Elliptic-Hyperbolic Partial Differential Equations

Mathematics Taught Course Centre
Michaelmas Term
22 October – 3 December 2013
Tuesday 14:00-16:00

By Prof. Gui-Qiang G. Chen
Dr. Wei Xiang

If you have not registered, please also email to

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In order to receive credits, you should write a miniproject (4-8 pages) after the end of the course on some (your favorite) topic which the course will cover.

Oxford grades: pass/fail, or distinction for particularly good work.

Course Homepage:
http://people.maths.ox.ac.uk/chengq/teach/tcc13/tcc-ehpde.html

Elliptic-Hyperbolic PDEs

arise naturally in Mechanics, Mathematical Physics, Analysis, Differential Geometry, Biology, Economics, …

The solution of some fundamental issues in the areas greatly requires a deep understanding of nonlinear mixed elliptic-hyperbolic PDEs:

Engineering, Physics, Biology, Economics, …

- **An introduction to some facets of techniques, approaches, and their applications to mixed PDEs.**
- **Prerequisite: Basic analysis and PDE theory.**
- **Some familiarity with nonlinear PDE theory, fluid mechanics, and differential geometry is desirable.**

Topics:

1. Introduction

2. Linear Degenerate Elliptic Equations
   [Elliptic Euler-Poisson-Darboux equations, …]

3. Nonlinear Degenerate Elliptic Equations

4. Fixed Point Theorems, Degree Theory, and
   Their Applications

5. Nonlinear Conservation Laws of Mixed Type and
   Shock Reflection-Diffraction Problems

6. Mathematics of Shock Reflection-Diffraction

7. Further Topics on PDEs of Mixed Elliptic-Hyperbolic Type

References:

1. *G.-Q. Chen and M. Feldman: **Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures,** Annals of Mathematics Studies, Princeton University Press, 2014.

2. R. Courant and D. Hilbert: **Methods of Mathematical Physics,** Vol. II. Reprint of the 1962 original. John Wiley & Sons, Inc.: New York, 1989.

3. C. M. Dafermos: **Hyperbolic Conservation Laws in Continuum Physics,** Third edition. Springer-Verlag: Berlin, 2010.

4. L. C. Evans: **Partial Differential Equations**, 2nd edition. AMS: 2010

5. *R. Courant and K. O. Friedrichs: **Supersonic Flow and Shock Waves,** 1948

6. *L. Bers: **Mathematical Aspects of Subsonic &Transonic Gas Dyn.,** 1958

7. *T. H. Otway: **The Dirichlet Problem for Elliptic-Hyperbolic Equations of Keldysh Type,** Lecture Notes in Mathematics, 2043, Springer: Berlin, 2012

8. D. Gilbarg and N. S. Trudinger: **Elliptic Partial Differential Equations of Second Order,** Springer: Berlin-Heidelberg-New York, 1998.

9. P. D. Lax: **Hyperbolic Differential Equations,** AMS: Providence, 2000

10. *A. Bressan, G.-Q. Chen, M. Lewicka, D. Wang: **Nonlinear Conservation Laws and Applications,** IMA Volume 153, Springer: New York, 2011.

Linear Partial Differential Equations

- Two of the Basic Types: Representatives

Hyperbolic: $\partial_{tt}u - \Delta_{\mathbf{x}}u = 0$ (Wave Equation)

Elliptic: $\Delta_{\mathbf{x}}u = 0$ (Laplace's Equation)

$$(\mathbf{x} = (x_1, \cdots, x_n), \quad \Delta_{\mathbf{x}} = \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j})$$

- Mixed Hyperbolic-Elliptic Type

Lavrentyev-Bitsadze Eq.: $\partial_{xx}u + \text{sign}(x)\partial_{yy}u = 0$

Tricomi Eq.: $\partial_{xx}u + x\partial_{yy}u = 0$ (hyperbolic degeneracy at $x = 0$)

Keldysh Eq.: $x\partial_{xx}u + \partial_{yy}u = 0$ (parabolic degeneracy at $x = 0$)

Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type I:

Shock Reflection-Diffraction Problems in Fluid Mechanics

Bow Shock in Space generated by a Solar Explosion

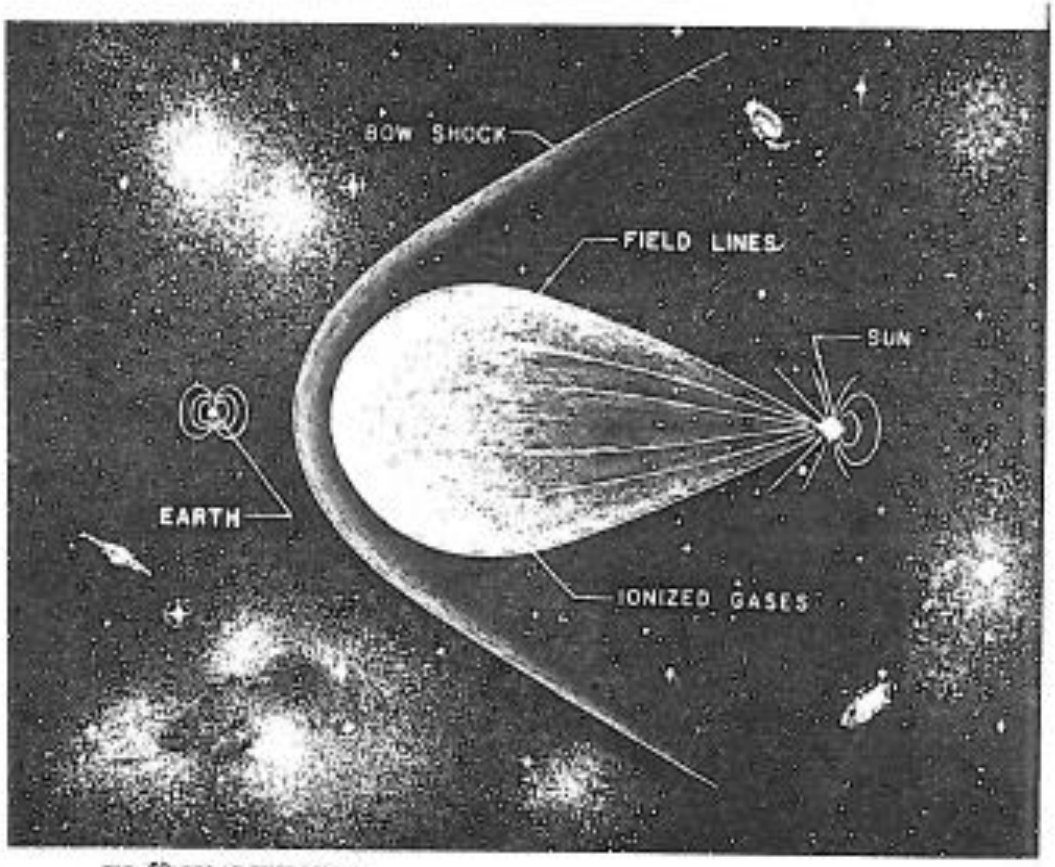


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Blast Wave from a TNT Surface Explosion

