# Mean variance hedging under default risk

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# Some references

We want to solve the classical mean variance hedging problem. This problem has been solved

- by a method of projection

  Schweizer(92) : "Mean-variance hedging for general claims"

  Gourieroux, Laurent and Pham(98) : "Mean-variance hedging and numéraire"

  Cesny and Kallsen(06) : "On the structure of general mean-variance hedging strategies"

# Some references

- by stochastic control

  Lim(04) : "Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market"

  Arai(05) : "An extension of mean-variance hedging to the discontinuous case"

  Kohlmann, Xiong and Ye(10) : "Mean variance hedging in a general jump model"

  Jeanblanc, Mania, Santacroce and Schweizer(11) : "Mean-Variance Hedging via Stochastic Control and BSDEs for General Semimartingales"

# Our model

- The assets are discontinous due to possible defaults.
- We use a progressive enlargement of filtrations.
- We solve the problem by stochastic control.

# Market information

On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ :

- Reference filtration $\mathbb{F}$ : default-free information
  $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ natural filtration generated of a Brownian motion $W$.
- $\tau$ a random time and $L$ a mark associated to $\tau$, $L$ taking values in $E \subset \mathbb{R}$.
- $\mathbb{D} = (\mathcal{D}_t)_{t\in[0,T]}$ the filtration generated by the associated jump process.
- $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ the enlarged progressive filtration $\mathbb{F} \vee \mathbb{D}$ : global market information taking into account the observation of the default when it occurs.

## Density hypothesis

There exists a $\mathbb{F}$-adapted process $\alpha$ such that for all $0 \leq t \leq T$,

- $(\omega, \theta, l) \to \alpha_t(\omega, \theta, l)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E)$-measurable
- for any Borel function $f$ on $\mathbb{R}^+ \times E$,

$$\mathbb{E}[f(\tau, L)|\mathcal{F}_t] \; = \; \int_{\mathbb{R}^+ \times E} f(\theta, l)\alpha_t(\theta, l)d\theta\eta(dl) \quad a.s.,$$

where $d\theta$ is the Lebesgue measure on $\mathbb{R}$, and $\eta(dl)$ is a Borel measure on $E$, with $\eta$ a nonnegative Borel measure on $E$.

# Structure of $\mathbb{G}$-adapted processes

(Jeulin, Yor)

- $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ and which makes $\tau$ a stopping time.
- Any $\mathbb{G}$-adapted process $\phi$ is written in the form

$$\phi_t^0 1_{t<\tau} + \phi_t^1(\tau, L) 1_{t\geq\tau},$$

where $\phi^0$ is $\mathbb{F}$-adapted
and $\phi^1(\tau, L)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable.

# Asset price model with counterparty default

The stock price $\mathbb{G}$-adapted process is given by

$$S_t = S_t^0 1_{t<\tau} + S_t^1(\tau, L) 1_{t \geq \tau}, \quad 0 \leq t \leq T,$$

where

- $S^0$ is **the price process in the market before default** :

$$dS_t^0 = S_t^0 \left( \mu_t^0 dt + \sigma_t^0 dW_t \right), \quad S_0^0 = S_{0_-}, \quad 0 \leq t \leq T,$$

$\mu^0$, $\sigma^0$ are $\mathbb{F}$-adapted processes, $\sigma^0 > 0$, satisfying

$$\int_0^T \left| \frac{\mu_t^0}{\sigma_t^0} \right|^2 dt + \int_0^T \left| \sigma_t^0 \right|^2 dt < \infty, \quad a.s.$$

# Asset price dynamics after default

- $\{S^1(\theta, l), \theta \in [0, T], l \in E\}$ is the family of **price process in the market after default** at $\theta$ with mark $l$ :

$$
\begin{aligned}
dS_t^1(\theta, l) &= S_t^1(\theta, l)\left(\mu_t^1(\theta, l)dt + \sigma_t^1(\theta, l)dW_t\right) \\
S_\theta^1(\theta, l) &= S_{\theta-}^1\left(1 + \gamma_\theta^0(l)\right) \quad \gamma_t^0 \in [-1, \infty)
\end{aligned}
$$

1. $\gamma^0$ is a $\mathbb{F}$-predictable process representing the proportional loss ($\gamma^0 < 0$) or gain ($\gamma^0 > 0$) on the stock price at the default.
2. $\mu_t^1(\theta, l), \sigma_t^1(\theta, l)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurables, $\sigma_t^1(\theta, l) > 0$ interpreted as change of regimes after default : typically, we should have $\sigma^1(\theta, l) > \sigma^0$.

