# High order discrete-time approximation for BSDEs

J-F Chassagneux (Imperial College London)
joint work with
D. Crisan (ICL)
and useful discussions with
K. Manolarakis (ICL)

Young Researchers' Meeting on BSDEs, Numerics and Finance,
04-07-2012

# Backward Stochastic Differential Equation

- Decoupled Forward Backward SDE:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s$$

$$Y_t = g(X_T) + \int_t^T f(Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s$$

$\hookrightarrow$ $b$, $\sigma$, $f$ and $g$ are 'nice' coefficients.

- We use the PDE representation

$$-L^{(0)}u = f(u, L^{(1)}u) \text{ and } u(T,.) = g$$

$L^{(0)}u = u^{(0)} := \partial_t u + \mathcal{L}_X u$ and $L^{(1)}u = u^{(1)} := \partial_x u.\sigma$
then

$$Y_t = u(t, X_t)\,,\; Z_t = u^{(1)}(t, X_t) \text{ and } f(Y_t, Z_t) = -u^{(0)}(t, X_t)$$

# Remarks

- $u$ is a **smooth** function: We can apply the operator $L^0$ and $L^1$ to u (and to $g$) many times.
- We do not approximate $X$: only the discrete-time error for $Y, Z$ is analysed on a grid $\pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\}$.
- The terminal condition of the schemes is given by:

$$Y_n = g(X_T) \text{ and } Z_n = \partial_x g(X_T)\sigma(X_T)$$

- The convergence error is given by:

$$\mathcal{E}^Y(\pi) := \max_{0 \le i \le n} \mathbb{E}\big[|Y_{t_i} - Y_i|^2\big] \text{ and } \mathcal{E}^Z(\pi) := \sum_{i=0}^{n} h\mathbb{E}\big[|Z_{t_i} - Z_i|^2\big] \ .$$

# Bouchard-Touzi-Zhang Scheme

*Notation*: For this talk, $t_{i+1} - t_i = h = T/n$.

- Implicit (or explicit) Euler Scheme for the $Y$ approximation

$$Y_i = \mathbb{E}_{t_i}[Y_{i+1} + hf(Y_i, Z_i)] \text{ or } Y_i = \mathbb{E}_{t_i}[Y_{i+1} + hf(Y_{i+1}, Z_{i+1})]$$

$$Z_i = \mathbb{E}_{t_i}\left[H_2^i Y_{i+1}\right], \text{ with } H_2^i := \frac{1}{h}(W_{t_{i+1}} - W_{t_i})$$

- Order-1 scheme (Gobert-Labart 2007)

$$\mathcal{E}^Y(\pi) + \mathcal{E}^Z(\pi) \le Ch^2 = Ch^{2\text{x}1}$$

# Crisan-Manolarakis Scheme

- A Crank-Nicholson scheme for the $Y$-approximation

$$Y_i = \mathbb{E}_{t_i}\Big[Y_{i+1} + \frac{h}{2}f(Y_i, Z_i) + \frac{h}{2}f(Y_{i+1}, Z_{i+1})\Big]$$
$$Z_i = \mathbb{E}_{t_i}\big[H_2^i(Y_{i+1} + hf(Y_{i+1}, Z_{i+1}))\big]\,,$$

with $H_2^i := \frac{1}{h}\int_{t_i}^{t_{i+1}}(4 - 6\frac{u-t_i}{h})\mathrm{d}W_s$

- Order-2 Scheme (Crisan-Manolarakis 10)

$$\mathcal{E}^Y(\pi) + \mathcal{E}^Z(\pi) \le Ch^4 = Ch^{2x2}$$

# How to retrieve higher order scheme ?

$\hookrightarrow$ Adapt numerical methods for ODEs to BSDEs setting.

- *Linear multi-step schemes*:
  $\hookrightarrow$ use the value computed at time $t_{i+j}$, $1 \leq j \leq r$ i.e.

$$Y_i = \mathbb{E}_{t_i}\left[ Y_{i+1} + h \sum_{j=0}^{r} \beta_j f(Y_{t_{i+j}}) \right]$$

  $\hookrightarrow$ up to now, convergence results only for driver of the type $f(Y)$ (Zhao-Zhang-Ju 10)
  $\hookrightarrow$ Question: how to initialize the scheme in an efficient way ?
- *Runge-Kutta schemes:* One-step schemes (use only the value at time $t_{i+1}$) but add intermediary computational steps between $t_i$ and $t_{i+1}$.