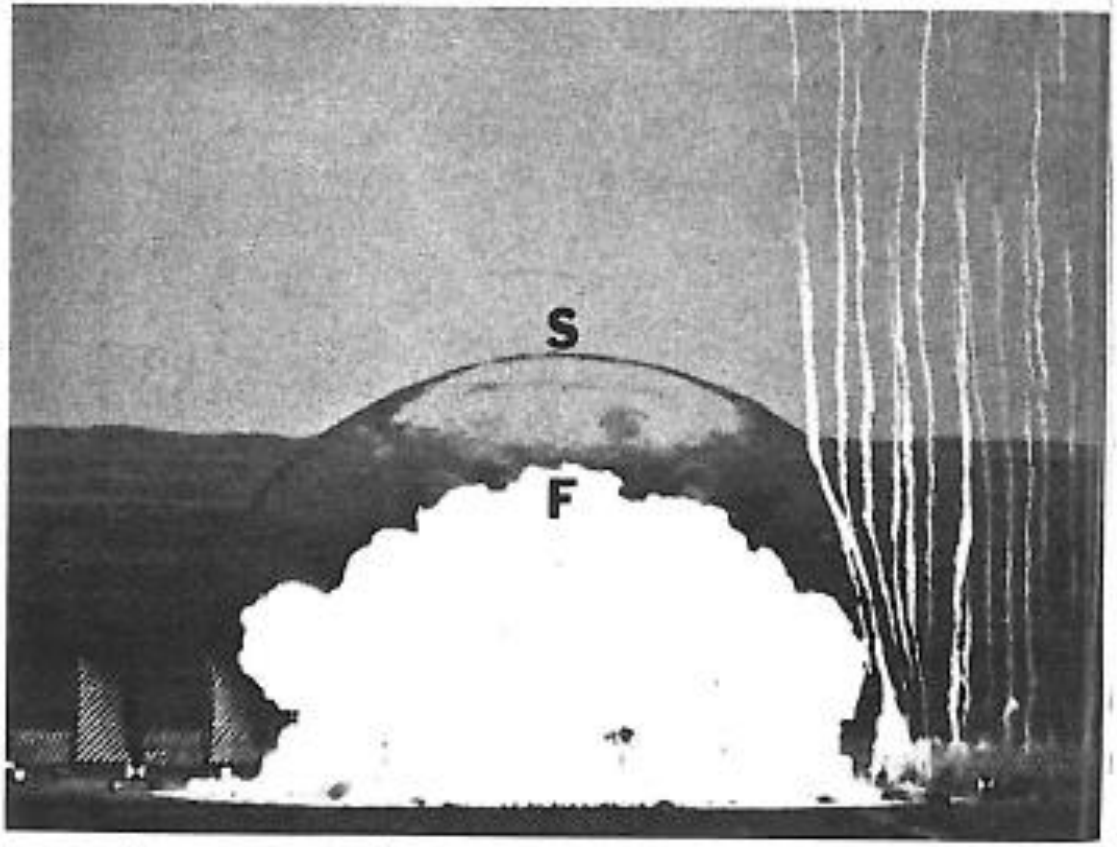
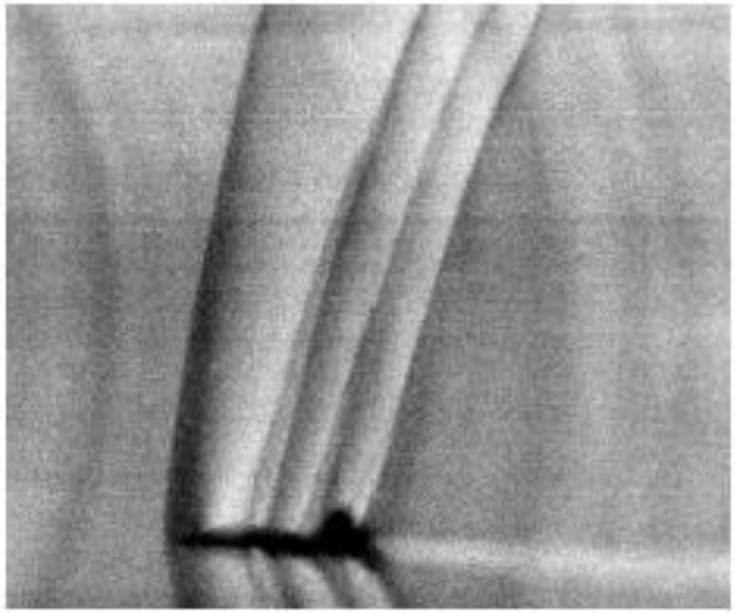


FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT

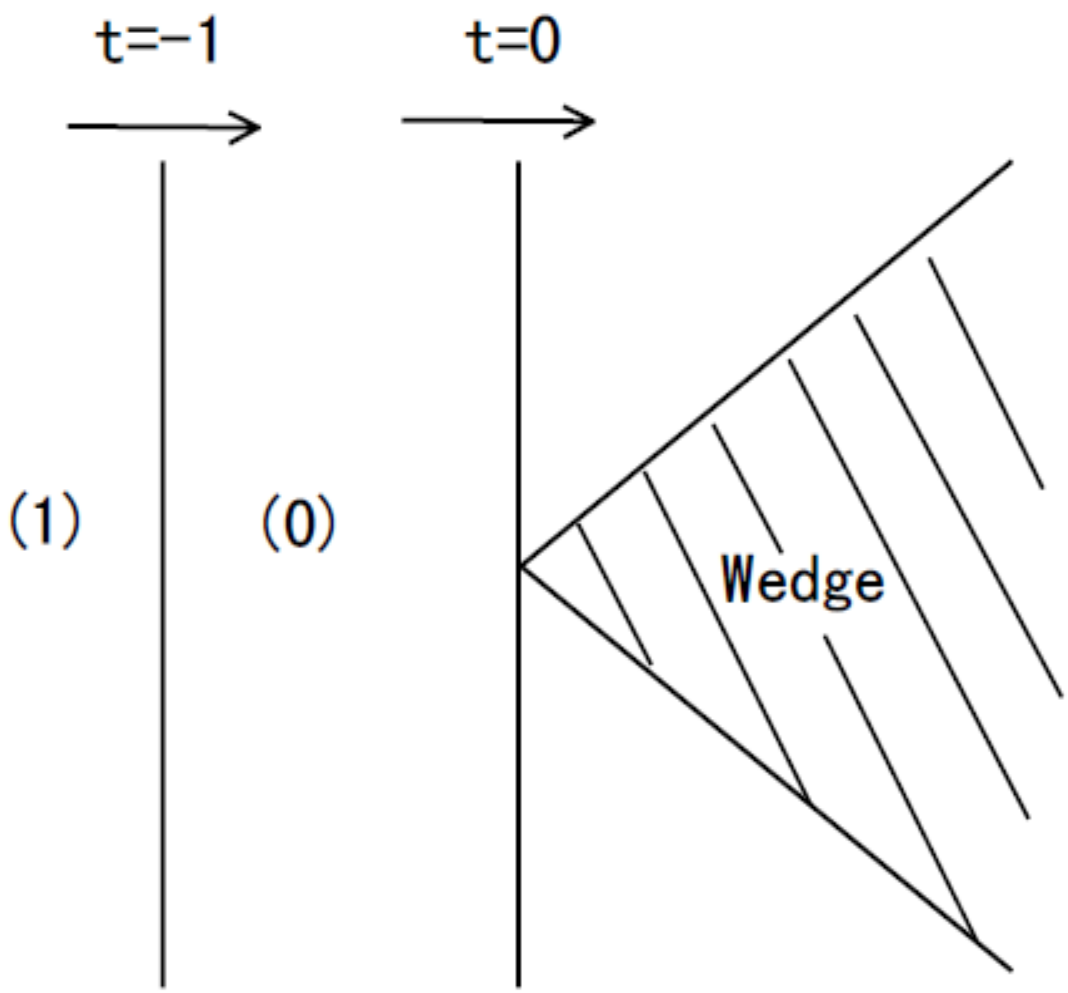
The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities.

