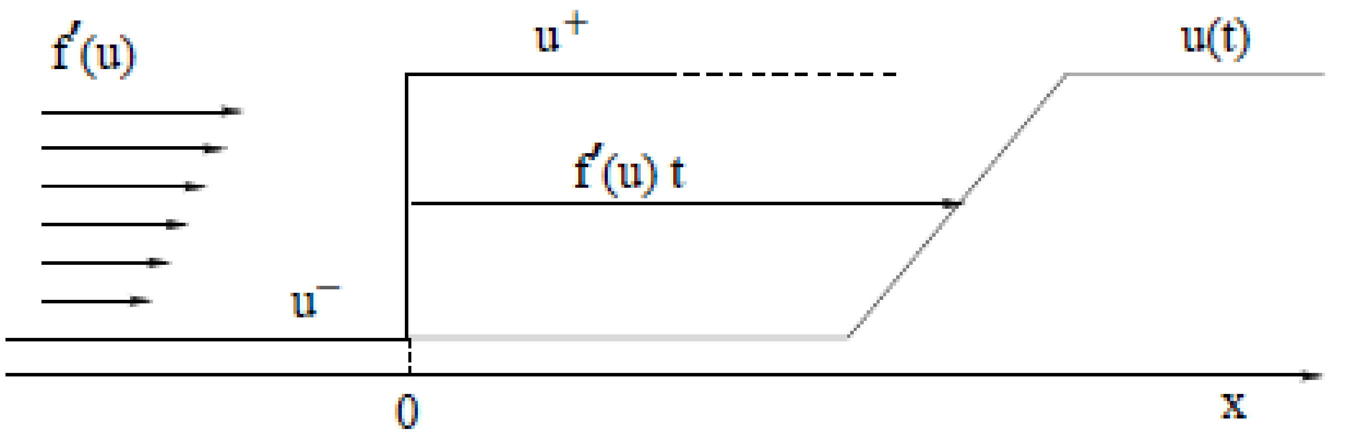
CASE 1: Linear flux: $f(u) = \lambda u$.

Jump travels with speed λ (contact discontinuity)

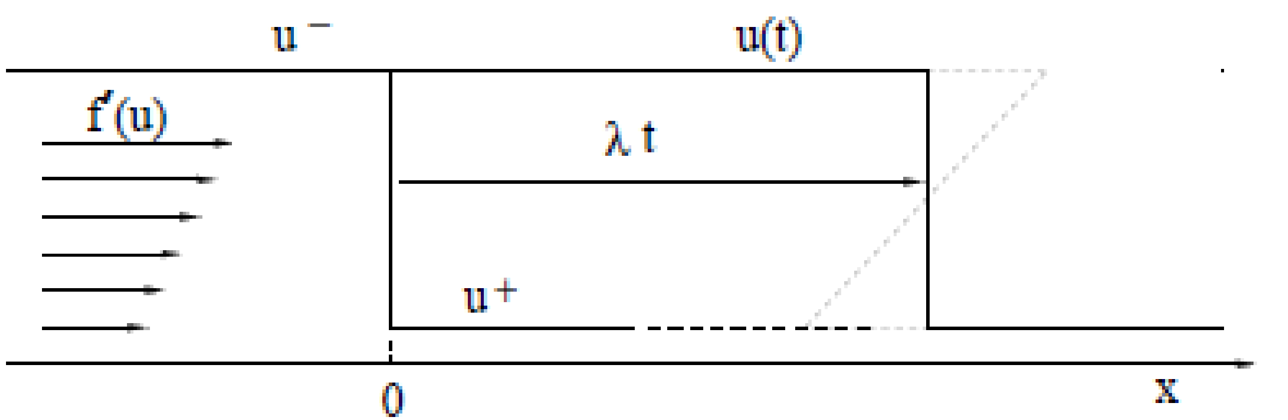


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \quad \Longrightarrow \quad$ centered rarefaction wave



$u^+ < u^- \quad \Longrightarrow \quad$ stable shock



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x \;=\; 0$$

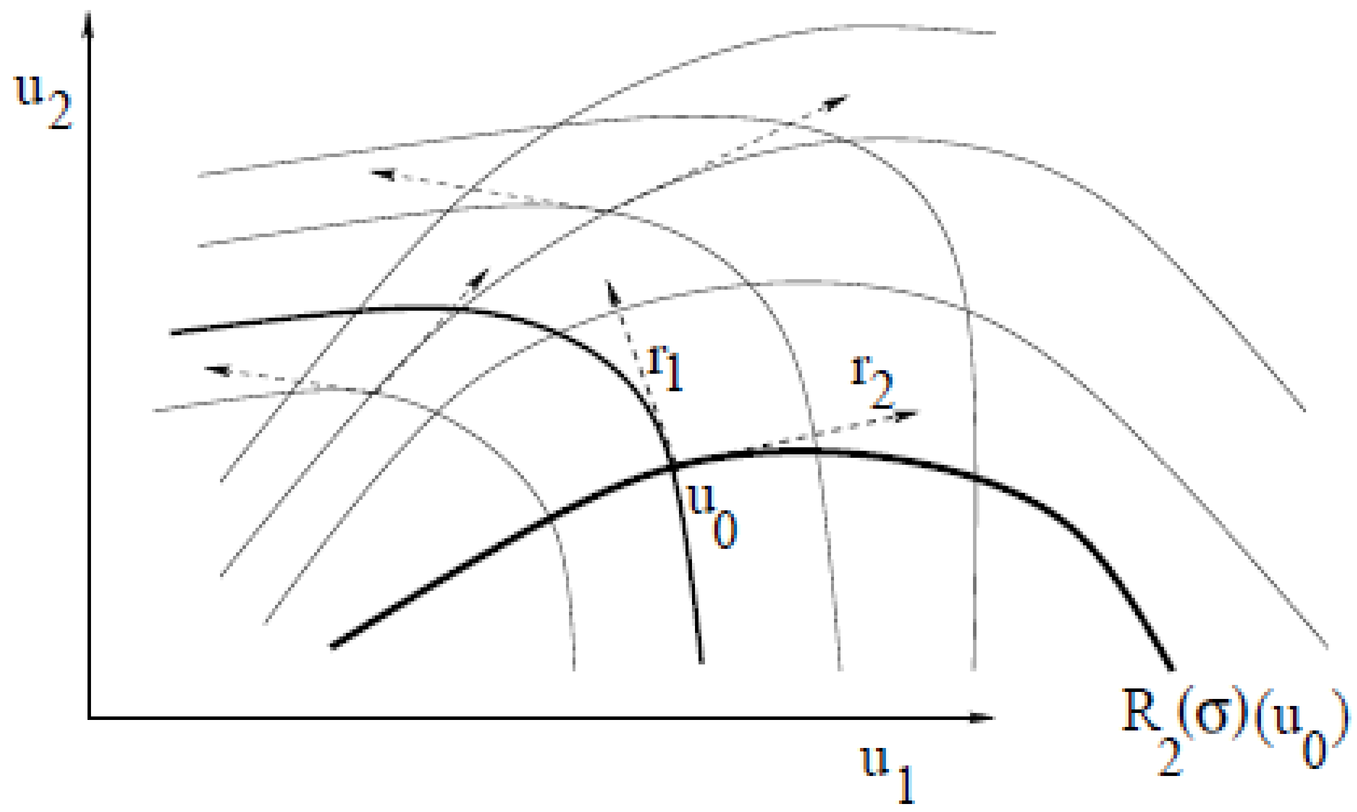
$$A(u) = Df(u) \qquad\qquad A(u)r_i(u) \;=\; \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i-th characteristic field is