# Main idea

Extremely popular for ODEs.

- between two time-steps of the grid $\pi$ introduce intermediary dates: $t_{i,j} = t_{i+1} - c_j h$ with

$$c_1 := 0 < c_2 \leq \cdots \leq c_q \leq c_{q+1} = 1$$

- Compute $q-1$ intermediary values : $Y_{i,j}$, $Z_{i,j}$ and $f(Y_{i,j}, Z_{i,j})$.
- Use them to finally compute $Y_i$ and $Z_i$.

# Example: two-stage schemes (RK2)

- at time $t_{i,2}$:

$$Y_{i,2} = \mathbb{E}_{t_{i,2}}[Y_{i+1} + c_2 h f(Y_{i+1}, Z_{i+1})]$$
$$Z_{i,2} = \mathbb{E}_{t_{i,2}}\Big[H_2^i\Big(Y_{i+1} + c_2 h f(Y_{i+1}, Z_{i+1})\Big)\Big]$$

- at time $t_i$:

$$Y_i = \mathbb{E}_{t_i}\Big[Y_{i+1} + h(1 - \frac{1}{2c_2})f(Y_{i+1}, Z_{i+1}) + \frac{h}{2c_2} f(Y_{i,2}, Z_{i,2})\Big]$$
$$Z_i = \mathbb{E}_{t_i}\Big[H_3^i\Big(Y_{i+1} + h f(Y_{i+1}, Z_{i+1})\Big)\Big].$$

# Special case $c_2 = 1$

- at time $t_{i,2}$ $(= t_i!)$:

$$Y_{i,2} = \mathbb{E}_{t_{i,2}}[Y_{i+1} + hf(Y_{i+1}, Z_{i+1})]$$

- at time $t_i$:

$$Y_i = \mathbb{E}_{t_i}\Big[Y_{i+1} + \frac{h}{2}f(Y_{i+1}, Z_{i+1}) + \frac{h}{2}f(Y_{i,2}, Z_{i,2})\Big]$$
$$Z_i = \mathbb{E}_{t_i}\Big[H_3^i\Big(Y_{i+1} + hf(Y_{i+1}, Z_{i+1})\Big)\Big] .$$

# General form

- at time $t_{i,j}$: ($Y_{i,1} = Y_{i+1}$ and $Z_{i,1} = Z_{i+1}$)

$$Y_{i,j} = \mathbb{E}_{t_{i,j}}\left[ Y_{i+1} + c_j h \sum_{k=1}^{j} a_{jk} f(Y_{i,k}, Z_{i,k}) \right]$$

$$Z_{i,j} = \mathbb{E}_{t_{i,j}}\left[ H^i_j Y_{i+1} + h \sum_{k=1}^{j-1} \alpha_{jk} H^i_{j,k} f(Y_{j,k}, Z_{j,k}) \right]$$

- at time $t_i$: ($Y_{i,q+1} = Y_i$ and $Z_{i,q+1} = Z_i$)

$$Y_i = \mathbb{E}_{t_i}\left[ Y_{i+1} + h \sum_{j=1}^{q+1} b_j f(Y_{i,j}, Z_{i,j}) \right]$$

$$Z_i = \mathbb{E}_{t_i}\left[ H^i_{q+1} Y_{i+1} + h \sum_{j=1}^{q} \beta_j H^i_{q+1,j} f(Y_{i,j}, Z_{i,j}) \right].$$

# Description of the $H$-coefficients

## Definition

(i) For $m \geq 0$, we denote by $\mathcal{B}^m_{[0,1]}$ the set of bounded measurable. function $\psi : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 \psi(u)\mathrm{d}u = 1 \text{ and if } m \geq 1, \int_0^1 \psi(u)u^k\mathrm{d}u = 0 \;,\; 1 \leq k \leq m.$$

(ii) Let $\psi \in \mathcal{B}^m_{[0,1]}$, for $t \in [0,T]$ and $h > 0$ s.t. $t + h \leq T$, we define,

$$H^{\psi}_{t,h} := \frac{1}{h}\int_t^{t+h} \psi(\frac{u-t}{h})\mathrm{d}W_u \;.$$

By convention, we set $H^{\psi}_{t,0} = 0$.

