On the long-term asymptotic exponential arbitrage

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4 July, 2012

Outline

1. Introduction of the notion
2. The conjecture in Föllmer and Schachermayer [FS07]
3. Our extension
4. Statement of the main theorem
5. Outline of its proof

Introduction of the notion

Market Model

- $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space
- the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions
- the price process $S = (S_t)_{t \geq 0}$ be any general $\mathbb{R}^d$-valued semimartingale.

Attainable Contingent Claims

$$\mathbf{K}^T := \left\{ \int_0^T H_s dS_s \;\middle|\; \int HdS \geq -a \;\text{ for some } a \in \mathbb{R}_+ \right\}.$$

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Definition: The price process $S = (S_t)_{t \geq 0}$ allows asymptotic exponential arbitrage with exponentially decaying failure probability if there exist $0 < \tilde{T} < \infty$ and constants $C, \gamma_1, \gamma_2 > 0$ such that for all $T \geq \tilde{T}$, there is $X_T \in \mathbf{K}^T$ with:

a) $X_T \geq -e^{-\gamma_1 T}$ P-a.s.

b) $P[X_T \leq e^{\gamma_1 T}] \leq Ce^{-\gamma_2 T}$.

- This form of a long-term asymptotic arbitrage was considered for the first time in Föllmer and Schachermayer [FS07].
- Such a name was given in Mbele Bidima and Rásonyi [MBR10].

If S has that property:

- we can find a profit X_T for any large enough maturity T :
  - $X_T \geq -e^{-\gamma_1 T} \Rightarrow$ exponentially decreasing potential loss
  - $P[X_T \leq e^{\gamma_1 T}] \leq Ce^{-\gamma_2 T} \Rightarrow$ exponentially growing up with an exponentially small probability of failure
- we get an explicit relation between any tolerance level of failure and the necessary time to reach this level.
- when $T \to \infty$, we get in the limit a riskless profit.

$\Rightarrow$ can be interpreted as a strong and quantitative form of long-term arbitrage.

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Let's have a short look at the relation between asymptotic exponential arbitrage with exponentially decaying failure probability and utility maximization.

We define

$$\mathbf{K}_x^T := \left\{ \int_0^T H_s dS_s \in \mathbf{K}^T \;\middle|\; \int_0^T H_s dS_s \geq -x \right\},$$

For any utility function U, we define for any fixed $T < \infty$ its primal value function

$$u_T(x) := \sup_{X_T \in \mathbf{K}_x^T} E\big[U(x + X_T)\big].$$

Proposition: Let $U : (0, \infty) \to \mathbb{R}$ be a utility function. If the price process S allows asymptotic exponential arbitrage, then for any $x > 0$,

$$\lim_{T \to \infty} u_T(x) = U(\infty).$$

Proposition: Let $U : (0, \infty) \to [0, \infty)$ be a positive utility function. If the price process S allows asymptotic exponential arbitrage with exponentially decaying failure probability
$\Rightarrow$ there is a constant $\gamma > 0$ such that for all $x > 0$, there exists $T_x < \infty$ such that for any $T \geq T_x$,

$$u_T(x) \geq \frac{1}{2} U(e^{\gamma T}).$$

Proofs and more details: See Föllmer and Schachermayer [FS07] and Mbele Bidima [MBR10].

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A Nature Question is whether we give any characterization on the process S which allows asymptotic exponential arbitrage with exponentially decaying failure probability

The conjecture in Föllmer and Schachermayer [FS07]

In Föllmer and Schachermayer [FS07], the authors considered an $\mathbb{R}^d$-valued diffusion process $\tilde{S} = (\tilde{S}_t)_{t\geq 0}$ of the form

$$d\tilde{S}_t = \sigma(\tilde{S}_t)\big(d\tilde{W}_t + \varphi(\tilde{S}_t)dt\big) \tag{1}$$

- $\mathbb{R}^N$-valued Brownian motion $(\tilde{W}_t)_{t\geq 0}$
- the market price of risk function $\varphi : \mathbb{R}^d \to \mathbb{R}^N$.
- $\varphi(\tilde{S}_t) \in \big(\ker(\sigma(\tilde{S}_t))\big)^{\perp}$ for all $t \geq 0$
- the process $\tilde{Z} := (\tilde{Z}_t)_{t\geq 0}$ defined by

$$\tilde{Z}_t := \exp\left(\int_0^t \varphi(\tilde{S}_s)d\tilde{W}_s - \frac{1}{2}\int_0^t \|\varphi(\tilde{S}_s)\|^2 ds\right)$$

is a strictly positive martingale, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$.

In Föllmer and Schachermayer [FS07], the authors introduced the following notion and conjecture:

Definition: The market price of risk function $\varphi(\cdot)$ of $\tilde{S}$ satisfies a large deviations estimate if there are constants $c_1, c_2 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(P\left[\frac{1}{T} \int_0^T \|\varphi(\tilde{S}_t)\|^2 dt \le c_1 \right] \right) < -c_2.$$

Conjecture:
- If the filtration $\tilde{\mathbb{F}}$ is the P-augmentation of the raw filtration generated by $(\tilde{W}_t)_{t\geq 0}$ and
- if the market price of risk function $\varphi(\cdot)$ of $\tilde{S}$ defined in (1) satisfies a large deviations estimate

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Existing Result In Mbele Bidima and Rásonyi [MBR10], the authors have proved this conjecture in a discrete-time version of the model (1) with bounded drift and volatility.