In the sequel, we will assume that :

**Assumption 1**

- $\forall t \in [0, T]$, $\mu_t^0$, $\sigma_t^0$, $\mu_t^1$, $\sigma_t^1$ and $\gamma_t^0$ and the family process $\{\alpha_t(\theta, l); (\theta, l) \in [0, T] \times E\}$ are uniformly bounded.
- The measure $\eta(dl)$ is uniformly bounded.

# Portfolio and wealth process

A trading strategy is a $\mathbb{G}$-predictable process $\pi$, representing the proportion of wealth invested in the risky asset.
By writing $\pi$ in the form : $\pi_t^0 1_{t \leq \tau} + \pi_t^1(\tau, L) 1_{t > \tau}$, this means that the wealth process is in the form

$$X_t = X_t^0 1_{t < \tau} + X_t^1(\tau, L) 1_{t \geq \tau}, \quad 0 \leq t \leq T$$

where :

- $X^0$ is **the wealth process in the market before default** :

$$dX_t^0 = \pi_t^0 \frac{dS_t^0}{S_t^0}, \quad X_0^0 = X_{0-} \quad 0 \leq t \leq T$$

# Portfolio and wealth process

- $X^1(\theta, l)$ is **the wealth process in the market after default** at $\theta$ with mark $l$ :

$$\begin{aligned} dX_t^1(\theta, l) &= \pi_t^1(\theta, l)\frac{dS_t^1(\theta, l)}{S_t^1(\theta, l)} \\ X_\theta^1(\theta, l) &= X_{\theta-}^0 + \pi_\theta^0 \gamma_\theta^0(l). \end{aligned}$$

# Formulation of the problem

Our problem of *mean variance hedging (MVH)* is then formulated as

$$V^0 = \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[(H_T - X_T)^2].$$

where

- $H \in \mathrm{L}^\infty(\Omega, \mathcal{G}_T)$ is a contingent claim such that $H = H^0 1_{\{\tau < T\}} + H^1(\tau, L) 1_{\{\tau > T\}}$.

# Admissible strategies

- $\mathcal{A}^0_{\mathbb{F}}$ denotes the sets of $(\mathcal{F}_t)_{(0<t\leq T)}$-adapted processes $\{\pi^0_t, 0 < t \leq T\}$ satisfying $\int_0^T |\pi^0_t|^2 dt < \infty$ a.s. and such that

$$\mathbb{E}\left[\int_0^T |\pi^0_s|^2 ds\right] < \infty$$

- $\mathcal{A}^1_{\mathbb{F}}$ denotes the sets of processes $\{\pi^1_t(\theta, l), \theta < t \leq T\}$ satisfying the same conditions on $[\theta, T]$.
- $\mathcal{A}_{\mathbb{G}} = (\mathcal{A}^0_{\mathbb{F}}, \mathcal{A}^1_{\mathbb{F}})$

# Definition of $V^1$

(Pham, Jiao, Kharroubi,11)

Let us introduce the value-function process of the "after-default" problem :

$$V^1(x,\theta,l) = \operatorname*{ess\,inf}_{\pi^1 \in \mathcal{A}^1_{\mathbb{F}}} J^{1,\theta,l}_\theta(x,\pi^1(\theta,l))$$

$$J^{1,\theta,l}_\theta(x,\pi^1(\theta,l)) = \mathbb{E}\left[(H^1_T - X^{1,x}_T(\theta,l))^2 \alpha_T(\theta,l) | \mathcal{F}_\theta\right]$$

# Decomposition

Denoting $G_T = \mathbb{P}[\tau > T|\mathcal{F}_T]$, we obtain that $V^0$ is a function of $V^1$ :

$$
\begin{aligned}
V^0(x) &= \inf_{\pi^0 \in \mathcal{A}^0_{\mathbb{F}}} \mathbb{E}\Big[(H^0_T - X^{0,x}_T)^2 G_T \\
&+ \int_0^T \int_E V^1(X^{0,x}_\theta + \pi^0_\theta \gamma^0_\theta(l), \theta, l) d\theta \eta(dl)\Big]
\end{aligned}
$$

# Dynamic programming

Let define the two sets of controls :