# Examples

(i) The function $\psi = \mathbf{1}_{[0,1]}$ belongs to $\mathcal{B}^0$.

(ii.a) The polynomial function $x \mapsto \psi(x) = 4 - 6x$ belongs to $\mathcal{B}^1$.

(ii.b) For $c \in (0,1)$, the function $\psi = \frac{1}{c(c-1)}\mathbf{1}_{[1-c,1]} + \frac{c-2}{c-1}\mathbf{1}_{[0,1]}$ belongs to $\mathcal{B}^1$.

(iii) For $c, c' \in (0,1)$, the following

$$\psi = (\frac{1}{c-c'} + \frac{1}{c'-1})\frac{1}{c}\mathbf{1}_{[1-c,1]} + (\frac{1}{c'-c} + \frac{1}{c-1})\frac{1}{c'}\mathbf{1}_{[1-c',1]} + (1 - \frac{1}{c-1} - \frac{1}{c'-1})\mathbf{1}_{[0,1]}$$

belongs to $\mathcal{B}^2$.

# Convergence analysis - Stability (1/2)

The scheme rewrites

$$\left\{\begin{array}{rcl} Y_i & = & \mathbb{E}_{t_i}\big[Y_{i+1} + h\Phi_i^Y(t_{i+1}, Y_{i+1}, Z_{i+1}, h)\big] \\ Z_i & = & \mathbb{E}_{t_i}\Big[H^i_{q+1} Y_{i+1} + h\Phi_i^Z(t_{i+1}, Y_{i+1}, Z_{i+1}, h)\Big] \end{array}\right.$$

We consider a pertubed scheme:

$$\left\{\begin{array}{rcl} \tilde{Y}_i & = & \mathbb{E}_{t_i}\Big[\tilde{Y}_{i+1} + h\Phi^Y(t_i, h, \tilde{Y}_{i+1}, \tilde{Z}_{i+1})\Big] + \zeta_i^Y \\ \tilde{Z}_i & = & \mathbb{E}_{t_i}\Big[H^i_{q+1} \tilde{Y}_{i+1} + h\Phi^Z(t_i, h, \tilde{Y}_{i+1}, \tilde{Z}_{i+1}, h)\Big] + \zeta_i^Z \end{array}\right.$$

# Convergence analysis - Stability (2/2)

The scheme is said to be $L^2$-stable if

$$\max_i \mathbb{E}\big[|\delta Y_i|^2\big] + h\sum_{i=0}^{n}\mathbb{E}\big[|\delta Z_i|^2\big] \le C\sum_{i=0}^{n-1} h\mathbb{E}\Big[\frac{1}{h^2}|\zeta_i^Y|^2 + |\zeta_i^Z|^2\Big]$$

i.e. the overall error is the sum of the error done at each step.

- (Sufficient Condition for $L^2$-Stability) If $f$ is Lipschitz-continuous, then the scheme is $L^2$-stable.

# Convergence analysis - Order (1/2)

Set
$$\begin{cases} \hat{Y}_{t_i} & := & \mathbb{E}_{t_i}\big[Y_{t_{i+1}} + h\Phi_i^Y(t_{i+1}, Y_{t_{i+1}}, Z_{t_{i+1}}, h)\big] \\ \hat{Z}_{t_i} & := & \mathbb{E}_{t_i}\Big[H^i_{q+1} Y_{t_{i+1}} + h\Phi_i^Z(t_{i+1}, Y_{t_{i+1}}, Z_{t_{i+1}}, h)\Big] . \end{cases}$$

then the true solution satisfies a pertubed scheme

$$\begin{cases} Y_{t_i} & := & \mathbb{E}_{t_i}\big[Y_{t_{i+1}} + h\Phi_i^Y(t_{i+1}, Y_{t_{i+1}}, Z_{t_{i+1}}, h) + \delta_i^Y\big] \\ Z_{t_i} & := & \mathbb{E}_{t_i}\Big[H^i_{q+1} Y_{t_{i+1}} + h\Phi_i^Z(t_{i+1}, Y_{t_{i+1}}, Z_{t_{i+1}}, h) + \delta_i^Z\Big] . \end{cases}$$