- either genuinely nonlinear, so that $\nabla\lambda_i \cdot r_i > 0$ for all u
- or linearly degenerate, so that $\nabla\lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x \;=\; 0 \qquad\qquad A(u) = Df(u)$$

i-rarefaction curve through u_0: $\quad \sigma \mapsto R_i(\sigma)(u_0)$

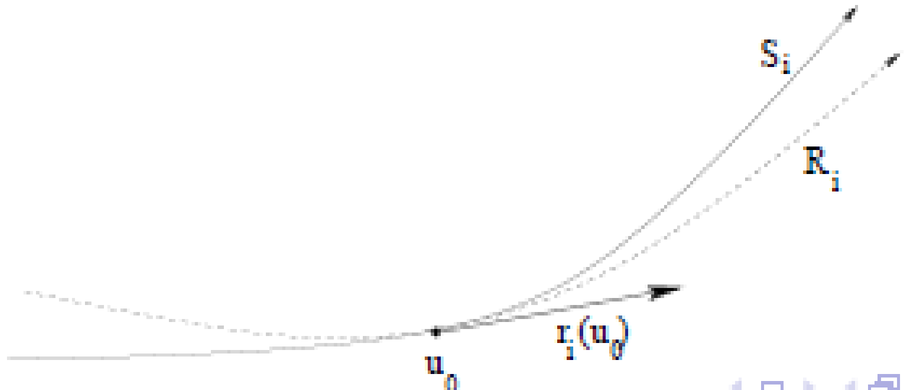
= integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \qquad\qquad u(0) = u_0$$

i-shock curve through u_0: $\quad \sigma \mapsto S_i(\sigma)(u_0)$

= set of points u connected to u_0 by an i-shock, so that

$u - u_0$ is an i-eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

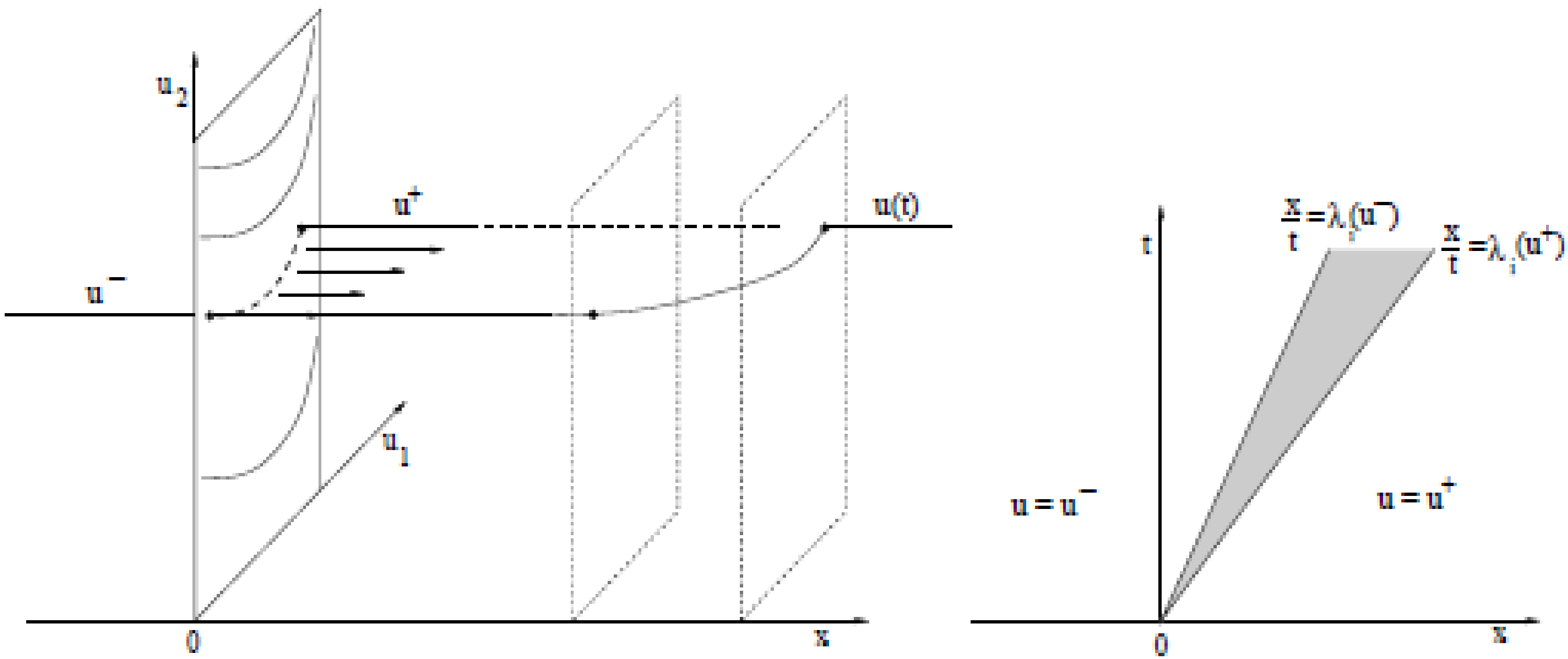
CASE 1 **(Centered rarefaction wave).** Let the i-th field be genuinely nonlinear.