Shock Waves generated when U.S. Navy F/A-18 Breaking the Sound Barrier: August 19, 2007

Shock Waves generated by Supersonic Aircrafts



Shock Reflection-Diffraction

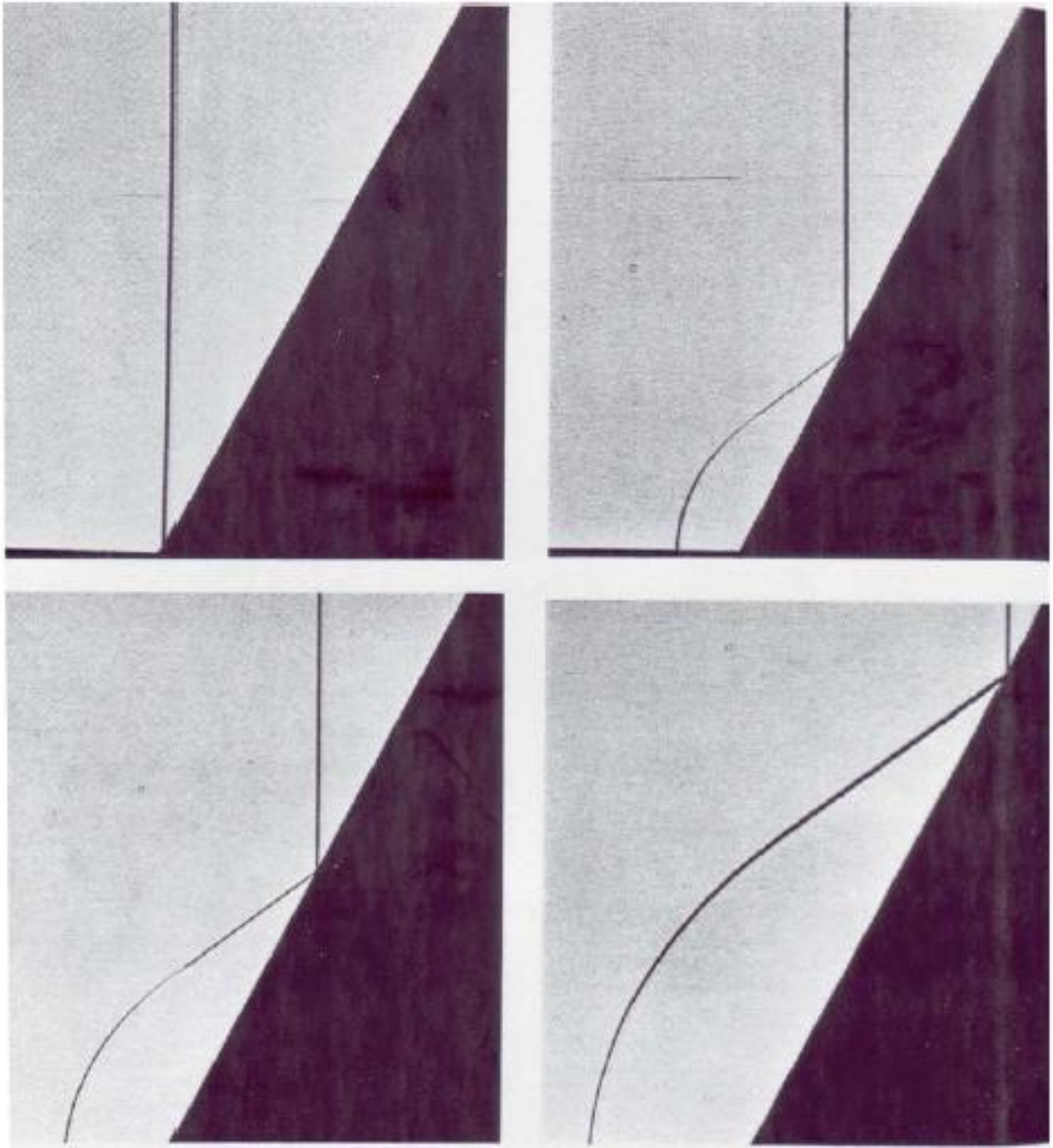


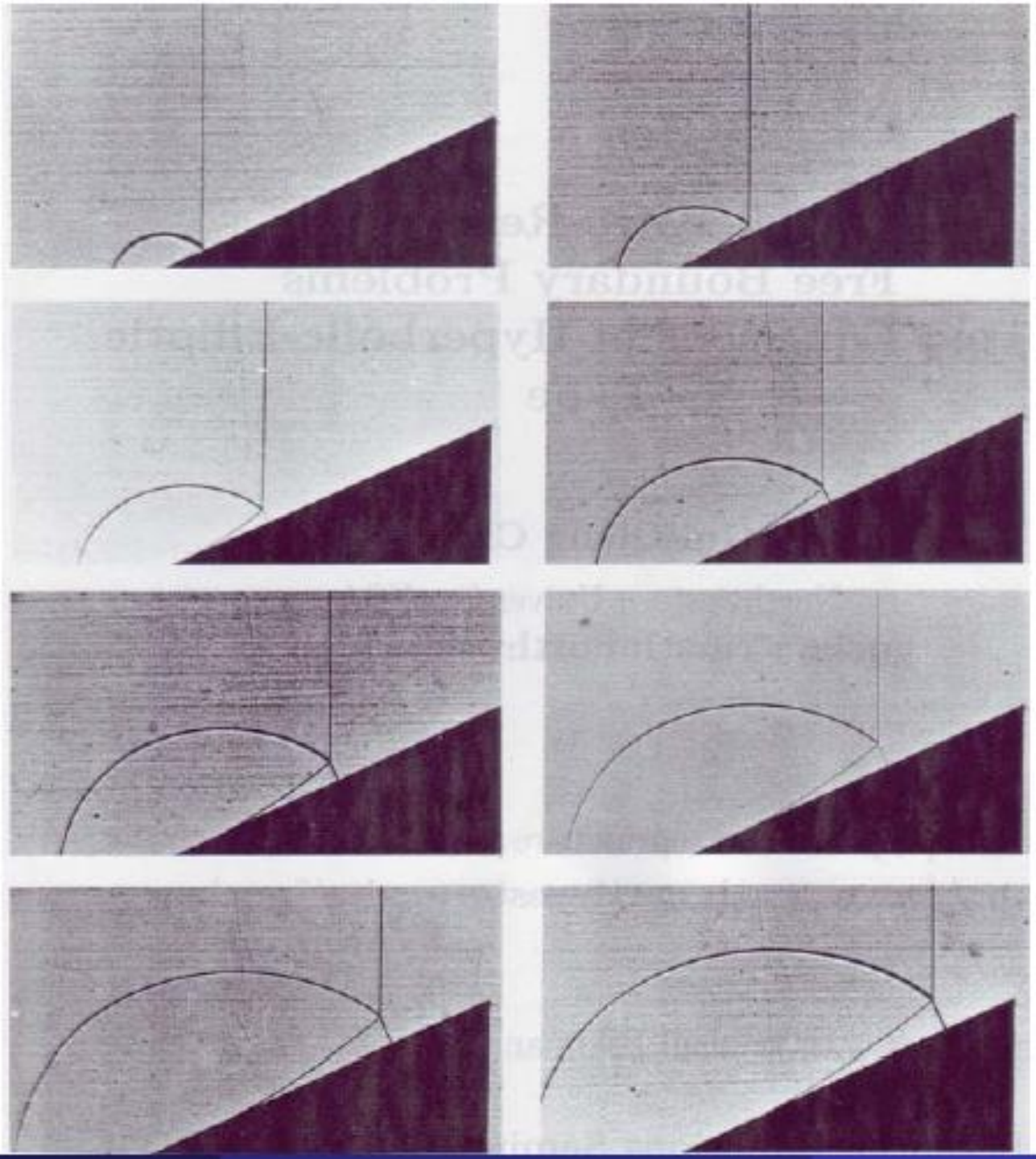
? **Shock Wave Patterns** around a Wedge (airfoils, inclined ramps, $\cdots$)

Ernst Mach: 1878

Complexity of Reflection-Diffraction Configurations:

Über den verlauf von funkenwellenin der ebene und im raume,
Sitzungsber. Akad. Wiss. Wien, **78** (1878), 819–838.





John von Neumann: 1943

1. **Oblique Reflection of Shocks**, Explos. Res. Rep. **12** (1943), Navy Dept., Bureau of Ordnance, Washington, DC., USA.
2. **Refraction, Intersection, and Reflection of Shock Waves**, NAVORD Rep. **203-45** (1945), Navy Dept., Bureau of Ordnance, Washington, DC, USA.