- $$\mathcal{A}^0_{\mathbb{F}}(t,\nu^0) = \{\pi^0 \in \mathcal{A}^0_{\mathbb{F}} : \pi^0_{.\wedge t} = \nu^0_{.\wedge t}\}$$
- $$\mathcal{A}^1_{\mathbb{F}}(t,\nu^1) = \{\pi^1 \in \mathcal{A}^1_{\mathbb{F}} : \pi^1_{.\wedge t} = \nu^1_{.\wedge t}\}.$$

# Dynamic programming

Let also define the dynamic version of the value process

- before the default by, for $t \in [0, T]$ :

$$V_t^0(x, \nu^0) = \operatorname*{ess\,inf}_{\pi^0 \in \mathcal{A}_{\mathbb{F}}^0(t,\nu^0)} \mathbb{E}\Big[(H_T^0 - X_T^{0,x})^2 G_T + \int_t^T \int_E V_\theta^1(X_\theta^{0,x} + \pi_\theta^0 \gamma_\theta^0(l), \theta, l) d\theta \eta(dl) \Big| \mathcal{F}_t\Big]$$

- after the default by, for $t \in [\theta, T]$ :

$$V_t^1(x, \theta, l, \nu^1) = \operatorname*{ess\,inf}_{\pi^1 \in \mathcal{A}_{\mathbb{F}}^1(t,\nu^1)} \mathbb{E}\Big[(H_T^1 - X_T^{1,x}(\theta, l))^2 \alpha_T(\theta, l) | \mathcal{F}_t\Big]$$

# Characterization of $V^1$

By the quadraticity of the problem and by the dynamic programming principle we obtain the following theorem :

**Theorem 1**

For any $\nu \in A^1_{\mathbb{F}}$, the value process $V^1(x, \theta, l, \nu)$ admits for all $t \in [\theta, T]$ the quadratic form decomposition :

$$V^1_t(x, \theta, l, \nu) = v^{1,\theta,l}_t (X^{1,x}_t - Y^{1,\theta,l}_t)^2$$

with $Y^{1,\theta,l} = \mathbb{E}^{Q(\theta,l)}[H^1|\mathcal{F}_t]$
where $Q(\theta, l)$ is the risk-neutral probability associated to $(\theta, l)$.

## Theorem 1

Besides, the couple $(v^{1,\theta,l}, Y^{1,\theta,l})$ satisfies the following BSDEs for all $t \in [\theta, T]$ :

$$\begin{aligned} \frac{dv_t^{1,\theta,l}}{v_t^{1,\theta,l}} &= \frac{\left(\mu^1(\theta,l) + \sigma^1(\theta,l)\beta_t^{1,\theta,l}\right)^2}{(\sigma^1(\theta,l))^2} dt - \beta_t^{1,\theta,l} dW_t, \quad & v_T^{1,\theta,l} = \alpha_T(\theta,l); \\ dY_t^{1,\theta,l} &= \frac{\mu^1(\theta,l)}{\sigma^1(\theta,l)} Z_t^{1,\theta,l} dt - Z_t^{1,\theta,l} dW_t, \quad & Y_T^{1,\theta,l} = H_T^1(\theta,l); \end{aligned}$$

# Value process before the default

We deduce the characterization of $V^0$ from the one of $V^1$ :

## Theorem 2

For any $\nu \in A^0_{\mathbb{F}}$, the value process $V^0(x,\nu)$ admits for all $t \in [0, T]$ the quadratic form decomposition :

$$V^0_t(x,\nu) = v^0_t(X^{0,x}_t - Y^0_t)^2 + \xi^0_t$$

## Theorem 2

Moreover, the triple $(v_t^0, Y_t^0, \xi_t^0)$ satisfies the following three BSDEs for all $t \in [0, T]$, where $\delta > 0$ :

$$\begin{cases} \frac{dv_t^0}{v_t^0} = -g_t^{0,(1)}(v_t^0, \beta_t^0)dt + \beta_t^0 dW_t, v_T^0 = G_T, v_t^0 \geq \delta \\ dY_t^0 \ \ = -g_t^{0,(2)}(Y_t^0, Z_t^0)dt + Z_t^0 dW_t, \ \ Y_T^0 = H_T^0 \\ \\ d\xi_t^0 = -g_t^{0,(3)}(\xi_t^0, R_t^0)dt + R_t^0 dW_t, \ \ \xi_T^0 = 0. \end{cases}$$

