Exact replication under portfolio constraints: a viability approach

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Motivation

Complete market with no interest rate and one stock : $dS_t = \sigma(S_t)dW_t$

Price and Hedge of a European option with regular payoff $h(S_T)$:

$$P_t = \mathbb{E}_t\left[h(S_T)\right] \qquad \Delta_t = \mathbb{E}_t\left[h'(S_T)\frac{\nabla S_T}{\nabla S_t}\right]$$

where ∇S is the tangent process with dynamics $d\nabla S_t = \sigma'(S_t)\,\nabla S_t dW_t$.

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$$h \text{ is increasing} \quad \Longrightarrow \quad \Delta_t \;=\; \mathbb{E}_t\left[h'(S_T)\tfrac{\nabla S_T}{\nabla S_t}\right] \;\geq\; 0$$

$\Longrightarrow$ If h is increasing, the super-replication price under no short sell constraints of $h(S_T)$ is the replication price.

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For which couple [model,constraints] is this property satisfied ?

Agenda

1. Super-replication under portfolio constraints
2. Easily tractable financial examples
3. Viability for BSDE or PDE
4. First order viability and portfolio constraints
5. Financial applications

Super-replication price

The market model

$$S_t = S_0 + \int_0^t \sigma(S_u) dW_u \,, \qquad 0 \le t \le T \,.$$

Portfolio process

$$X_s^{t,x,\Delta} = x + \int_t^s \Delta_u dS_u = x + \int_t^s \Delta_u \sigma(S_u) dW_u \,, \qquad 0 \le t \le s \le T \,.$$

In addition to classical admissibility conditions, we impose

$$\Delta \in \mathcal{A}_t^K := \{\Delta \in \mathcal{A} \text{ such that } \Delta_s \in K \;\; \mathbf{P}-a.s. \,, \quad t \le s \le T\} \,,$$

where K is a closed convex set.

The super-replication price of $h(S_T)$ at time t under K-constraints defines as

$$p_t^K[h] := \inf \left\{ x \in \mathbb{R} \,, \;\; \exists\, \Delta \in \mathcal{A}_t^K \text{ such that } X_T^{t,x,\Delta} \ge h(S_T) \;\; \mathbf{P}-a.s. \right\}$$

Condition at maturity : Facelift transform

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At maturity T, we need $\Delta_T \in K$.

$\Longrightarrow$ Need to change the terminal condition.

$\Longrightarrow$ Smallest function above h whose "derivatives" belong to K.

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Definition of the facelift operator :

$$\mathrm{F}_K[h](x) \quad := \quad \sup_{y \in \mathbb{R}^d} h(x+y) - \delta_K(y)\,, \qquad x \in \mathbb{R}^d\,,$$

where $\delta_K : y \mapsto \sup_{z \in K} \langle y, z\rangle$ is the support function of K.

$\mathrm{F}_K[h]$ identifies as the smallest viscosity super-solution of

$$\min\left\{u - h,\ \inf_{|\zeta|=1} \delta_K(\zeta) - \langle \zeta, \partial_x u\rangle\right\} = 0$$

Characterizations of the super-replication price

Direct PDE characterization

$p_t^K[h] = v^K[h](t, S_t)$ where $v^K[h]$ is the unique viscosity solution of the PDE

$$\min\left\{-\mathcal{L}^\sigma u,\ \inf_{|\zeta|=1} \delta_K(\zeta) - \langle \zeta, \partial_x u\rangle\right\} = 0 \quad \text{for } t < T \qquad \text{and} \qquad u(T,x) = F^K[h]\,,$$

with $\mathcal{L}$ the Dynkin operator of the diffusion S.

Dual representation in terms of pricing measure :

$$v^K[h](t,x) = \sup_{\nu \text{ s.t. } \delta_K(\nu)<\infty} \mathbb{E}^{\mathbb{Q}^\nu_{t,x}}\left[h(X_T^{t,x}) - \int_t^T \delta_K(\nu_s)ds\right]\,,$$

with $\mathbb{Q}^\nu$ the equivalent measure for which $W_t - \int_0^t \nu_s ds$ is a Brownian motion.