3. **Collected Works**, Vol. **6**, Pergamon Press, 1963.

Richard Courant and Kurt Otto Friedrichs: 1948

Supersonic Flow and Shock Waves,
Springer-Verlag: New York, 1948. xvi+464 pp.

Experimental Analysis: 1940s–

Walker Bleakney: Palmer Physical Laboratory
Princeton University, USA

Irvine Israel Glass: Institute for Aerospace Studies
University of Toronto, Canada

LeRoy Freame Henderson: School of Aerospace, Mechanical and
Mechatronic Engineering, University of Sydney, Australia

Tatiana V. Bazhenova: Joint Institute of High Temperatures
Russian Academy of Sciences, Moscow, Russia

Kazuyoshi Takayama: Institute of Fluid Science
Tohoku University, Japan

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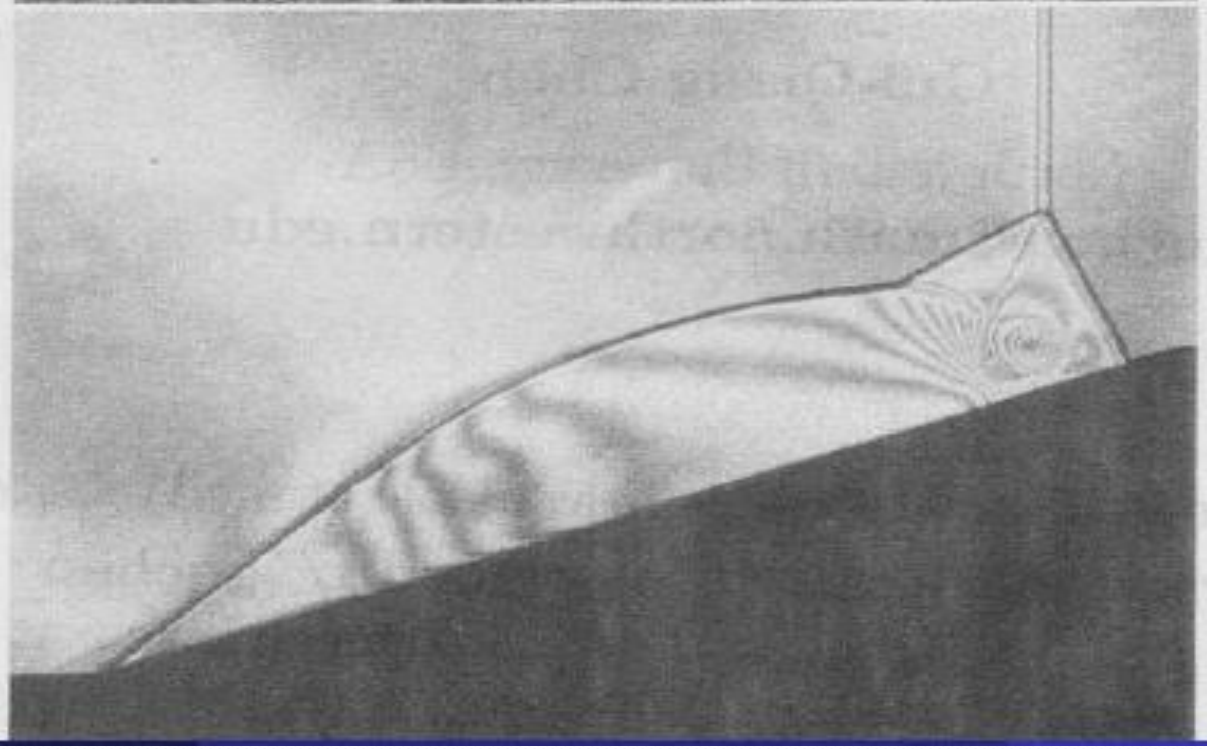
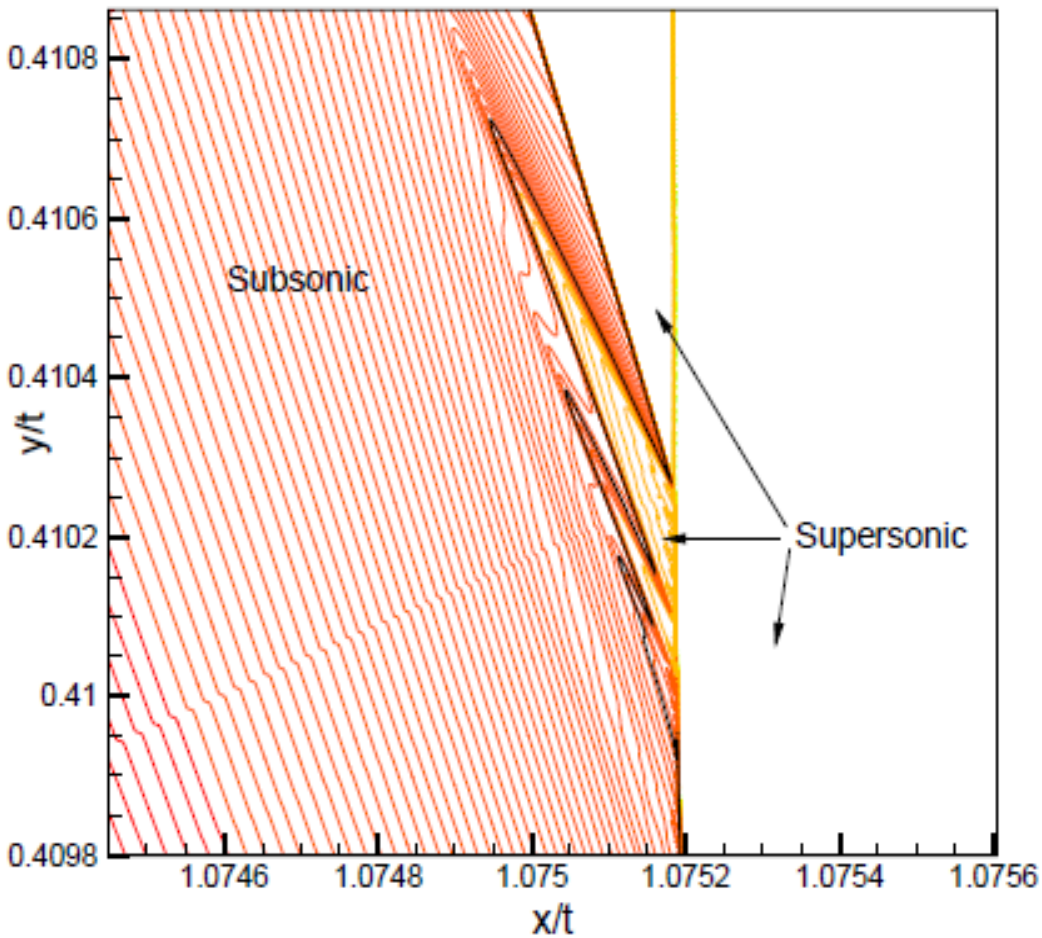
a)

