Minimal Supersolutions of Backward Stochastic Differential Equations and Robust Hedging

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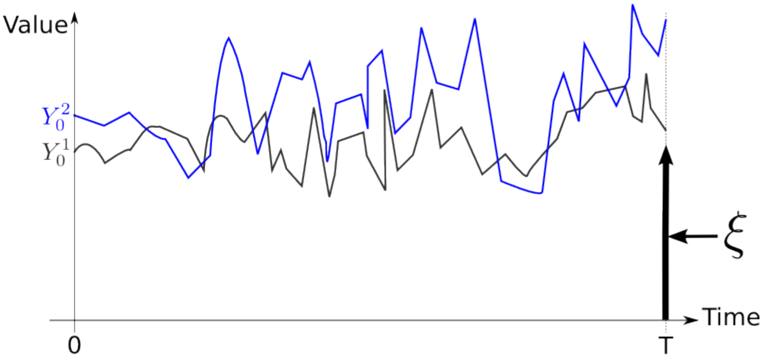
Supersolutions of BSDE

Motivation

Superhedging problem

$$Y_0 + \underbrace{\int_0^T Z_u dS_u}_{\text{trading gains}} \geq \xi$$

- Y_0= superhedging price of ξ
- Z= superhedging strategy

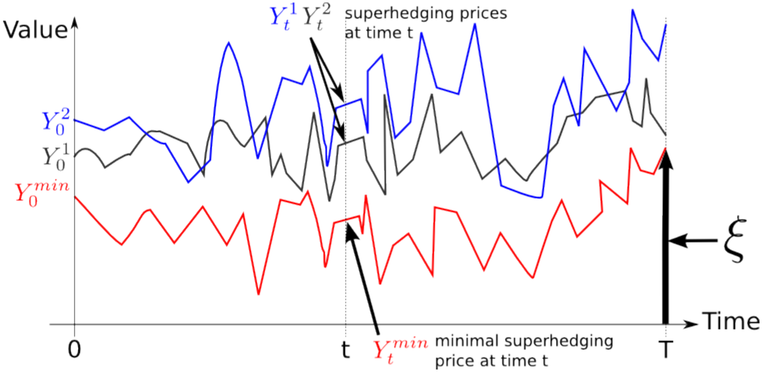


Motivation

Superhedging problem

$$Y_t + \underbrace{\int_t^T Z_u dS_u}_{\text{trading gains}} \geq \xi$$

- Y_t= superhedging price of ξ at t
- Z= superhedging strategy



Definition

(Y, Z) is a supersolution of the Backward Stochastic Differential Equation with driver g and terminal condition ξ if

$$Y_t - \underbrace{\int_t^T g(Y_u, Z_u)du}_{\text{drift part}} + \underbrace{\int_t^T Z_u dW_u}_{\text{martingale part}} \geq \xi \quad \forall t \in [0, T]$$

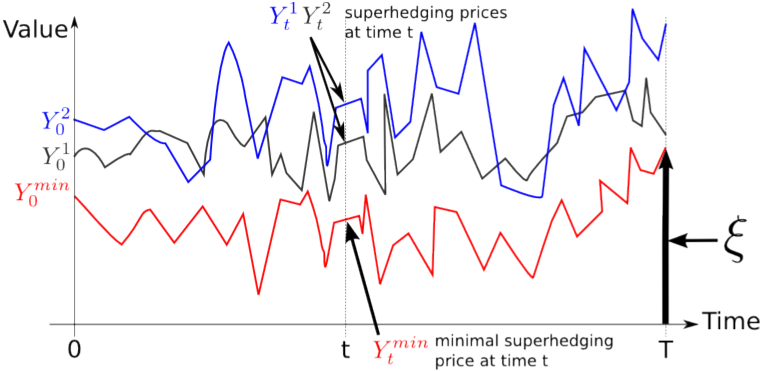
- Y = value process
- Z = control process

Equality instead of inequality: (Y, Z) is solution of the BSDE.

Extensively studied:
$\rightsquigarrow$ Bismut, Pardoux, Peng, Ma, Protter, Yong, Briand, Hu, Kobylanski, Touzi, Delbaen, Imkeller, El Karoui, ...

Applications in utility maximization, stochastic games, stochastic equilibria, . . .