BSDE characterization :

Minimal solution of the Z-constrained BSDE

$$Y_t = F^K[h](S_T) - \int_t^T Z_s dW_s + \int_t^T dL_s\,, \qquad \text{with} \quad Z_t \in K\sigma(S_t)$$

The question of interest

We always have

$$\begin{array}{c}\text{super-replicate } h(S_T) \text{ under } K\text{-constraints} \\ \Updownarrow \\ \text{super-replicate } F^K[h](S_T) \text{ under } K\text{-constraints}\end{array}.$$

When do we have

$$\begin{array}{c}\text{super-replicate } h(S_T) \text{ under } K\text{-constraints} \\ \Updownarrow \\ \text{replicate } F^K[h](S_T) \text{ without constraints}\end{array} ?$$

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In the Black Scholes model : [Broadie, Cvitanic, Soner]

True for intervals in dimension 1

True for any convex set K and money or wealth proportion constraints

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For general local volatility model : [Our contribution]

A necessary and sufficient condition for the previous property to hold for a large class of payoff functions h.

Intervals in dimension 1

Dimension 1 stock : $dS_t = \sigma(S_t)dW_t$ with σ regular.

Interval convex constraint $K := [a, b]$.

Let h be a payoff function such that $F^K[h]$ is differentiable.

Do we have $p_t^K[h] = p_t[F^K[h]]$?

The unconstrained hedging strategy of $F^K[h]$ at time t is

$$\Delta_t := \mathbb{E}_t\left[\nabla F^K[h](S_T)\ \frac{\nabla S_T}{\nabla S_t}\right], \qquad \text{with } d\nabla S_t = \sigma'(S_t)\nabla S_t dW_t\ .$$

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$\Longrightarrow$ ∇S interprets as a probability change and we can find a proba $\mathbb{Q}$ s. t.

$$\Delta_t := \mathbb{E}_t^{\mathbb{Q}}\left[\nabla F^K[h](S_T)\right] \in K\ , \qquad 0 \leq t \leq T\ ,$$

since $\nabla F^K[h]$ is valued in the convex K.

$\Longrightarrow$ Revisit and generalize this known result for the Black Scholes model.

Hypercubes for d stocks with separate dynamics

Dimension d stock with separate dynamics : $dS^i_t = \sigma^i(S^i_t)dW_t \ , \quad 1 \le i \le d \ .$

Hypercube constraints $K := \Pi_{i=1}^d [a_i, b_i]$.

Let h be a payoff function such that $F^K[h]$ is differentiable.

Do we have $p^K_t[h] = p_t[F^K[h]]$?

The unconstrained hedging strategy of $F^K[h]$ at time t is

$$\Delta^i_t \quad := \quad \mathbb{E}_t \left[(\nabla F_K[h](S_T))^i \ \ \frac{\nabla S^i_T}{\nabla S^i_t} \right] \qquad \text{with } d\nabla S^i_t = \nabla\sigma^i(S^i_t)^\top \nabla S^i_t dW_t \ .$$

$\Longrightarrow$ Since $\nabla F^K[h]$ is valued in the hypercube K, $\Delta \in K$ because

$$a_i \ = \ a_i \ \mathbb{E}_t \left[\frac{\nabla S^i_T}{\nabla S^i_t} \right] \ \le \ \Delta^i_t \ \le \ b_i \ \mathbb{E}_t \left[\frac{\nabla S^i_T}{\nabla S^i_t} \right] \ = \ b_i \ , \qquad 0 \le t \le T \ ,$$

Does it generalize to any convex set or any model ?