b)





A New Mach Reflection-Diffraction Pattern:

A. M. Tesdall and J. K. Hunter: TSD, 2002

A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008

B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114

Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor** Shock Wave Reflection Phenomena
  Springer-Verlag: New York, 307 pages, 1992.

  Experimental results before 1991
  Various proposals for transition criteria

- **Peter O. K. Krehl** History of Shock Waves, Explosions and Impact
  A Chronological and Biographical Reference
  2009, XXII, 1288 p. 1200 illus., 300 in color.

- **Milton Van Dyke** An Album of Fluid Motion
  The parabolic Press: Stanford, 176 pages, 1982.

  Various photographs of shock wave reflection phenomena

Scientific Issues

- **Structure of the Shock Reflection-Diffraction Patterns**
- **Transition Criteria among the Patterns**
- **Dependence of the Patterns on the Parameters**
  wedge angle θ_w, adiabatic exponent $\gamma \geq 1$
  incident-shock-wave Mach number M_s
- $\cdots\cdots$

Interdisciplinary Approaches:

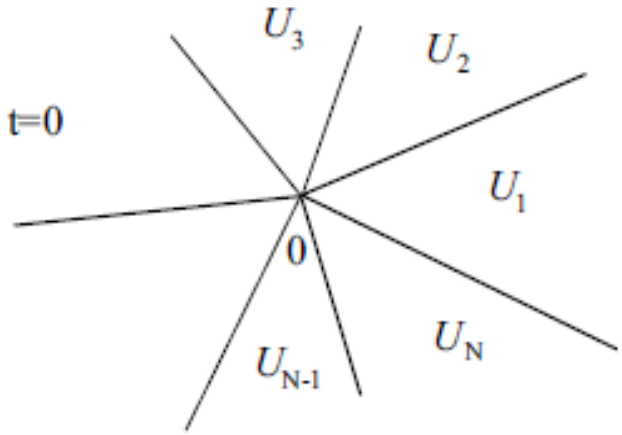
- **Experimental Data and Photographs**
- **Large or Small Scale Computing**
  Colella, Berger, Deschambault, Glass, Glaz, Woodward,....
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  Yu. Dem'yanov, Panasenko, ...
- **Asymptotic Analysis**
  Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian...
  Morawetz: CPAM 1994
- **Rigorous Mathematical Analysis??** **(Global Solutions)**
  Existence, Stability, Regularity, Bifurcation, $\cdots\cdots$

2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

or

$$\partial_t \mathbf{A}(U, U_t, \nabla_{\mathbf{x}} U) + \nabla_{\mathbf{x}} \cdot \mathbf{B}(U, U_t, \nabla_{\mathbf{x}} U) = 0$$



Books and Survey Articles:
Chang-Hsiao 1989, Glimm-Majda 1991, Li-Zhang-Yang 1998, Zheng 2001
Chen-Wang 2002, Serre 2005, Chen 2005, Dafermos 2010, $\cdots$

Numerical Solutions: Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Lax-Liu 1998, Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000
Kurganov-Tadmor 2002, $\cdots$