- Supersolutions are typically not unique.
- Find a minimal supersolution $(Y^{\min}, Z^{\min})$!
  That is $Y^{\min} \leq Y$ for any other supersolution (Y, Z).



Minimal Supersolutions

Filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, filtration generated by a Brownian motion W satisfying the usual conditions.

$$\begin{cases} Y_s - \int_s^t g(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t, & 0 \leq s \leq t \leq T \\ Y_T \geq \xi \end{cases} \tag{0.1}$$

1. ξ is $\mathcal{F}_T$-measurable.
2. Y is $(\mathcal{F}_t)$-adapted and càdlàg $\rightsquigarrow \mathcal{S}$
3. Z is $(\mathcal{F}_t)$-progressive, such that $\int_0^T Z_u^2 du < +\infty$ and !

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3. Z is $(\mathcal{F}_t)$-progressive, such that $\int_0^T Z_u^2 du < +\infty$ and Z is admissible, i.e. $\int ZdW$ is a supermartingale ($\rightarrow$ Dudley and Harrison/Pliska) $\rightsquigarrow \mathcal{L}$

The set of supersolutions with driver g and terminal condition ξ

$$\mathcal{A} := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : (Y, Z) \text{ fulfills } (0.1)\}$$

Minimal Supersolutions

A generator is a lower semicontinuous function

$$g : \mathbb{R} \times \mathbb{R}^d \to]-\infty, \infty].$$

Additional properties:

(Pos) $g(y, z) \in [0, +\infty]$ for all (y, z).

(Conv) $z \mapsto g(y, z)$ is convex.

(Mon) $g(y, z) \geq g(y', z)$ for all $y \geq y'$.

(Mon') $g(y, z) \leq g(y', z)$ for all $y \geq y'$.

Minimal Supersolutions

A natural candidate for the value process of a minimal supersolution:

$$\hat{\mathcal{E}}_t = \operatorname{ess\,inf}\left\{ Y_t : (Y, Z) \in \mathcal{A} \right\}, \quad t \in [0, T]$$

Question: Does there exist a càdlàg modification $\mathcal{E}$ of $\hat{\mathcal{E}}$ and a control process $Z \in \mathcal{L}$ such that $(\mathcal{E}, Z)$ is a supersolution ?

Minimal Supersolutions

A natural candidate for the value process of a minimal supersolution:

$$\hat{\mathcal{E}}_t = \operatorname{ess\,inf} \{ Y_t : (Y, Z) \in \mathcal{A} \}, \quad t \in [0, T]$$

Theorem:

Assume (*Pos*), (*Conv*) and either (*Mon*) or (*Mon'*). Suppose $\xi^- \in L^1$ and $\mathcal{A} \neq \emptyset$. Then

$$\mathcal{E}_t := \hat{\mathcal{E}}_t^+ = \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s$$

is the value process of the unique minimal supersolution, that is, there exists a unique control process Z such that $(\mathcal{E}, Z) \in \mathcal{A}$.

- Compactness (Delbaen and Schachermayer) versus fixpoint.
- Drop positivity for **(Pos')** $g(y, z) \geq az + b$. (utility maximization)
- Gregor Heyne, Michael Kupper and Christoph Mainberger, drop convexity in z for $g(y, 0) = 0$. (Barlow and Protter).

- Any sequence (x_n) in $\mathbb{R}^d$ such that $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ has a subsequence (x_{n_k}) converging to some $x \in \mathbb{R}^d$.

- Let (X_n) be a sequence of random variables in $L^2(\Omega, \mathcal{F}, P)$ such that $\sup_{n\in\mathbb{N}} E[X_n^2] < \infty$. Then there exists a sequence $Y_n \in \mathrm{conv}(X_n, X_{n+1}, \dots)$ such that $Y_n \to Y$ in $L^2(\Omega, \mathcal{F}, P)$.

- (Delbaen/Schachermayer) Let $(\int H^n dW)$ be a $\mathcal{H}^1$-bounded sequence of martingales. Then there exist $K^n \in \mathrm{conv}\{H^n, H^{n+1}, \dots\}$ and a localizing sequence of stopping times (τ^n) such that $(\int K^n dW)^{\tau^n} \to \int K dW$ in $\mathcal{H}^1$.