General convex set K and model dynamics σ

Consider

- A model dynamics : σ Lipschitz, differentiable and invertible
- Portfolio constraints : K closed convex set with non empty interior

Problem of interest :

Is there a structural condition on the coupe $[K, \sigma]$ under which for any payoff h in a given class , we have $p^K[h] \;=\; p[F^K[h]]$?

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First, simplified version :

Is there a structural condition on the couple $[K, \sigma]$ under which For any payoff $h \in C^1_K$, we have $p^K[h] = p[h]$?

where C^1_K denotes the class of C^1_b functions with derivatives valued in K.
(i.e. regular and stable under F^K)

BSDE representation for the Δ

For any payoff $h \in C^1_K$, the unconstrained price $(p(t,S_t))_{0\le t\le T}$ of $h(S_T)$ is solution of the BSDE

$$Y_t \quad = \quad h(S_T) - \int_t^T Z_r dW_r \ , \quad 0 \le t \le T \ .$$

The corresponding hedging strategy Δ^h_t identifies to $\nabla_x p(t,S_t) = \nabla Y_t (\nabla X_t)^{-1}$.

Hence Δ satisfies the (linear) BSDE :

$$\Delta^h_t \quad = \quad \nabla h(S_T) + \int_t^T \sum_{j=1}^d [\partial_x \sigma^j(S_r)]^\top \Gamma^h_r \sigma(S_r) dr - \int_t^T \Gamma^h_r \sigma(S_r) dW_r \ ,$$

We know that $\nabla h(S_T) \in K$.

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We know that $\nabla h(S_T) \in K$.

$\Longrightarrow$ End up on a viability problem :

For any ∇h valued in K, does the solution Δ^h of the BSDE remains in K ?

Viability property for BSDE

[Buckdahn, Quincampoix, Rascanu] provide a Necessary and Sufficient condition for viability property on BSDEs (or PDEs) :

For any terminal condition $\xi \in K$, the solution of the BSDE

$$Y_t = \xi + \int_t^T F(Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

satisfies $Y_t \in K \;\; \mathbf{P} - a.s.$, for $0 \leq t \leq T$.

$$\Updownarrow$$

There exists $C > 0$ such that

$$2\langle y - \pi_K(y), F(y,z)\rangle \leq \frac{1}{2}\langle \partial^2_{xx}[d^2_K(y)]z, z\rangle + C d^2_K(y) \;, \quad \forall (y,z) \in \mathbb{R}^d \times M^d$$

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$\Longrightarrow$ This provides a sufficient condition for our problem.

Is it necessary ?

Revisiting the condition of [BQR] for "regular" convex set K

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$$\Updownarrow$$

Denoting by n the unit normal vector to K, there exists $C > 0$ s.t.

$$2\langle y - \pi_K(y), F(y,z)\rangle \quad \leq \quad \frac{1}{2}\langle n(y)z, n(y)z\rangle + C d^2_K(y)\ , \quad \forall y \notin Int(K), \forall z \in M^d$$

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$$\langle n(y),F(y,z)\rangle \quad \leq \quad 0\ , \quad \forall (y,z)\in\partial K\times M^d \quad \text{s.t. } n(y)^\top z=0$$

Adapting the condition to our framework

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rewrites

$$2\langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(x)]^\top \gamma \sigma(x)\rangle \quad = \quad 0\,, \;\; \forall (y,\gamma) \in \partial K \times M^d \quad \text{s.t. } n(y)^\top \gamma = 0$$

Condition too strong in our context.

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But γ is symmetric and we shall work under the condition :

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Technical point : What about points with multiple normal vectors ?

$\Longrightarrow$ Need to restrict to border points $\widetilde{\partial K}$ with unique normal vector.