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks

Riemann Solutions I

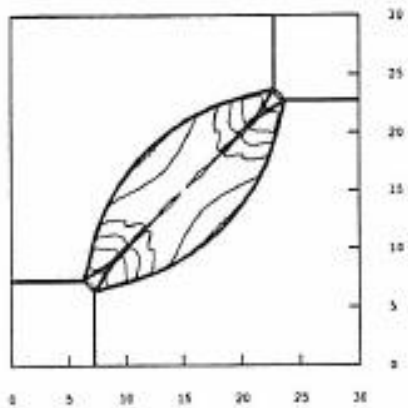


FIG. 5.5A
Density contour curves

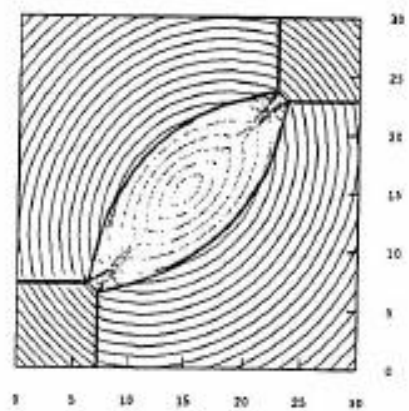


FIG. 5.5B
Self-Mach number contour curves

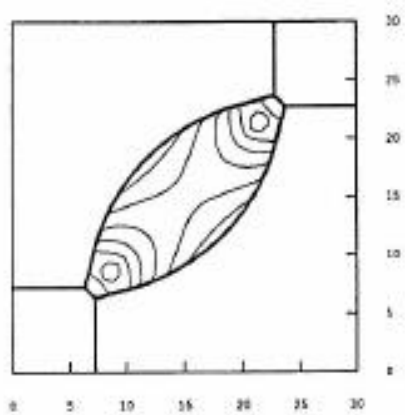


FIG. 5.5C. Pressure contour curves

Riemann Solutions II

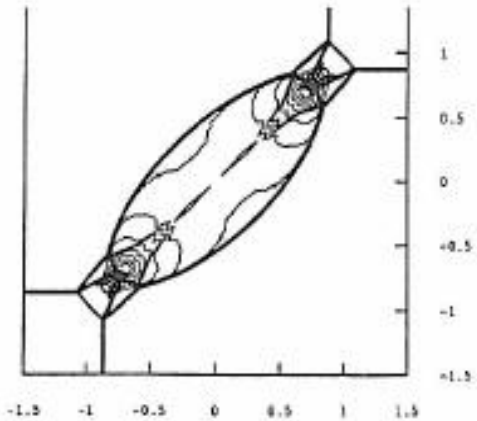


FIG. 5.6A
Density contour curves

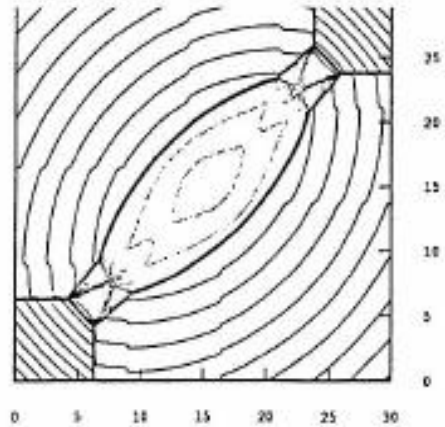


FIG. 5.6B
Self-Mach number contour curves

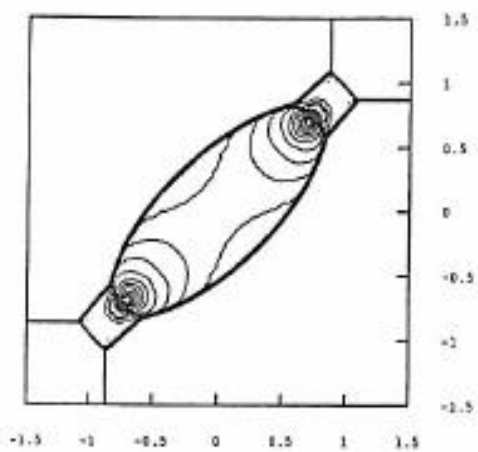


FIG. 5.6C. Pressure contour curves

Full Euler Equations: $(t, \mathbf{x}) \in \mathbb{R}^3_+ := (0, \infty) \times \mathbb{R}^2$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\ \partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\ \partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 & \text{(conservation of energy)} \end{cases}$$

Constitutive Relations: $p = p(\rho, e)$

- ρ–density, $\mathbf{v} = (v_1, v_2)^\top$–fluid velocity, p–pressure
- e–internal energy, θ–temperature, S–entropy

For a polytropic gas: $p = (\gamma - 1)\rho e, \quad e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v} > 1$

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = e(\rho, S) \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v},$$

- $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- $c_v > 0$ is the specific heat at constant volume
- $\gamma > 1$ is the adiabatic exponent, $\kappa > 0$ is any constant under scaling

Euler Equations for Potential Flow: $\mathbf{v} = \nabla\Phi$, $\gamma > 1$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0, & \text{(Conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, & \text{(Bernoulli's law)} \end{cases}$$

or, equivalently,

$$\partial_t \rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi\big) = 0,$$

with

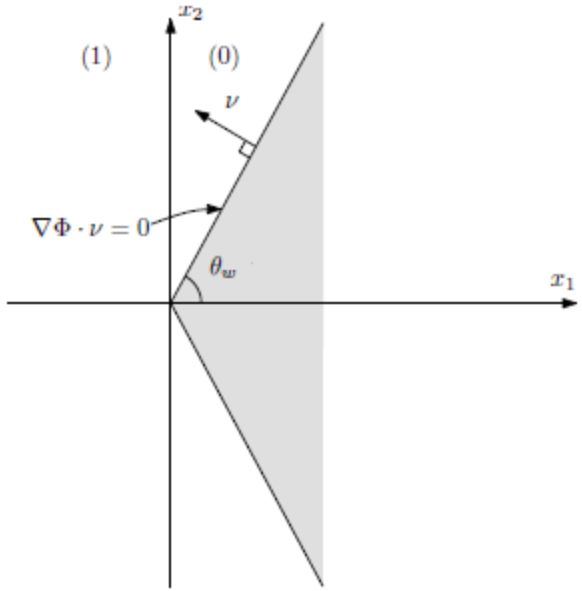
$$\rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

- The potential flow equations and the full Euler equations coincide in important regions of the solution and are very close each other in the other regions in the configuration of regular shock reflection-diffraction.
- Aerodynamics/Gas Dynamics: Fundamental PDE
- J. Hadamard: Leçons sur la Propagation des Ondes, Hermann: Paris 1903
  Morawetz: CPAM 1994, Majda-Thomann: CPDE 1987, $\cdots\cdots$

Initial-Boundary Value Problem: $0 < \rho_0 < \rho_1,\ u_1 > 0$

Initial condition: $(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0), & |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (\rho_1, u_1 x_1), & x_1 < 0; \end{cases}$

Slip boundary condition on the wedge boundary: $\nabla\Phi \cdot \nu = 0$



Invariant under the Self-Similar Scaling:

$$(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \longrightarrow (\rho, \Phi/\alpha) \qquad \alpha \neq 0$$

Seek Self-Similar Solutions: $(\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t})$

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\big(\varphi(\xi, \eta) + \tfrac{1}{2}(\xi^2 + \eta^2)\big)$$

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \Longrightarrow \Delta\varphi + 2 = 0$
- Elliptic: $|\nabla\varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$
- Hyperbolic: $|\nabla\varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$

Second-order nonlinear equations of mixed hyperbolic-elliptic type

Near the Sonic Circle with rescaled coordinates (x,y):

$$\left(2x - (\gamma+1)\psi_x\right)\psi_{xx} + \frac{1}{c_s^2}\psi_{yy} - \psi_x \sim o(x^2)$$