The *local truncation error* is defined as:

$$\eta_i := h\mathbb{E}\left[\frac{1}{h^2}|\delta_i^Y|^2 + |\delta_i^Z|^2\right] ,\ \delta_i^Y := Y_{t_i} - \hat{Y}_{t_i} ,\ \delta_i^Z := Z_{t_i} - \hat{Z}_{t_i} .$$

It measures: how accurately the true solution satisfies the scheme *or* the error done in one step provided no error was done before.

# Convergence analysis - Order (2/2)

- A method is said to be of order $m$ if (for all grid, $u$ smooth)

$$\sum_i \eta_i \le Ch^{2m}$$

- Study of $\delta_i^Y$ and $\delta_i^Z$ based on

  Let $m \ge 0$, then for a function $v$ smooth enough

  (i) $\mathbb{E}_t[v(t+h, X_{t+h})] = v_t + hv_t^{(0)} + \frac{h^2}{2} v_t^{(0,0)} + \cdots + \frac{h^m}{m!} v_t^{(0)_m} + O_t(h^{m+1})$

  (ii) For $\psi \in \mathcal{B}^m_{[0,1]}$

$$\mathbb{E}_t\Big[H^{\psi}_{t,h} v(t+h, X_{t+h})\Big] = v_t^{(1)} + hv_t^{(1,0)} + \cdots + \frac{h^m}{m!} v_t^{(1)*(0)_m} + O_t(h^{m+1})$$

# Convergence analysis - End

Provided that the method is stable, we have

$$\mathcal{E}^Y(\pi) + \mathcal{E}^Z(\pi) \le C \sum_i \eta_i$$

which leads to

$$\mathcal{E}^Y(\pi) + \mathcal{E}^Z(\pi) \le Ch^{2m}$$

if the method is of order $m$.

# Convergence results

Denoting $s$ the number of stage and $o$ the order of the scheme.

- Explicit Methods:
  -$o = 1$, $s = 1$ (e.g BTZ )
  -$o = 2$, $s = 2$,
  -$o = 3$, $s = 3$,
  -$o = 4$, $s > 4$: (order barrier)

  Remark: For ODEs ($o = 4$, $s = 4$) and then ($o = 5$, $s > 5$).

- Implicit Methods:
  - $s = 1$, $o = 2$ (Crisan-Manolarakis)
  - $s = 2$, $o = 2$ if $\partial_z f \neq 0$ (implicit barrier)

# Order 3 condition

$c_2 \neq 1,\ c_2 \neq c_3$

$$b_1 + b_2 + b_3 = 1$$
$$b_2 c_2 + b_3 c_3 = \frac{1}{2}$$
$$b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3}$$
$$b_3 a_{32} c_2 = b_3 \alpha_{32} c_2 = \frac{1}{6}$$

and

$$\beta_1 + \beta_2 + \beta_3 \mathbf{1}_{\{c_3<1\}} = 1$$
$$\beta_2 c_2 + \beta_3 c_3 \mathbf{1}_{\{c_3<1\}} = \frac{1}{2}$$