## Theorem 2

$$g_t^{0,(1)} = \int_E (1+v_t^{J,l})\eta(dl) - \frac{\left(\mu_t^0 + \sigma_t^0\beta_t^0 + \int_E (1+v_t^{J,l})\gamma_t^0(l)\eta(dl)\right)^2}{(\sigma_t^0)^2 + \int_E (1+v_t^{J,l})(\gamma_t^0(l))^2\eta(dl)}$$

$$g_t^{0,(2)} = \beta_t^0 Z_t^0 + \int_E U_t^{J,l,0}(1+v_t^{J,l})\eta(dl) + \left(-\int_E U_t^{J,l,0}(1+v_t^{J,l})\gamma_t^0(l)\eta(dl) - \sigma_t^0 Z_t^0\right)\frac{\mu_t^0 + \sigma_t^0\beta_t^0 + \int_E (1+v_t^{J,l})\gamma_t^0(l)\eta(dl)}{(\sigma_t^0)^2 + \int_E (1+v_t^{J,l})(\gamma_t^0(l))^2\eta(dl)}$$

$$g_t^{0,(3)} = v_t^0\left[\int_E (U_t^{J,l,0})^2(1+v_t^{J,l})\eta(dl) + (Z_t^0)^2 - \frac{\left(-\int_E (1+v_t^{J,l})U_t^{J,l,0}\gamma_t^0(l)\eta(dl) - \sigma_t^0 Z_t^0\right)^2}{(\sigma_t^0)^2 + \int_E (1+v_t^{J,l})(\gamma_t^k(l))^2\eta(dl)}\right]$$

where $1+v_t^{J,l} = \frac{v^{1,\theta,l}}{v_t^0}$ and $U_t^{J,l,0} = Y_t^{1,\theta,l} - Y_t^0$.

## Remarks

- The process $v^0$ does not depend on the payoff $H$.
- $$v_0^0 = \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[(X_T^{1,\pi})^2] > 0$$
- The process $Y^0$ is linked to the quadratic approximation price of the contingent claim $H$.
- The process $\xi^0$ represents the incompleteness of this market.
- The generators of the BSDEs are $\mathbb{F}$-adapted.
- $g^{0,(1)}$ is quadratic in $\beta^0$ and $g^{0,(2)}$ is linear in $Z^0$

# Sketch of proof

- We suppose that $V^0$ has a quadratic form in $X^0$.
- By Itô calculus and the dynamic programming principle we obtain the BSDEs.
- We conclude our results by a verification theorem.

## Proposition

Under Assumption 1, there exists a pair $(v_t^0, \beta_t^0) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of the first BSDE :

$$dv_t^0 = v_t^0 \left[ -g_t^{0,(1)}(v_t^0, \beta_t^0)dt + \beta_t^0 dW_t \right], \text{ with } \; v_T^0 = G_T$$

# Sketch of proof

Let define the modified BSDE :

$$dv_t^0 = -\overline{g_t^{0,(1)}}(v_t^0, \overline{\beta_t^0})dt + \overline{\beta_t^0} dW_t$$

where the coefficient of the BSDE is defined as :

$$\begin{aligned} \overline{g_t^{0,(1)}} &= \int_E v_t^{1,\theta,l} \eta(dl) \\ &- \frac{\left(\mu_t^0 |v_t^0| + \sigma_t^0 \overline{\beta}_t^0 + \int_E v_t^{1,\theta,l} \gamma_t^0(l) \eta(dl)\right)^2}{(\sigma_t^0)^2 |v_t^0| + \int_E v_t^{1,\theta,l} (\gamma_t^0(l))^2 \eta(dl)} \end{aligned}$$

- We obtain that

$$|\overline{g_t^{0,(1)}}| \leq C\left[1 + |v_t^0| + |\bar{\beta}_t^0|^2\right]$$

- Therefore the coefficient follows a quadratic growth with respect to $\bar{\beta}^0$ and a linear growth with respect to $v^0$.
- Then by Kobylanski's Theorem [7], there exists a pair $(v^0, \bar{\beta}^0) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of this modified BSDE.

We then find a lower bound $\bar{f}_t$ of the coefficient $\overline{g^{0,(1)}}$ such that the BSDE

$$d\bar{Y}_t = -\bar{f}_t dt + \bar{Z}_t dW_t, \quad Y_T = G_T.$$

has a solution $(\bar{Y}, \bar{Z})$ with $\bar{Y} \geq \delta$.
Therefore we get the expected result by comparison theorem of Kobylanski [7].