Ellipticity: $\psi_x \leq \frac{2x}{\gamma+1}$

Apriori Estimate: $|\psi_x| \leq \frac{4x}{3(\gamma+1)}$

Asymptotics: ψ "$\sim$" $\frac{x^2}{2(\gamma+1)} + h.o.t.$ when $x \approx 0$

Important Nonlinear Models and Equations of Mixed Type

- **Transonic Small Disturbance Equation**:

$$((u-x)u_x + \frac{u}{2})_x + u_{yy} = 0$$

or, for $v = u - x$,

$$v\, v_{xx} + v_{yy} + \text{l.o.t.} = 0$$

Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, $\cdots$

- **Nonlinear Wave System**: Canic-Keyfitz-Kim, $\cdots$
  Pressure Gradient System: Y. Zheng, $\cdots$

- **Steady Potential Flow Equation of Aerodynamics**

$$\nabla \cdot (\rho(\nabla\varphi, \rho_0)\nabla\varphi) = 0$$

  - **Elliptic:** $|\nabla\varphi| < c_*(\rho_0) := \sqrt{\frac{2}{\gamma+1}\rho_0^{\gamma-1}}$
  - **Hyperbolic:** $|\nabla\varphi| > c_*(\rho_0)$

Steady Potential Flow Equation of Aerodynamics

$$\nabla \cdot (\rho(\nabla\varphi, \rho_0)\nabla\varphi) = 0$$

- **Pure Elliptic Case: Subsonic Flow past an Obstacle**
  Shiffman, L. Bers, Finn-Gilbarg, G. Dong, $\cdots$
- **Degenerate Elliptic Case: Subsonic-Sonic Flows**
  Shiffman, Chen-Dafermos-Slemrod-Wang, Elling-Liu, Xin, $\cdots$
- **Pure Hyperbolic Case (even Full Euler Eqs.)**:
  Gu, Li, Schaeffer, S. Chen, Xin-Yin, Y. Zheng, $\cdots$
  T.-P. Liu-Lien, S. Chen-Zhang-Wang, Chen-Zhang-Zhu, $\cdots$
- **Elliptic-Hyperbolic Mixed Case**
  Transonic Nozzles: Chen-Feldman, S. Chen, J. Chen, Yuan, Xin-Yin,...
  Wedge or Conical Body: S. Chen, B. Fang, Chen-Fang, $\cdots$
  Transonic Flow past an Obstacle: Morawetz, Chen-Slemrod-Wang,...

Mach Reflection: Self-Similar Solutions for the Full Euler Equations $(u,v,p,\rho)(t,\mathbf{x}) = (u,v,p,\rho)(\xi,\eta)$, $(\xi,\eta) = (\frac{x_1}{t}, \frac{x_2}{t})$

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U,V) = (u - \xi, v - \eta)$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$(repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,

where $c = \sqrt{\gamma p/\rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$, the system consists of

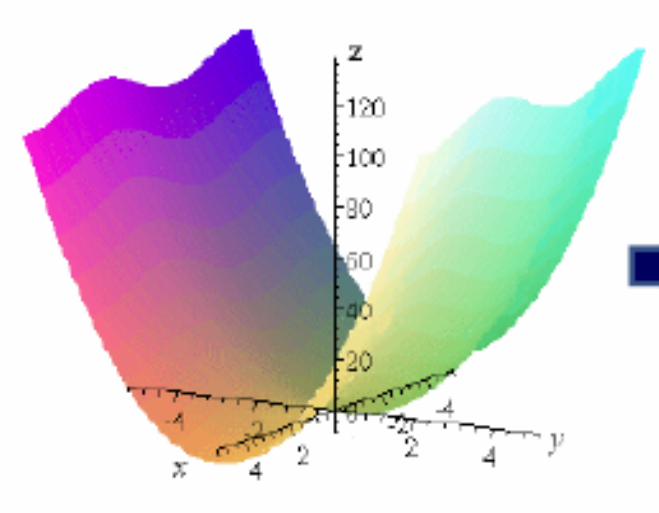
- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type: Two kinds of transonic flow: Transonic shocks and sonic curves

Nonlinear PDEs
of Mixed Elliptic-Hyperbolic Type II:

Isometric Embedding Problems

in Differential Geometry

Isometric Embedding Problems



g_{ij} ⟷ **Metric**

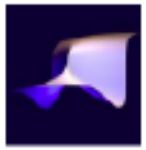
(First Fundamental Form: $\mathrm{I} = \sum g_{ij} dx^i dx^j$)

h_{ij} ⟷ **Curvatures**

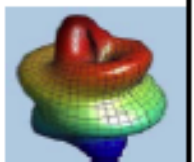
(Second Fundamental Form: $\mathrm{II} = \sum h_{ij} dx^i dx^j$)

Given a metric g_{ij} and certain curvatures

Inverse Problem: CAN we find a surface in our real world with this metric g_{ij} and corresponding curvatures?



Realization Question?



Question: CAN *we produce even more sophisticated surfaces or thin sheets?*

Fundamental

- Mathematics: Differential Geometry, Topology, ……
- Understanding evolution of sophisticated shapes of surfaces or thin sheets in nature, including

  --Elasticity, Materials Science, ……

  --Biology and Algorithmic Origami: Protein Folding, ……

  *US DARPA's 10th question of the 23 Challenge Questions in the Sciences [US Defense Advanced Research Project Agency]:

  Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.
- Design, Visual Arts, ……

History: *Schlaefli (1873), Darboux (1894), Hilbert (1901), Weyl (1916), Janet (1926-27), Cartan (1926-27), Lewy (1936), Nash (1954-56), Kuiper (1955), Yau (1980's, 1990's), Gromov (1970, 1986), Günther (1989), Poznyak (1973), Levi (1908), Heinz (1962), Alexandroff (1938, 1942), Pogorelov (late 1940's, 1972), Nirenberg (1953, 1963), Efimov (1963), Bryant-Griffiths-Yan (1983), Lin (1985-86), Hong (1991,1993), ……*

Nash Isometric Embedding Theorem

(C^k embedding theorem, $k \geq 3$)

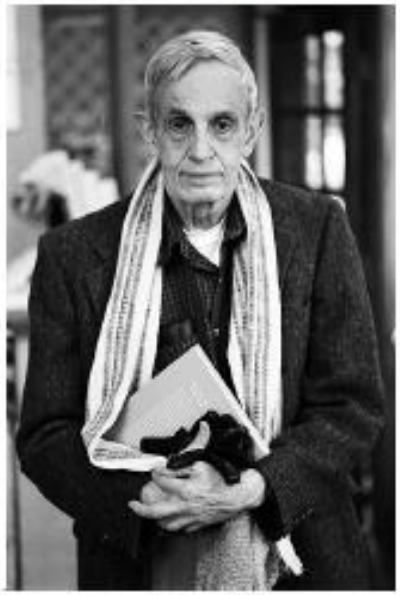
Every n-Dimensional Riemannian manifold (analytic or $C^k, k \geq 3$) can be C^k isometrically imbedded in the Euclidean space $\mathbb{R}^N$:

Compact Case: $N = 3s_n + 4n$

Noncompact Case: $N = (n+1)(3s_n + 4n)$

Gromov (1986): $N = s_n + 2n + 3$

Günther (1989): $N = \max\{s_n + 2n, s_n + n + 5\}$



the Subject of the Hollywood Movie *A Beautiful Mind*

Open Problems Important for Applications

Lowest Target Dimension? Janet-D: $N = s_n = \frac{n(n+1)}{2}$?