# Framework

Goal: try to compare the order of different schemes

- We consider the process

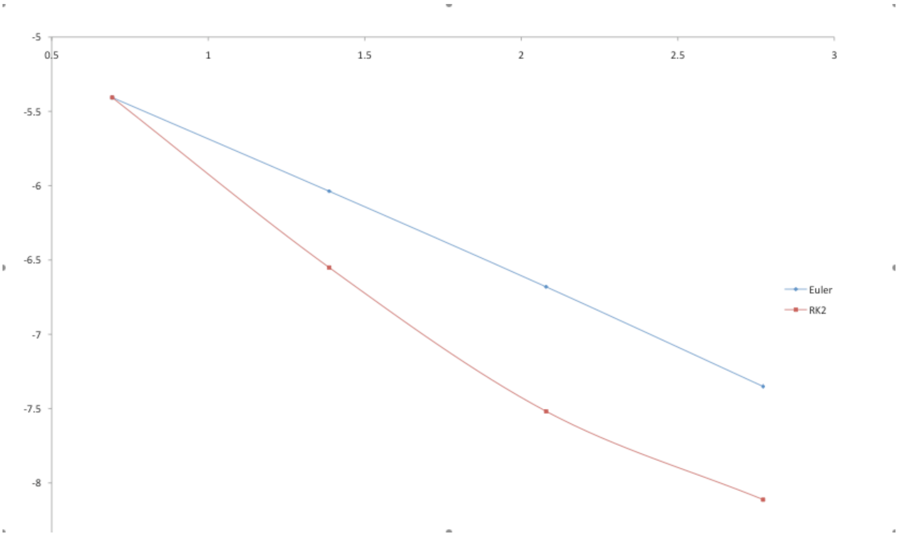
$$(X_t, Y_t, Z_t) = \left(W_t, \frac{1}{1+\exp(-W_t - \frac{t}{4})}, \frac{\exp(-W_t - \frac{t}{4})}{(1+\exp(-W_t - \frac{t}{4}))^2}\right)$$

  Solution of a BSDE with driver:

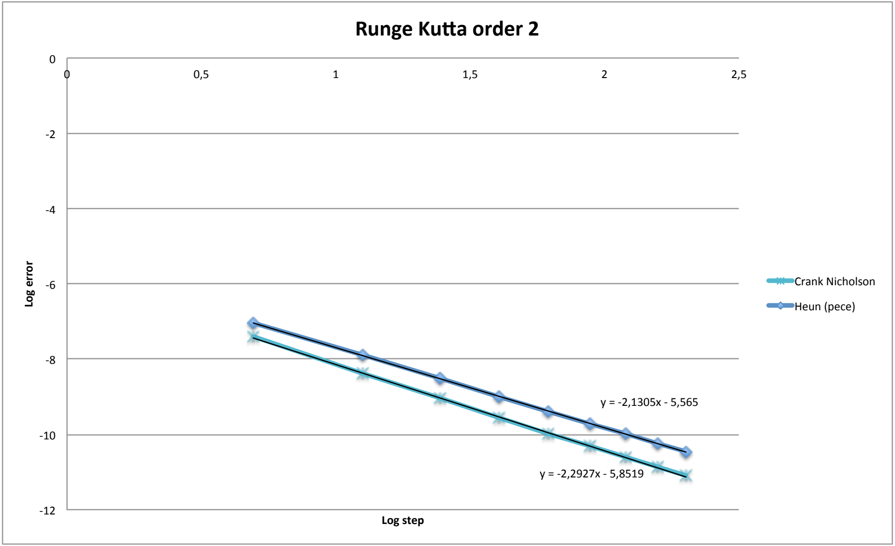
$$f(y) = -y(1-y)(\frac{3}{4} - y) \text{ or } f(y,z) = -z(\frac{3}{4} - y)$$

- schemes : BTZ, Crank-Nicholson, RK2 ($c_2 = \frac{1}{2}$ )
- Simulation/Regression approach : HC (fine grid) + a lot of simulation ($10^6$)
  or tree method for BM

# Order of the methods (Simulation/Regression)



# Order of the methods (Tree)



# General form

Methods with $r > 0$ steps: these methods use the value computed at time $t_{i+j}$, $1 \leq j \leq r$ to compute the value at time $t_i$.

$$\left\{ \begin{array}{rcl} Y_i & = & \mathbb{E}_{t_i}\Big[\sum_{j=1}^r a_{i,j} Y_{i+j} + h \sum_{j=0}^r b_{i,j} f(Y_{i+j}, Z_{i+j})\Big] \\ Z_i & = & \mathbb{E}_{t_i}\Big[\sum_{j=1}^r \alpha_{i,j} H^Y_{i,j} Y_{i+j} + h \sum_{j=1}^r \beta_{i,j} H^f_{i,j} f(Y_{i+j}, Z_{i+j})\Big] \end{array} \right.$$