1) Paste strategies between stopping times $\rightsquigarrow$ construct $(Y^n, Z^n) \subset \mathcal{A}$ with,

$$\hat{\mathcal{E}}_{t_k^n} \geq Y^n_{t_k^n} - 1/n, \quad \text{and} \quad Y^n_t \geq Y^{n+1}_t,$$

2) $Y = \lim_{n\to\infty} Y^n$ and $\hat{\mathcal{E}}$ are supermartingales $\rightsquigarrow \mathcal{E} := \hat{\mathcal{E}}^+ = Y^+$.
3) Show that $\hat{\mathcal{E}}_t \geq \mathcal{E}_t$.
4) There is a localizing sequence (σ_k) such that

$$\left(\int Z^n dW\right)^{\sigma_k}$$

is bounded in $\mathcal{H}^1$.
5) Delbaen and Schachermayer $\rightsquigarrow$ convex combinations such that

$$\int_0^t \tilde{Z}^n_s dW_s \xrightarrow[n\to+\infty]{} \int_0^t Z_s dW_s.$$

6) Verification with $(\mathcal{E}, Z)$ is based on Helly's theorem and Fatou's lemma.

Minimal Supersolutions

$\Updownarrow$

Nonlinear Expectations

($\rightsquigarrow$ Peng's g-expectations)

Stability results

For any “nice” generator g the mapping

$$\mathcal{E}^g : \xi \mapsto \text{ minimal supersolution with terminal condition } \xi$$

satisfies

	$\xi \mapsto \mathcal{E}^g_0(\xi)$	$E[\xi] = \int_\Omega \xi(\omega) P(d\omega)$
(N)	$\mathcal{E}^g_0(m) = m$	$E[m] = m$
(T)	$\mathcal{E}^g_0(\xi + m) = \mathcal{E}^g_0(\xi) + m$	$E[\xi + m] = E[\xi] + m$
(TC)	$\mathcal{E}^g_0(\xi) = \mathcal{E}^g_0\left(\mathcal{E}^g_t(\xi)\right)$	$E[\xi] = E\left[E[\xi \mid \mathcal{F}_t]\right]$
Linearity:	—	$E[\lambda\xi^1 + \xi^2] = \lambda E[\xi^1] + E[\xi^2]$

$\rightsquigarrow$ nonlinear expectation

Stability results

The nonlinear expectation $\mathcal{E}_0^g(\cdot)$ satisfies

- Monotone convergence:
  $0 \leq \xi^n \uparrow \xi$ implies $\mathcal{E}_0^g(\xi) = \lim_n \mathcal{E}_0^g(\xi^n)$
- Fatou's lemma: $\mathcal{E}_0^g(\liminf_n \xi^n) \leq \liminf_n \mathcal{E}_0^g(\xi^n)$
- is $\sigma(L^1, L^\infty)$-lower semicontinuous.

If g is independent of y, by convex duality:

$$\mathcal{E}_0^g(\xi) = \sup_{Q \ll P} \{E_Q[\xi] - \alpha_{\min}(Q)\}$$

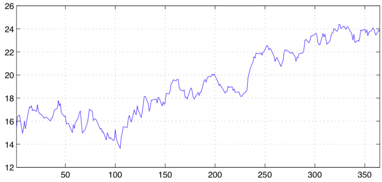
$=$ representation of a convex risk measure

Model Uncertainty and Robust Hedging

($\rightsquigarrow$ Peng's G-expectation)

The role of the probability measure P

The probability P defines the dynamics of the process



$P \quad \longleftrightarrow \quad$ probabilistic model, e.g. on $C([0,T];\mathbb{R})$

What is the probability measure P ?

The role of the probability measure P

P can only partially be identified by statistical methods

$$\Downarrow$$

Take into account a family $\mathcal{P}$ of probability measures (models)

$\rightsquigarrow$ **Model Uncertainty**

Remark: The probability measures are typically singular!

$$\mathcal{P}(A) := \sup_{P \in \mathcal{P}} P[A] = \text{capacity}$$

$\rightsquigarrow$ Denis, Martini, Peng, Hu, Bion-Nadal, Soner, Touzi, Zhang, Nutz,...