Optimal or Assigned Regularity?

$C^{1,1}$ Isometric Embedding? What about $BV(C^1)$?

Current Research Activities, ...

Efimov's Example (1966): No C^2 Isometric Embedding when $n = 2, s_n = 3$.

Gauss-Codazzi System: Compatibility/Constraint

Fundamental Theorem in Differential Geometry:
There exists a surface in $\mathbb{R}^3$ with 1st and 2nd fundamental form coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$, $\{g_{ij}\}$ being positive definite, provided that the coefficients satisfy the Gauss-Codazzi system.

*This theorem holds even when $h_{ij} \in L^p$ (Mardare 2003–05)

Given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Eqs. (Compatibility):

$$\begin{cases} \partial_x M - \partial_y L = L\Gamma_{22}^{(2)} - 2M\Gamma_{12}^{(2)} + N\Gamma_{11}^{(2)}, \\ \partial_x N - \partial_y M = -L\Gamma_{22}^{(1)} + 2M\Gamma_{12}^{(1)} - N\Gamma_{11}^{(1)}, \end{cases}$$

satisfying the Gauss Equation (Constraint):

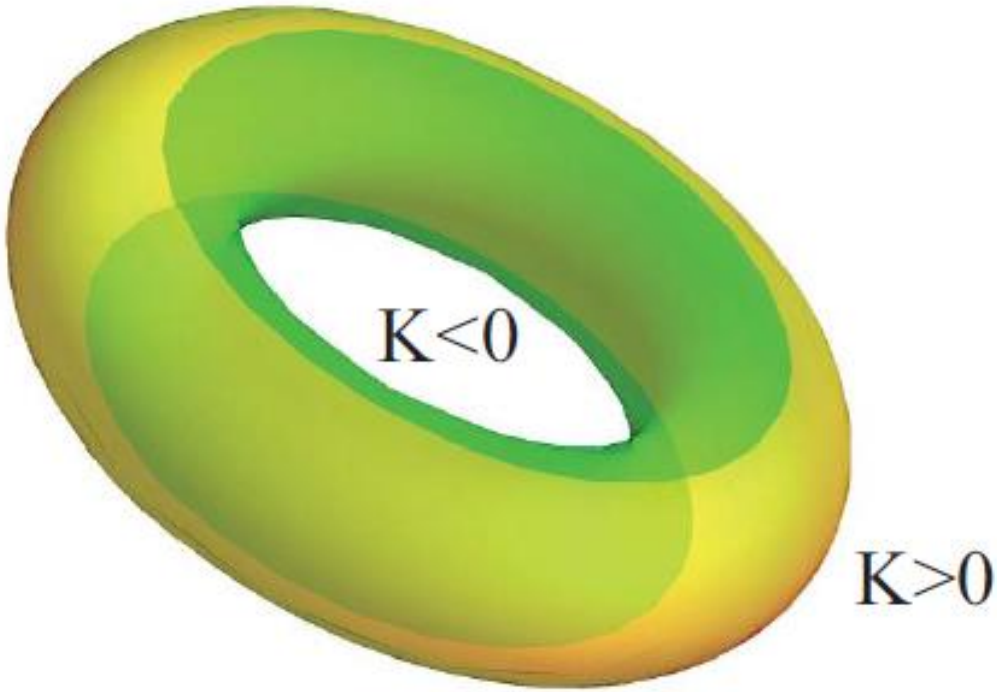
$$LN - M^2 = K,$$

where $\quad L = \frac{h_{11}}{\sqrt{|g|}}, \; M = \frac{h_{12}}{\sqrt{|g|}}, \; N = \frac{h_{22}}{\sqrt{|g|}}, \; |g| = g_{11}g_{22} - g_{12}^2$

$\Gamma_{ij}^{(k)}$—Christoffel symbols, depending on g_{ij} up to their 1st derivatives
$K(x,y)$—Gauss curvature, determined by g_{ij} up to their 2nd derivatives

***Nonlinear PDEs of Mixed Elliptic-Hyperbolic Equations: Sign of K**

Surfaces with Gauss Curvature of Changing Sign



Gauss Curvature K on a Torus:
Toroidal Shell or Doughnut Surface

Fluid Dynamics Formalism for Isometric Embedding

Set $\quad L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$
Choose p as the Chaplygin type gas: $\quad p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation:

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

$c^2 > q^2$ and the "flow" is subsonic when $K > 0$,
$c^2 < q^2$ and the "flow" is supersonic when $K < 0$,
$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

?? **Existence/Continuity of Isometric Embedding**

$\Leftarrow$ **Compensated Compactness and Entropy Analysis**

Chen-Slemrod-Wang: Commun. Math. Phys. 2010

Gauss-Codazzi-Ricci System for Isometric Embedding of d-D Riemannian Manifolds into $\mathbb{R}^N$: $d \geq 3$

Gauss equations: $h^a_{ji} h^a_{kl} - h^a_{ki} h^a_{jl} = R_{ijkl}$

Codazzi equations:

$$\frac{\partial h^a_{lj}}{\partial x^k} - \frac{\partial h^a_{kj}}{\partial x^l} + \Gamma^m_{lj} h^a_{km} - \Gamma^m_{kj} h^a_{lm} + \kappa^a_{kb} h^b_{lj} - \kappa^a_{lb} h^b_{kj} = 0$$

Ricci equations:

$$\frac{\partial \kappa^a_{lb}}{\partial x^k} - \frac{\partial \kappa^a_{kb}}{\partial x^l} - g^{mn}\left(h^a_{ml} h^b_{kn} - h^a_{mk} h^b_{ln}\right) + \kappa^a_{kc}\kappa^c_{lb} - \kappa^a_{lc}\kappa^c_{kb} = 0$$

*R_{ijkl} is the Riemann curvature tensor, $\kappa^a_{kb} = -\kappa^b_{ka}$ is the coefficients of the connection form (torsion coefficients) on the normal bundle; the indices a, b, c run from 1 to N, and i, j, k, l, m, n run from 1 to $d \geq 3$.

***The Gauss-Codazzi-Ricci system has no type, neither purely hyperbolic nor purely elliptic for general Riemann curvature tensor R_{ijkl}**

*Bryant-Griffiths-Yang (**1983**): Duke Math. J., 102 pages.

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega)\, \partial_\xi \mathbf{w} = 0$

?? **Existence** of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m \times m}\, \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega)\, \mathbf{r}_j(\mathbf{u}, \omega), \ 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features:

Finiteness of Propagation Speeds;
Discontinuities of Solutions, $\cdots\cdots$
Well-Posedness: Existence, Uniqueness, Stability, $\cdots$