# Some Linear multi-step method

(i) $r = 2$: Nystrom (a.k.a Leap-frog) method (for the $Y$-part), Order 2.

$$\begin{cases} Y_i &= \mathbb{E}_{t_i}[Y_{i+2} + 2hf(Y_{i+1}, Z_{i+1})] \\ Z_i &= \mathbb{E}_{t_i}\Big[H_i\Big(Y_{i+2} + 2hf(Y_{i+1}, Z_{i+1})\Big)\Big]. \end{cases}$$

(ii) Adams methods: Adams-Moulton methods for the $Y$-part (implicit) and Adams-Bashforth methods for the $Z$-part (always explicit).

$$\begin{cases} Y_i &= \mathbb{E}_{t_i}\Big[Y_{i+1} + h\sum_{j=0}^{r} b_j f(Y_{i+j}, Z_{i+j})\Big] \\ Z_i &= \mathbb{E}_{t_i}\Big[H_i Y_{i+1} + h\sum_{j=1}^{r} \beta_j H_{i,j} f(Y_{i+j}, Z_{i+j})\Big] \end{cases}$$

# Framework

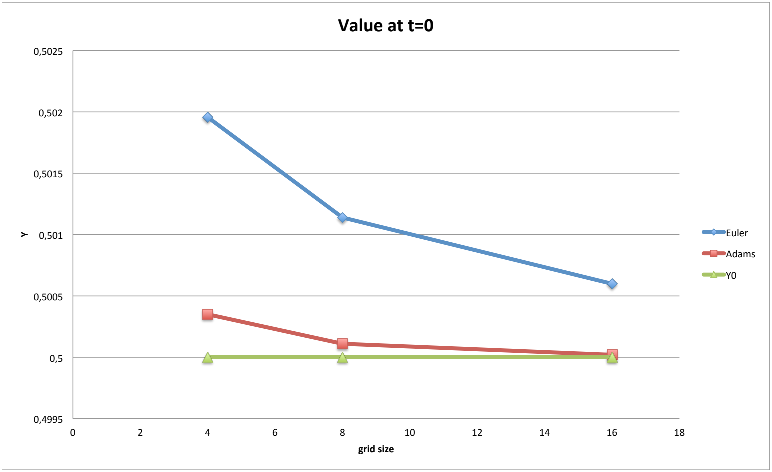
Goal: try to compare the order of different schemes

- We consider the process

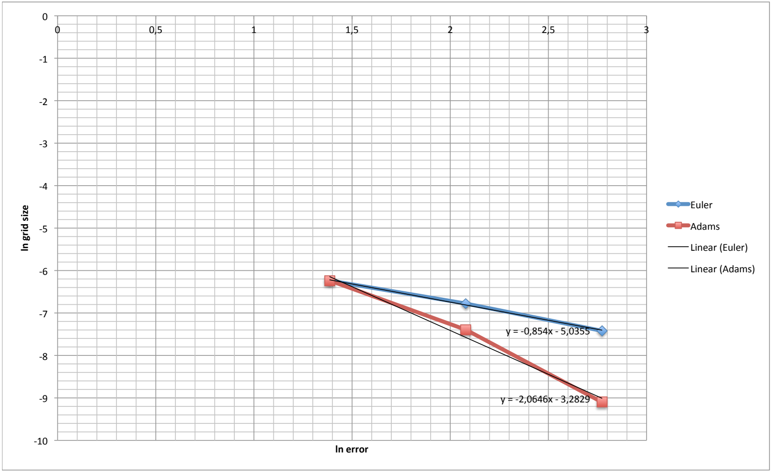
$$(X_t, Y_t, Z_t) = \left( W_t, \frac{1}{1 + \exp(-W_t - \frac{t}{4})}, \frac{\exp(-W_t - \frac{t}{4})}{(1 + \exp(-W_t - \frac{t}{4}))^2} \right)$$

- Schemes: BTZ, Adams (exp o2, imp o3) and Milne (o2)
- - Simulation/Regression approach : HC (fine grid) + a lot of simulation ($10^6$)
  - tree method (BM)

# Convergence (Simulation/Regression)



# Order of the methods (Simulation/Regression)



# Order of the methods (Tree)