E.g. $$\frac{dS_t(\theta)}{S_t(\theta)} = \mu dt + \theta dW_t, \qquad \underline{\theta} \leq \theta \leq \overline{\theta}$$

- Θ is a set of volatility processes:

$$\theta : \Omega \times [0, T] \to \mathbb{R}_{++} \quad (\mathbb{S}_d^{>0}).$$

- Our state space:

$$\tilde{\Omega} := \Omega \times \Theta$$

- Driving process: $\tilde{W} : \tilde{\Omega} \times [0, T] \to \mathbb{R}$, where

$$\tilde{W}(\theta) = \int \theta dW, \quad \theta \in \Theta$$

- Progressively learning about the volatility

$$\rightsquigarrow \quad \tilde{\mathcal{F}}_t := \sigma(\tilde{W}_s : s \leq t), \quad t \in [0, T]$$

In general not right-continuous.

Let μ^θ be the probability measure induced by $\tilde{W}(\theta)$ on $C([0,T],\mathbb{R}^d)$ with the Borel σ-algebra. These probability measures are singular to each others. There is no dominating probability measures!

$$\tilde{P}[A] := \sup_{\theta\in\Theta} P\left[A(\theta)\right], \quad A \in \tilde{\mathcal{F}}_T$$

Properties (like equality and inequalities) holds quasi-surely if the event B, where they do not hold is a polar set, i.e., $B \in \tilde{\mathcal{F}}_T$ with $\tilde{P}[B] = 0$.

We assume that $\{\mu^\theta : \theta \in \Theta\}$ is weakly compact.

$\rightarrow$ Denis, Martini, Peng, Hu, Bion-Nadal, Soner, Touzi, Zhang, Nutz

For all $\theta \in \Theta$,

$$\begin{cases} Y_\sigma(\theta) - \int_\sigma^\tau g_u(Y_u(\theta), Z_u(\theta))du + \int_\sigma^\tau Z_u(\theta)d\tilde{W}_u(\theta) \geq Y_\tau(\theta), \\ Y_T(\theta) \geq \xi(\theta) \end{cases} \tag{0.2}$$

where σ, τ are $(\mathcal{F}_t)$-stopping times with $0 \leq \sigma \leq \tau \leq T$.

1. ξ is $\tilde{\mathcal{F}}_T$-measurable.
2. Y is làdlàg and $Y_t \in L^1_b(\tilde{\mathcal{F}}_t) \cap C(\tilde{\mathcal{F}}_t)$, $Y(\theta)$ is optional $\rightsquigarrow \tilde{\mathcal{S}}$
3. Z is $(\tilde{\mathcal{F}}_t)$-predictable, such that $\int_0^T Z_u^2(\theta)\theta_u^2 du < +\infty$ and $\int Z(\theta)d\tilde{W}(\theta)$ is a supermartingale for all $\theta \in \Theta \rightsquigarrow \tilde{\mathcal{L}}$
4. $g : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$, such that $g(\theta)$ is a generator as before for all $\theta \in \Theta$.

The set of supersolutions with driver g and terminal condition ξ

$$\mathcal{A} := \left\{ (Y, Z) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{L}} : (Y, Z) \text{ fulfills (0.2)} \right\}$$

As a usual, our natural candidate for the minimal supersolution

$$\hat{\mathcal{E}}_t = \operatorname{ess\,inf}\{Y_t : (Y, Z) \in \mathcal{A}\}$$

However, there is no reference probability measure!
$\rightarrow$ Bion-Nadal, Nutz, ...

We consider the infimum:

$$\hat{\mathcal{E}}_t = \inf\{Y_t : (Y, Z) \in \mathcal{A}\}$$

Our existence Theorem reads as follows

Theorem:

Assume (*Pos*), (*Conv*) and either (*Mon*) or (*Mon*′). Suppose $\xi^- \in L^1_b(\tilde{\mathcal{F}}_T)$, $\mathcal{A} \neq \emptyset$ and $\hat{\mathcal{E}}_t \in C(\tilde{\mathcal{F}}_t)$. Then there exists a làdlàg modification $\mathcal{E}$ of $\hat{\mathcal{E}}$, which is the value process of the unique minimal supersolution, that is, there exists a unique control process Z such that

$$(\mathcal{E}, Z) \in \mathcal{A}.$$

We give conditions under which the $\hat{\mathcal{E}}$ fulfills these assumptions (Markovian Setting).

Thank You!