If $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, then

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x = t\lambda_i(s) \quad s \in [0,\sigma] \\ u^+ & \text{if} \quad x > t\lambda_i(u^+) \end{cases}$$

is a weak solution of the Riemann problem

A centered rarefaction wave



CASE 2 (**Shock or contact discontinuity**). Assume that

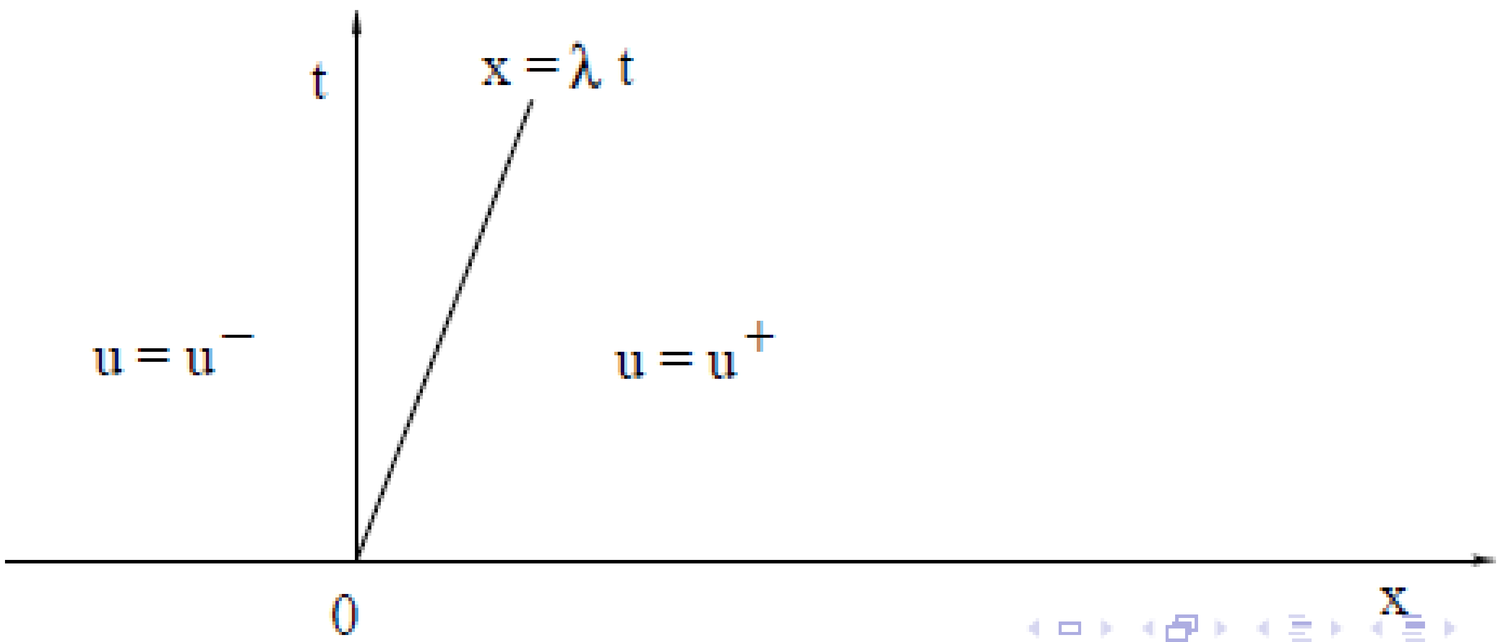
$u^+ = S_i(\sigma)(u^-)$ for some i, σ. Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

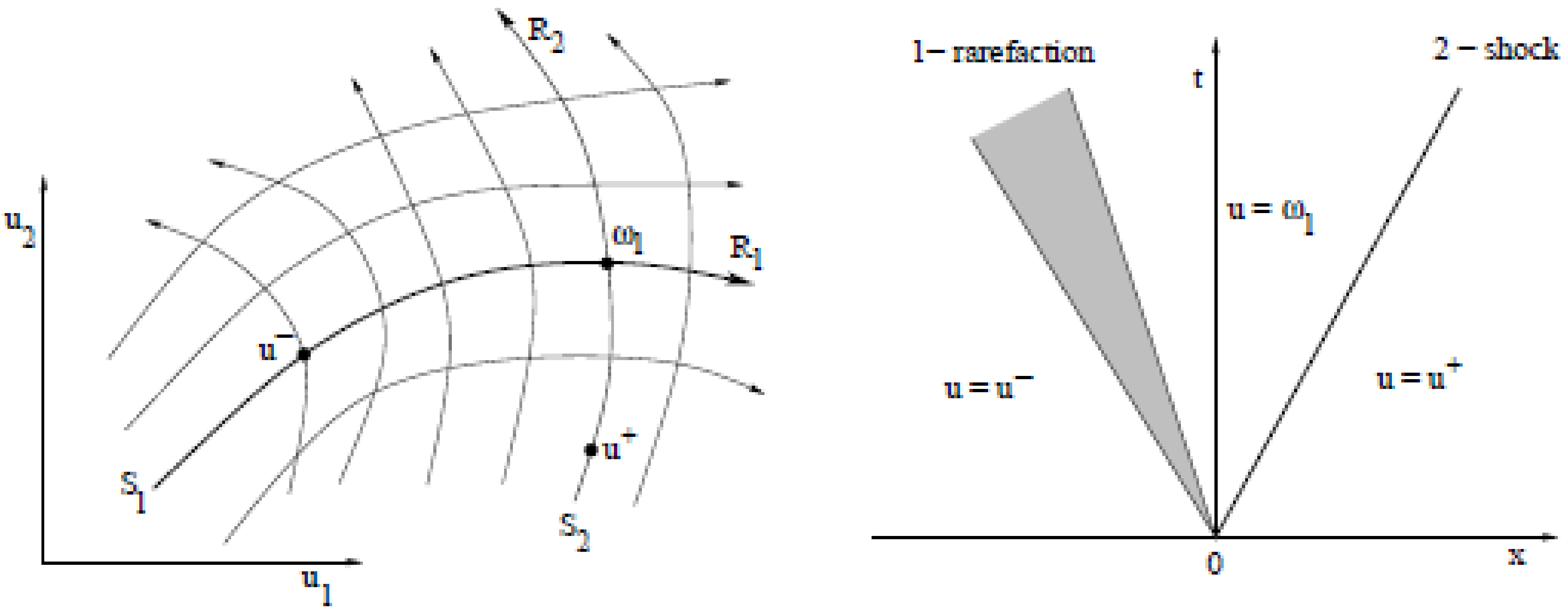
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem



Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

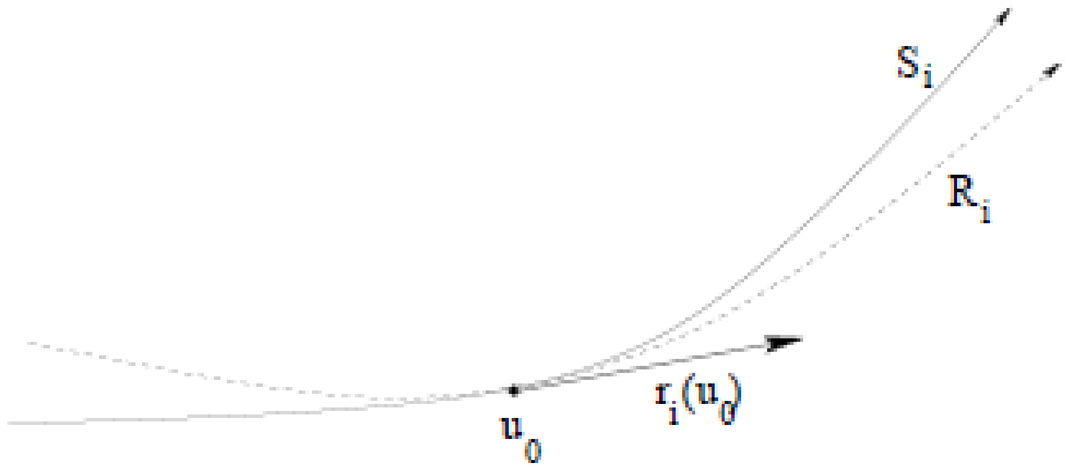
Problem: Find states $\omega_0, \omega_1, \cdots, \omega_m$ such that

$$\omega_0 = \mathbf{u}^- \qquad \omega_m = \mathbf{u}^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

define: $\Psi_i(\sigma)(u) \;=\; \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$