Our Extension

Our aim is to extend this conjecture to the continuous semimartingale case and prove it (is positive)

Let's go back to our Market Model

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Equivalent Martingale Measure

$$\mathbf{M}_m^{T,e} := \Big\{ Q \text{ p.m. on } \mathcal{F}_T \;\Big|\; Q \approx P|_{\mathcal{F}_T} \text{ and } (S_t)_{0 \leq t \leq T} \text{ is a local } Q\text{-martingale} \Big\}.$$

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Assumption: Assume that:

- $\mathbf{M}_m^{T,e} \neq \emptyset$ for any $0 < T < \infty$
- the filtration $\mathbb{F}$ is continuous,
  i.e. every local martingale with respect to $\mathbb{F}$ is continuous.

$\Rightarrow$ more general setting than in Föllmer and Schachermayer [FS07].

Two Questions.

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2. How to formulate the notion of a large deviations estimate in the general framework ?

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Lemma 1. Under the above Assumption, there exists an $\mathbb{R}^d$-valued process $\lambda = (\lambda_t)_{t \geq 0} \in L^2_{loc}(M)$, such that for any $T < \infty$ and any $Q \in \mathbf{M}_m^{T,e}$, the density process $Z^Q = (Z_t^Q)_{0 \leq t \leq T}$ of Q with respect to $P|_{\mathcal{F}_T}$ is

$$Z^Q = \mathcal{E}\left(\int -\lambda dM + N^Q \right) =: \mathcal{E}(L^Q) \qquad P \times dt \text{ -a.s} \quad \text{on } [0, T],$$

where M is the continuous local martingale coming from the canonical decomposition of S and $N^Q := (N_t^Q)_{0 \leq t \leq T}$ is a continuous local martingale with $N^Q \perp M^T$.

As a consequence, we have

$$\langle L^Q \rangle_t \geq \int_0^t \lambda_s^2 \, d\langle M \rangle_s \quad P\text{-a.s. for each} \quad t \in [0, T].$$

Proof: Based on Theorem 1 in Schweizer [Sch95].

Definition: We call λ a market price of risk of the price process S.

Remark: The market price of risk λ does not need to be unique.

However, the process $\int \lambda dM$ does not depend on the choice of a market price of risk λ.

We extend the notion of satisfying a large deviations estimate to our framework.

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We extend the notion of satisfying a large deviations estimate to our framework.

Definition: A market price of risk $\lambda := (\lambda_t)_{t \geq 0}$ of the price process S satisfies a large deviations estimate if there exist constants $c_1, c_2 > 0$ such that

$$\limsup_{T \to \infty} \frac{1}{T} \log \left(P\left[\frac{1}{T} \int_0^T \lambda_s^2 \, d\langle M \rangle_s \leq c_1 \right] \right) < -c_2.$$

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Main result

Theorem. Under the above Assumption, suppose that a market price of risk $\lambda = (\lambda_t)_{t \geq 0}$ of the price process S satisfies a large deviations estimate.

$\Rightarrow$ S allows asymptotic exponential arbitrage with exponentially decaying failure probability.

Corollary. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ be a filtered probability space where the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ is the $\tilde{P}$-augmentation of the raw filtration generated by an $\mathbb{R}^n$-valued BM $\tilde{W}$.
Moreover, let $\tilde{S}$ be the diffusion process defined in (1).
Suppose that the market price of risk function $\varphi(\cdot)$ satisfies a large deviations estimate,

$\Rightarrow \tilde{S}$ allows asymptotic exponential arbitrage with exponentially decaying failure probability.

$\Longrightarrow$ Gives us the proof of the conjecture in Föllmer and Schachermayer [FS07]

Main idea of the proof of the Theorem

An Auxiliary Lemma

Lemma 2. Fix any $T < \infty$ and let $0 < \varepsilon_1, \varepsilon_2 < 1$ be such that for each $Q \in \mathbf{M}_m^{T,e}$, there is a set $A_T^Q \in \mathcal{F}_T$ with $P[A_T^Q] \leq \varepsilon_1$ and $Q[A_T^Q] \geq 1 - \varepsilon_2$

$\Rightarrow$ For any $0 < \tilde{\varepsilon}_1, \tilde{\varepsilon}_2 < 1$ with

$2^{1+\alpha} \max(\varepsilon_1, \varepsilon_2^\alpha) \leq \tilde{\varepsilon}_1 \tilde{\varepsilon}_2^\alpha$ for some $0 < \alpha < \infty$,

there exists $X_T \in \mathbf{K}^T$ such that

a) $X_T \geq -\tilde{\varepsilon}_2$ P-a.s.

b) $P[X_T \geq 1 - \tilde{\varepsilon}_2] \geq 1 - \tilde{\varepsilon}_1$.

Proof: Direct consequence of Proposition 2.3 in Föllmer and Schachermayer [FS07].

Then it remains to prove

Lemma 3. Under the same Assumptions in the Theorem: price of risk $\lambda = (\lambda_t)_{t\geq 0}$ of the price process S satisfies a large deviations estimate.

$\Rightarrow \exists$ constants $\gamma_1, \gamma_2 > 0$ and $T_0 < \infty$ such that for all $T \geq T_0$, we have for any $Q \in \mathbf{M}_m^{T,e}$ a set $A_T^Q \in \mathcal{F}_T$ with

$$P[A_T^Q] \leq 2e^{-\gamma_1 T} \quad \text{and} \quad Q[A_T^Q] \geq 1 - e^{-\gamma_2 T}.$$

Proof. The key point is a time-change argument (applying Dambis-Dubins-Schwarz theorem).

References

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Thank you for your attention!