The main result

For a closed convex set K s.t. $Int\ K \neq \emptyset$ and an elliptic volatility σ, we have :

For any payoff $h \in C^1_K$, the hedging strategy of $h(S_t)$ belongs to K,
i.e. $p^K(h) = p(h)$

$$\Updownarrow$$

$$\langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(x)]^\top \gamma \sigma(x) \rangle = 0\ ,\quad \forall (x,y,\gamma) \in \mathbb{R}^d \times \widetilde{\partial K} \times \mathcal{S}^d \quad \text{s.t. } n(y)^\top \gamma = 0$$

This provides a structural condition on the couple $[K,\sigma]$

under which portfolio restrictions have no effect on payoff functions

whose derivatives satisfy the constraint.

Sketch of proof

- Half-space decomposition of K

$K = \cap_{y \in \widetilde{\partial K}} H_y$ with H_y half-space containing K and tangent to K at y

Due to the linearity of the driver, we observe

$$K \text{ is viable} \quad \Leftrightarrow \quad \text{any half-space } H_y \text{ is viable}$$

$\Longrightarrow$ need to verify that each half-space H_y with normal vector $n(y)$ is viable iff

$$\langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(x)]^\top \gamma \sigma(x) \rangle = 0 \,, \quad \forall (x, \gamma) \in \mathbb{R}^d \times \mathcal{S}^d \quad \text{s.t. } n(y)^\top \gamma = 0$$

- Focus on the dynamics of $\langle n(y), \Delta_t \rangle$

For Δ solution of the BSDE with $\Delta_T \in H_y$, Ito's formula gives

$$\langle n(y), \Delta_t \rangle \quad \leq \quad 0 + \int_t^T \langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(X_r)]^\top \Gamma_r \sigma(X_r) \rangle dr - \int_t^T \langle n(y), \Gamma_r \sigma(X_r) dW_r \rangle$$

Probability change $\Longrightarrow$ the condition is sufficient

Terminal condition $\Delta_T = \gamma(X_T - x)$ $\Longrightarrow$ the condition is necessary

The constrained super replication problem under constraints

What happens if the payoff needs to be facelifted ?

For any payoff $h \in \mathcal{H}$, the hedging strategy of $F^K[h](S_t)$ belongs to K,

i.e. $p^K(h) = p(F^K[h])$

$$\Updownarrow$$

$$\langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(x)]^\top \gamma \sigma(x) \rangle = 0 \ , \ \ \forall (x, y, \gamma) \in \mathbb{R}^d \times \widetilde{\partial K} \times \mathcal{S}^d \quad \text{s.t. } n(y)^\top \gamma = 0$$

where $\mathcal{H}$ it the class of lower semi continuous, bounded from below payoffs s.t.

$$\mathbb{E}|F^K[h](S_T^{t,x})|^2 < \infty \ , \ \forall (t,x) \in [0,T] \times \mathbb{R}^d .$$

When K is bounded, we can restrict to lower semi continuous functions.

The Necessary and sufficient condition

$$\langle n(y), \sum_{j=1}^{d} [\partial_x \sigma^j(x)]^\top \gamma \sigma(x) \rangle = 0 \,, \quad \forall (x, y, \gamma) \in \mathbb{R}^d \times \widetilde{\partial K} \times \mathcal{S}^d \quad \text{s.t. } n(y)^\top \gamma = 0$$

For a fixed y, let introduce $(n(y), \bar{n}_2(y), \dots, \bar{n}_d(y))$ an orthonormal basis of $\mathbb{R}^d$.

The family $(e_{k\ell})_{2 \le k \le \ell \le d}$ of $n(n-1)/2$ elements given by

$$e_{k\ell} \quad = \quad \bar{n}_\ell(y) \bar{n}_k(y)^\top + \bar{n}_k(y) \bar{n}_\ell(y)^\top \,, \qquad 2 \le k \le \ell \le d \,.$$

is an orthonormal basis of $\left\{ \gamma \in S_d \,, s.t. \;\; n(y)^\top \gamma = 0 \right\}$.