$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \;\mapsto\; \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

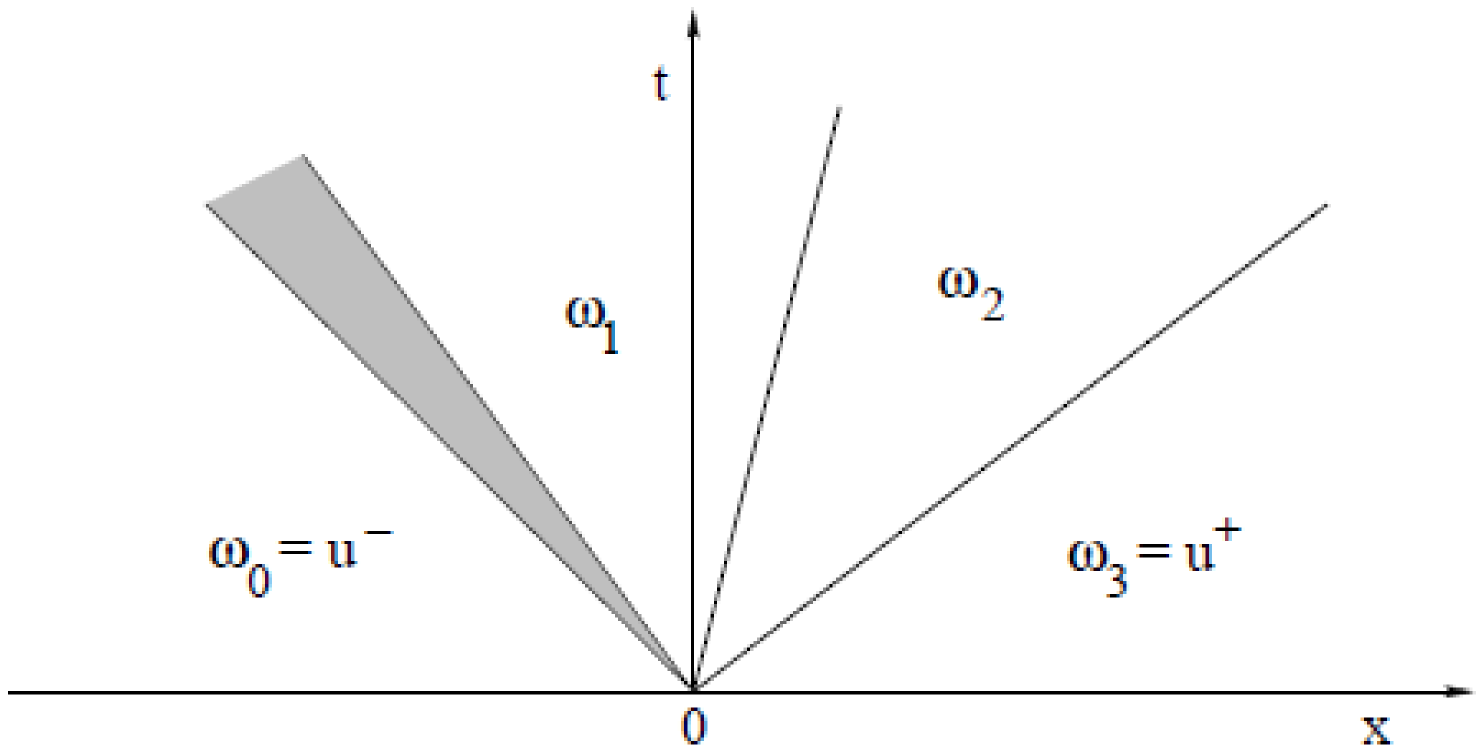
Jacobian matrix at the origin: $J \doteq \left(r_1(u^-) \middle| r_2(u^-) \middle| \cdots \middle| r_n(u^-) \right)$

always has full rank

If $|u^+ - u^-|$ is small, then the implicit function theorem yields existence and uniqueness of the intermediate states $\omega_0, \omega_1, \ldots, \omega_n$

General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)



Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm, 1965).

Assume:

- *system is strictly hyperbolic*
- *each characteristic field is either linearly degenerate or genuinely nonlinear*

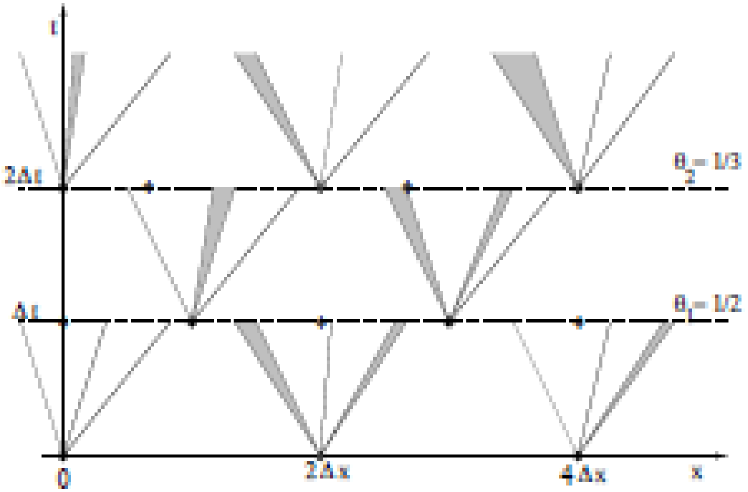
Then there exists a constant $\delta > 0$ such that, for every initial condition $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ with

$$\text{Tot.Var.}(\bar{u}) \leq \delta,$$

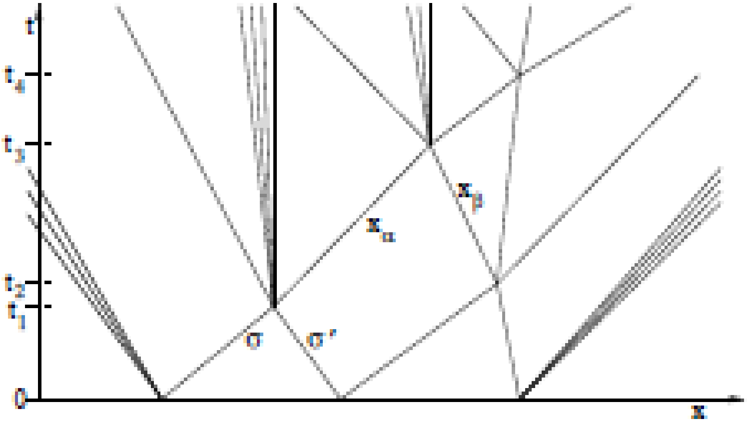
the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ defined for all $t \geq 0$.

Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in t-x plane (Glimm scheme)

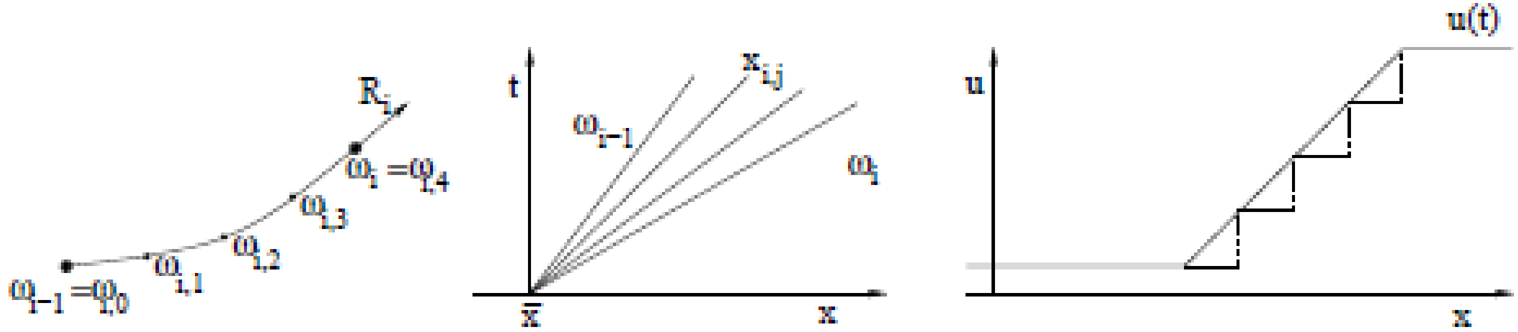


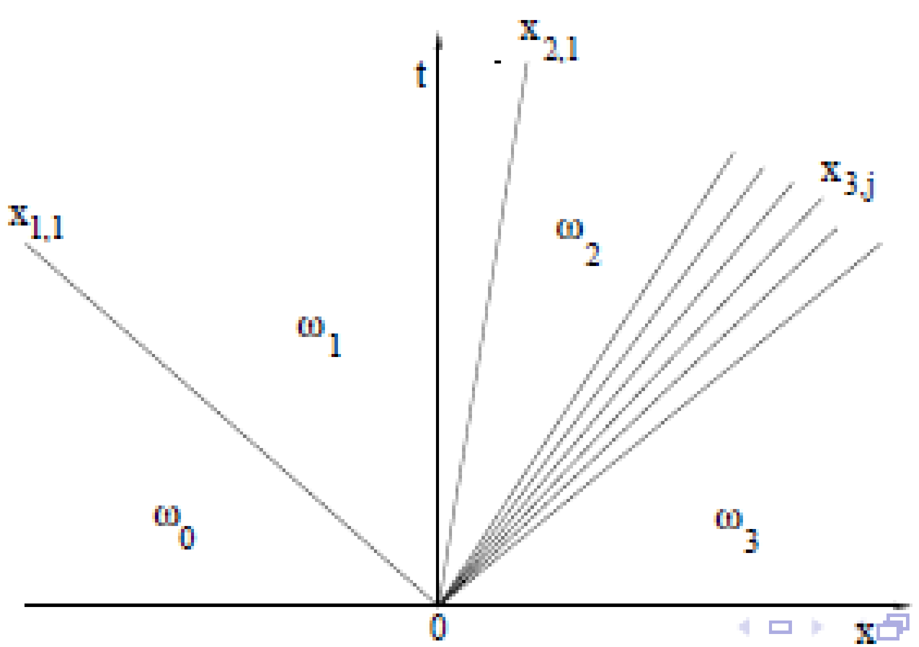
- at points where fronts interact (front tracking)



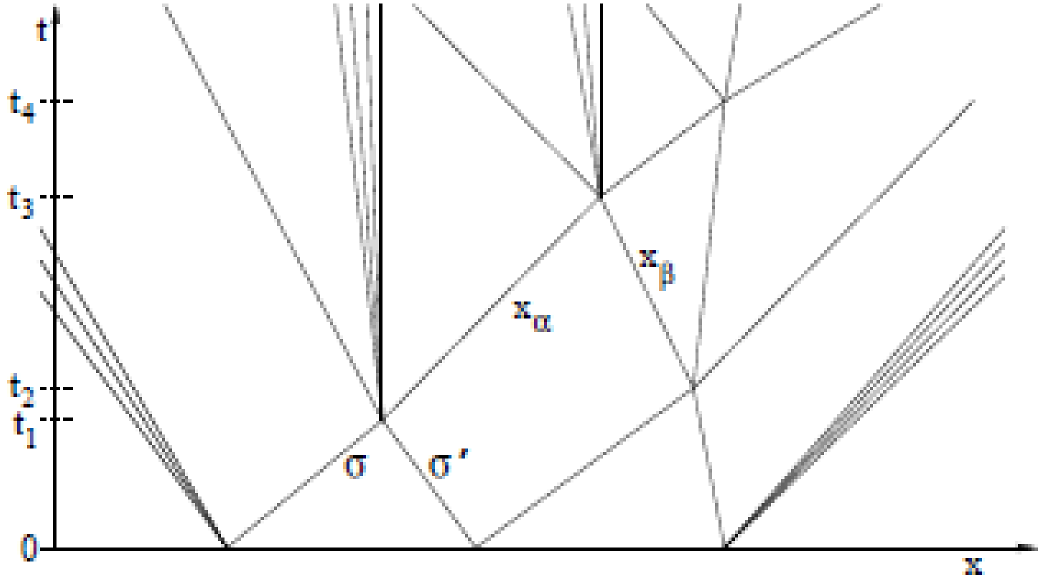
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans





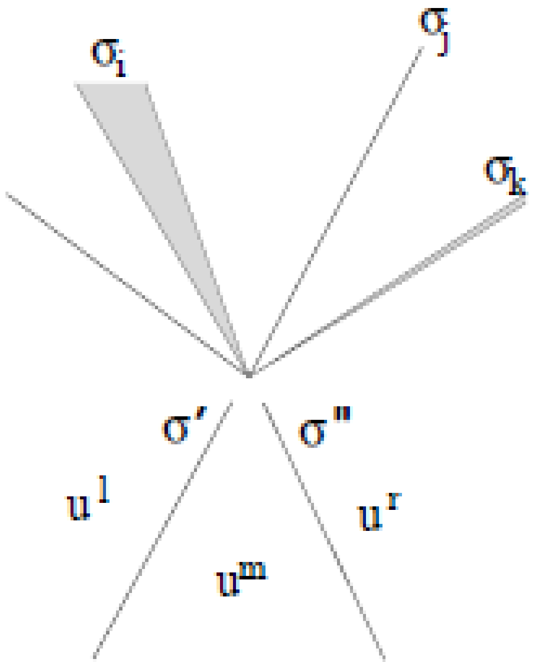
Front Tracking Approximations



- Approximate the initial data $\bar{u}$ with a piecewise constant function
- Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$
- at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...
- NEED TO CHECK: $\begin{cases} \text{- total variation remains small} \\ \text{- number of wave fronts remains finite} \end{cases}$

Interaction estimates

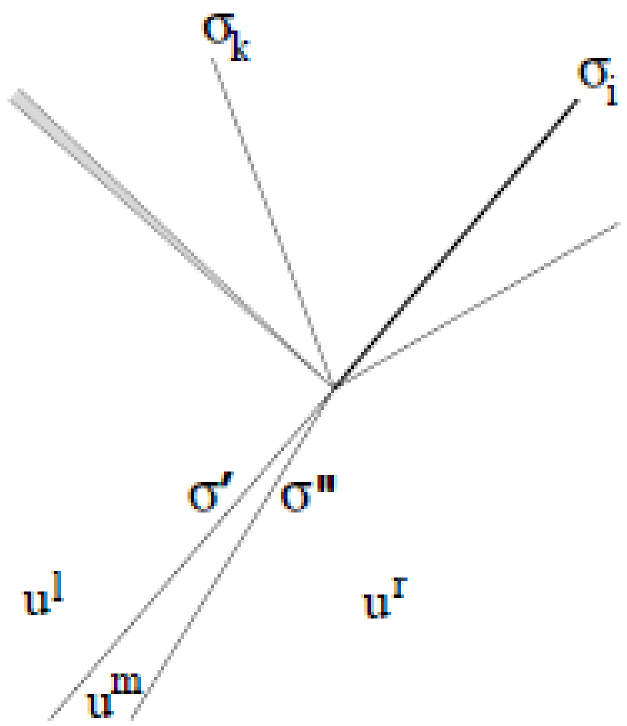
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j-wave of strength σ' and an i-wave of strength σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$



Incoming: two i-waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

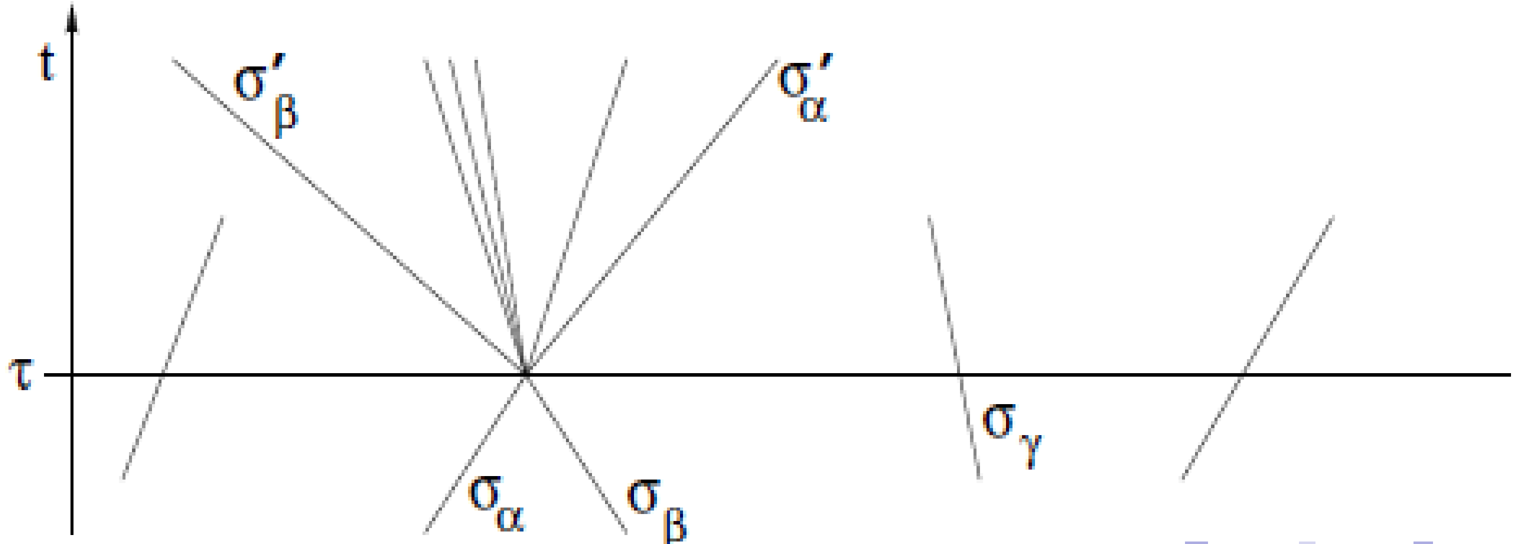
$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| \; = \; \mathcal{O}(1) \cdot |\sigma' \sigma''| \Big(|\sigma'| + |\sigma''| \Big)$$

Glimm functionals

Total strength of waves: $V(t) \doteq \sum_{\alpha} |\sigma_\alpha|$

Wave interaction potential: $Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|$

$\mathcal{A} \doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) \;=\; \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) \;=\; -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

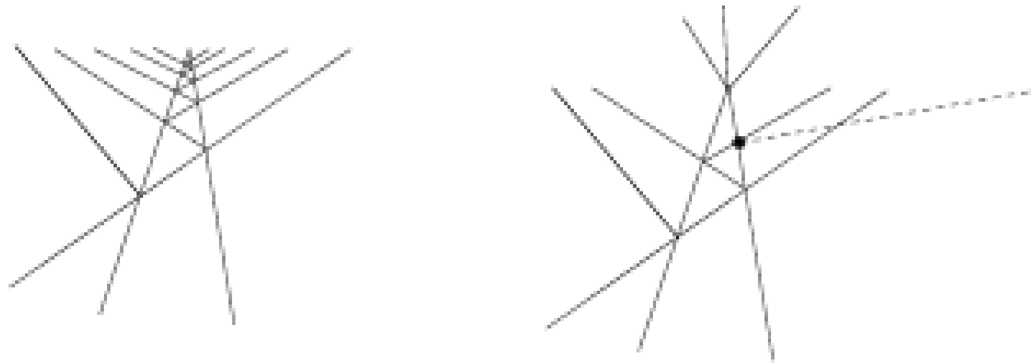
is nonincreasing, as long as V remains small