We can now prove the existence of the second BSDE, since the solution of the first one exists.
Given the solution of the first BSDE, the coefficient of the second one is linear.
Therefore, we can deduce the existence of the solution from the following proposition.

## Proposition

Under Assumption 1 there exists a pair $(Y^0, Z^0) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of the second BSDE :

$$dY_t^0 = -(a_t Z_t^0 + \kappa_t Y_t^0 + \Lambda_t)dt + Z_t^0 dW_t, \;\; Y_T^0 = H_T^0 \in \mathrm{L}^\infty$$

for some BMO processes $\kappa$ and $a$. Moreover,

$$Y_t^0 = \mathbb{E}\left[\frac{1}{\Gamma_t}\left(\Gamma_T H_T^0 + \int_t^T \Gamma_s \Lambda_s ds\right)|\mathcal{F}_t\right], \quad t \leq T$$

where the process $\Gamma$ verifies :

$$\frac{d\Gamma_t}{\Gamma_t} = \kappa_t dt + a_t dW_t, \quad \Gamma_0 = 1.$$

We consider a special case where

- $\mu^0$, $\sigma^0$ and $\gamma^0$ are constants.
- $\mu^1(\theta, l)$ and $\sigma^1(\theta, l)$ are deterministic functions of $\theta$ such that

$$\forall \theta \in [0, T], \mu^1(\theta, l) = \mu^0 \frac{\theta}{T} \quad \text{and} \quad \sigma^1(\theta, l) = \sigma^0 \left(2 - \frac{\theta}{T}\right).$$

- $\tau$ is independent of $\mathbb{F}$ and follows an exponential law of parameter $\lambda > 0$.
- There is no mark.
- $H^0$ and $H^1$ are two constants such that $H^0 > H^1$.

# BSDE $\rightarrow$ ODE

In this case, we obtain

$$v_t^{1,\theta} = \lambda \exp\left(-\lambda t + (T-\theta)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta}-1)}\right)^2\right)$$

. Besides, $v^0$ follows this ODE :

$$\frac{\partial v_t^{0,t}}{\partial t} = -v_t^{1,t} + \frac{\left(\mu^0 v_t^{0,t} + \gamma^0 v_t^{1,t}\right)^2}{(\sigma^0)^2 v_t^{0,t} + (\gamma^0)^2 v_t^{1,t}}, \quad v_T^{0,T} = e^{-\lambda T}$$

For the simulations, we take $\mu^0 = 0.2$, $\sigma^0 = 0.05$, $H^0 = 1.2$, $H^1 = 0.9$ and maturity $T = 1$.

We recall that

$$v_0^0 = \min_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[(X_T^{1,\pi})^2\right].$$

Therefore $v^0$ is related to the minimal variance of a portfolio investment on the asset $S$ with initial wealth $x = 1$.

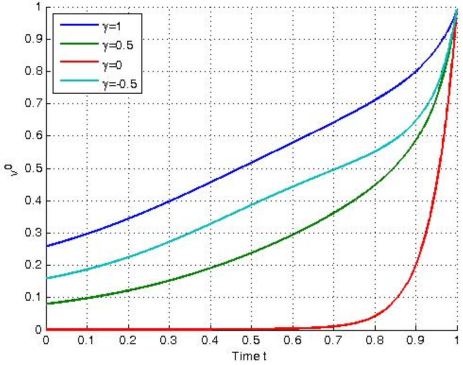


FIGURE: $v_t^0$ in function of time $t \in [0, T]$ with $T = 1$ and $\lambda = 0.01$ for different values of $\gamma$.

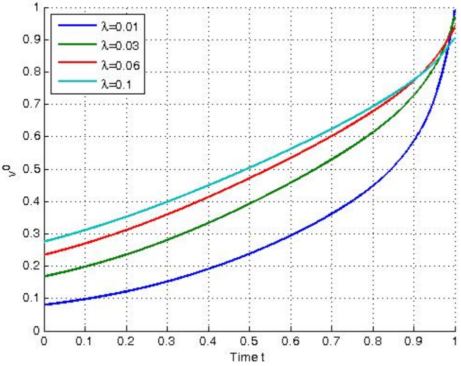


FIGURE: $v_t^0$ in function of time $t \in [0, T]$ with $T = 1$ and $\gamma = 0.5$ for different values of $\lambda$.

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