The Necessary and Sufficient condition rewrites

$$\left\langle n(y), \partial_x \left[\sum_{j=1}^{d} \langle \bar{n}_k(y), \sigma^{\cdot j}(x) \rangle \langle \bar{n}_\ell(y), \sigma^{\cdot j}(x) \rangle \right] \right\rangle \; = \; 0 \,, \quad \forall y \in \widetilde{\partial K} \,, \;\; 2 \le k, \ell \le d \,.$$

No short Sell on Asset 1

- In dimension 2

No short sell on Asset 1 : $n(y)^\top = (1,0)$, hence $\bar{n}^\top = (0,1)$ and the condition rewrites

$$\partial_1 \left[|\sigma^{21}|^2 + |\sigma^{22}|^2\right] \quad = \quad 0$$

The quadratic variation of asset 2 does not depend on asset 1.

- In dimension d

No short sell on Asset 1 : $n(y)^\top = (1,0,\ldots,0)$, hence $\bar{n}_j^\top = (1_{\{i=j\}})_i$ and the condition rewrites

$$\partial_1 \left[\sigma^{\ell 1}\sigma^{k1} + \ldots + \sigma^{\ell d}\sigma^{kd}\right] \quad = \quad 0\,, \qquad 2 \le \ell \le k \le d\,.$$

The quadratic covariation between other assets does not depend on asset 1.

Asset 1 non tradable

- In dimension 2

Asset 1 not tradable : $n(y)^\top = (1,0)$, hence $\bar{n}^\top = (0,1)$ and the condition rewrites

$$\partial_1 \left[|\sigma^{21}|^2 + |\sigma^{22}|^2 \right] \quad = \quad 0$$

The quadratic variation of asset 2 does not depend on asset 1.

Same conditions as for the no short sell case since only the border of the convex set K matters.

Bound on the number of allowed positions

Bound of the form $|\Delta_1| + |\Delta_2| \leq C$.

The convex set is a losange and we have two type of normal vectors.

First $n(y) = (1,1)$ so that $\bar{n}(y) = (-1,1)$ and the condition rewrites

$$\partial_1 \left[|\sigma^{11} - \sigma^{21}|^2 + |\sigma^{12} - \sigma^{22}|^2\right] + \partial_2 \left[|\sigma^{11} - \sigma^{21}|^2 + |\sigma^{12} - \sigma^{22}|^2\right] = 0$$

Second $n(y) = (-1,1)$ so that $\bar{n}(y) = (1,1)$ and the condition rewrites

$$\partial_1 \left[|\sigma^{11} + \sigma^{21}|^2 + |\sigma^{12} + \sigma^{22}|^2\right] - \partial_2 \left[|\sigma^{11} + \sigma^{21}|^2 + |\sigma^{12} + \sigma^{22}|^2\right] = 0$$

Conditions on quadratic variations in normal directions

Other applications in dimension 2

- Which convex sets work for the Black Scholes model ?

Only the hypercube ones.

- Which model dynamics works for any convex set ?

For assets with separate dynamics, the condition is equivalent to

$$\partial_1 \sigma^{11} = \partial_2 \sigma^{21} \qquad \text{and} \qquad \partial_1 \sigma^{12} = \partial_2 \sigma^{22} .$$

Hence, the only possible models are of the form

$$\begin{aligned} dS_t^1 &= \sigma^{11}(S_t^1) dB_t^1 + \sigma^{12}(S_t^1) dB_t^2 , \\ dS_t^2 &= [\sigma^{11}(S_t^2) + \lambda_1] dB_t^1 + [\sigma^{12}(S_t^2) + \lambda_2] dB_t^2 , \end{aligned}$$

Conclusion

- Necessary and sufficient condition ensuring that in order to super-replicate under constraints, the facelifting procedure of the payoff is sufficient.
- We can adapt the form of the model to anticipated portfolio constraints.
- US options.

- Portfolio constraints in terms of money amount or wealth proportion ?
- How can we compute numerically the solution whenever the condition is not satisfied ?