Total variation initially small $\quad\Longrightarrow\quad$ global BV bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \;\leq\; V(t) \;\leq\; V(0) + C_0 Q(0)$$

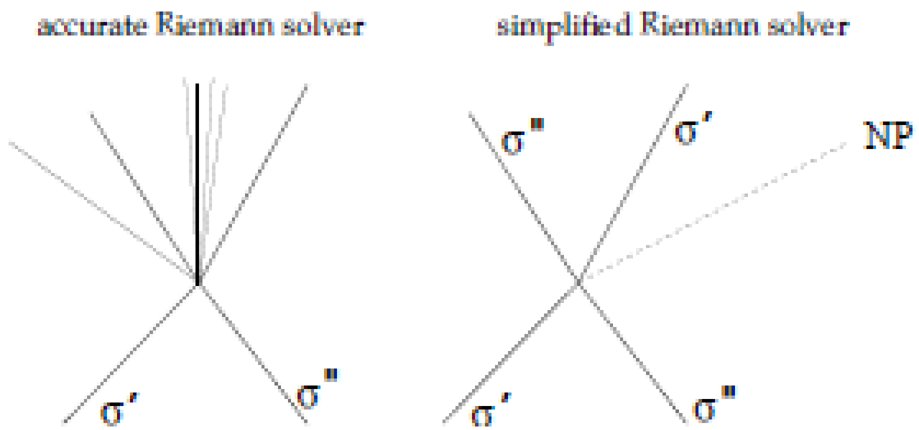
Front tracking approximations can be constructed for all $t \geq 0$

Keeping finite the number of wave fronts



At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

The total number of fronts can become infinite in finite time



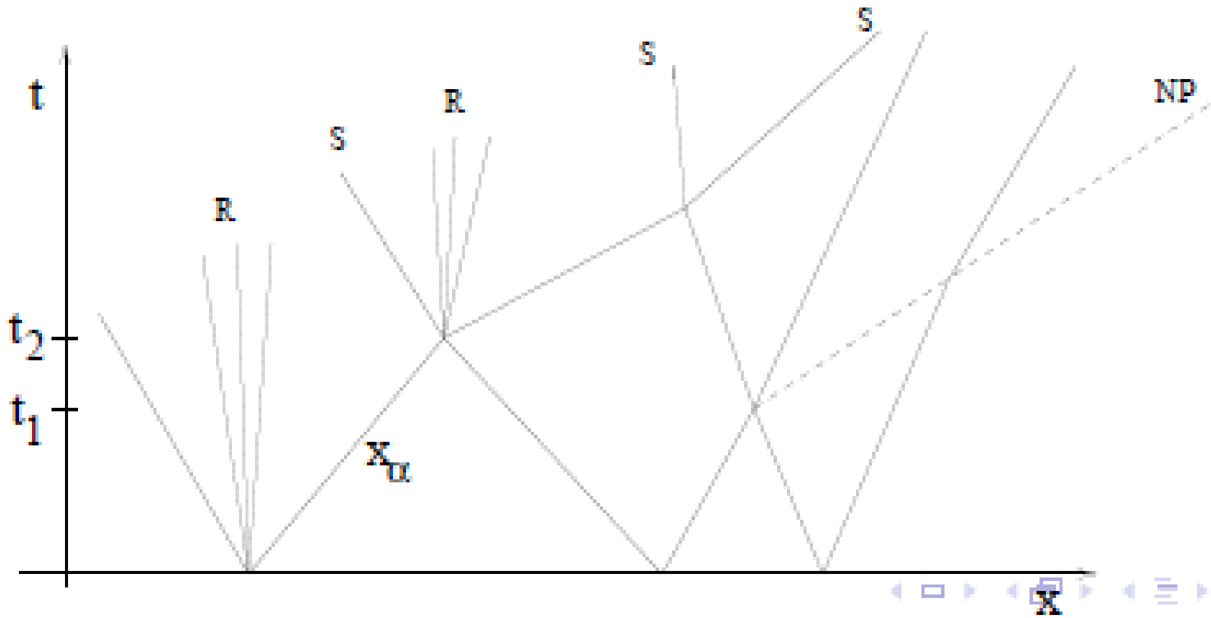
Need: a **Simplified Riemann Solver**, producing only one *"non-physical"* front

A sequence of approximate solutions

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

$(u_\nu)_{\nu \geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon_\nu \rightarrow 0$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t,\cdot)$ is $\leq \varepsilon_\nu$



Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C$$

$$\begin{aligned} \big\|u_\nu(t) - u_\nu(s)\big\|_{\mathbf{L}^1} &\leq (t-s)\cdot[\text{total strength of all wave fronts}]\cdot[\text{maximum speed}] \\ &\leq L\cdot(t-s) \end{aligned}$$

Helly's compactness theorem $\Longrightarrow$ a subsequence converges

$$u_\nu \to u \qquad\qquad \text{in } \mathbf{L}^1_{loc}$$

Claim: $u = \lim_{\nu \to \infty} u_\nu$ is a weak solution

$$\iint \left\{ \phi_t u + \phi_x f(u) \right\} dxdt = 0 \qquad \phi \in \mathcal{C}_c^1\Big( ]0, \infty[\times \mathbb{R} \Big)$$

Need to show:

$$\lim_{\nu \to \infty} \iint \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} dxdt = 0$$

$$\int_0^{\infty}\int_{-\infty}^{\infty}\left\{\phi_t(t,x)u_\nu(t,x)+\phi_x(t,x)f\left(u_\nu(t,x)\right)\right\}dxdt$$
$$=\sum_j\int_{\partial\Gamma_j}\Phi_\nu\cdot\mathbf{n}\,d\sigma$$

$$\limsup_{\nu\to\infty}\left|\sum_j\int_{\partial\Gamma_j}\Phi_\nu\cdot\mathbf{n}\,d\sigma\right|$$

$$\leq\ \limsup_{\nu\to\infty}\left|\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}}\left[\dot{x}_\alpha(t)\cdot\Delta u_\nu(t,x_\alpha)-\Delta f\left(u_\nu(t,x_\alpha)\right)\right]\phi(t,x_\alpha(t))\right|$$

$$\leq\ \left(\max_{t,x}\left|\phi(t,x)\right|\right)\cdot\limsup_{\nu\to\infty}\left\{\mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{R}}\varepsilon_\nu|\sigma_\alpha|\ +\ \mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{NP}}|\sigma_\alpha|\right\}$$

$$=\ 0$$

The Glimm scheme

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0,1]$

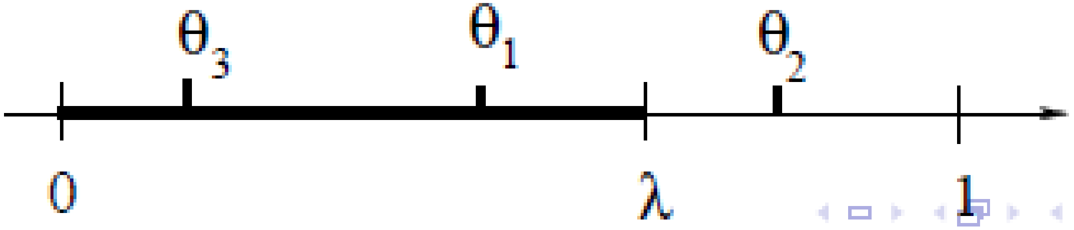
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \qquad \tau = 2Mt$$

Choose:

- a grid in the t-x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over $[0,1]$

$$\lim_{N\to\infty} \frac{\#\{j\,;\ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda \in [0,1].$$

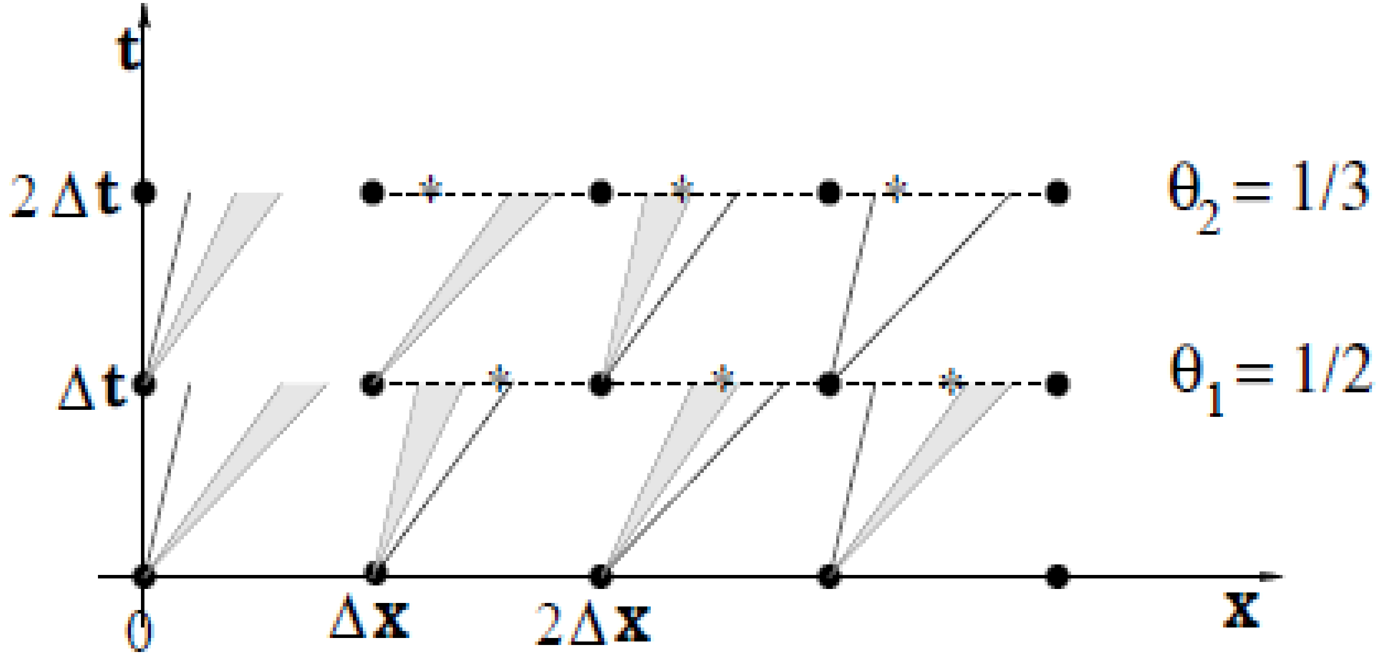


Glimm approximations

Grid points : $x_j = j \cdot \Delta x, \quad t_k = k \cdot \Delta t$

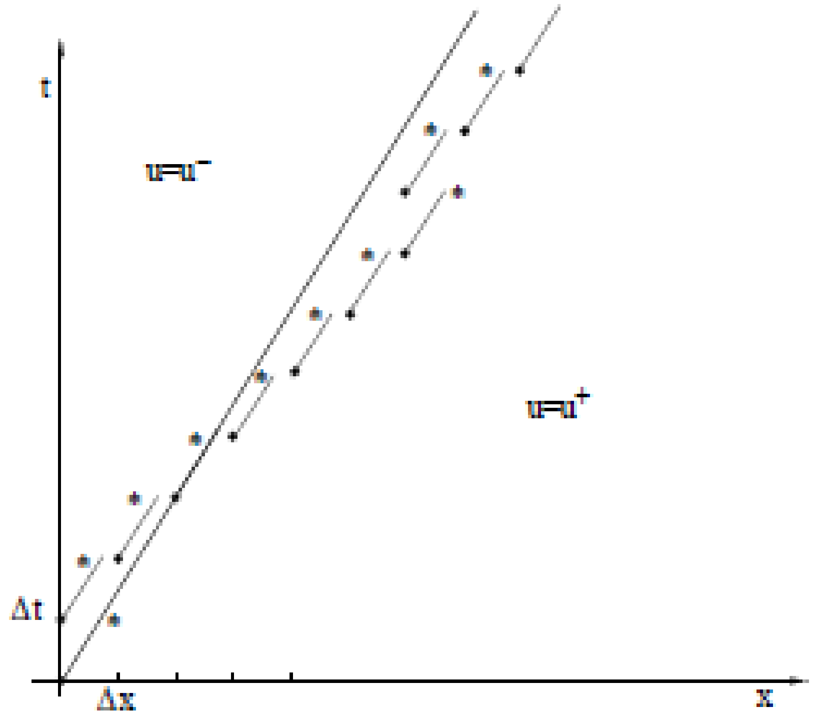
• for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j. The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$

• at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t \\ u^- & \text{if} \quad x < \lambda t \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) = \#\{j\ ;\ \ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \} \cdot \Delta t$$

$$= \frac{\#\{j\ ;\ \ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} \cdot T \ \to\ \lambda T \qquad \text{as} \ \ N \to \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k\geq 1}$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1\leq j\leq N,\ \ \theta_j\in[0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda\in[0,1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1\,, \quad \ldots \quad , \quad \theta_{759} = 0.957\,, \quad \ldots \quad , \quad \theta_{39022} = 0.22093\,, \quad \ldots$$

Convergence rate:

$$\lim_{\Delta x\to 0} \frac{\big\| u^{\rm Glimm}(T,\cdot) - u^{\rm exact}(T,\cdot)\big\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot|\ln \Delta x|} = 0$$

(A.Bressan & A.Marson, 1998)

Bressan, A.: **Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem**. Oxford University Press: Oxford, 2000.