The Utility Indifference Price of the Defaultable Bond in a Jump-Diffusion Model

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1 Introduction

2 Problem Formulation

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- Value Function as Viscosity Solution
- Comparison Principle

4 Numerical Results

Credit Risk

The global financial tsunami triggered by the U.S. subprime mortgage crisis drives the world economy into recession. Today the European debt crisis is getting more and more deteriorative, which makes the investors have no confidence in the national credit of sovereign state. One of the main reasons resulting in all of these is that the participants in financial markets lack enough awareness for credit risk.

Credit Risk

Credit risk, an important component consisting of financial risks, is an investors risk of loss arising from counterparties who cant or would not like to fulfill their obligations on time.

Credit Risk Models

Structural Model Merton(1974), Black and Cox(1976)

Shortcoming

Credit spread verges to zero in the limit of zero maturity.

Intensity-based Model Ramaswamy and Sundaresan(1986), Litterman and Iben(1991), Jarrow and Turnbull(1995).

Feature

Characterize credit risk as an exogenous stochastic impact, thus avoiding the aforementioned defect.

Financial Derivative Pricing

No-arbitrage Valuation: Black and Scholes(1973)

Shortcomings

- The framework is unable to capture and explain high premiums observed in credit derivatives markets for unlikely events.
- Most credit derivatives markets are OTC. There are many constraints on transaction. Some assets are illiquid or even non-tradable.

Financial Derivative Pricing

Utility Indifference Valuation: Hodges and Neuberger(1984)

Features

- Take the risk aversion of investors into account.
- The methodology is dependent on the optimal portfolio theory.
- The methodology is raised from utility indifference theory in economics.

Principle

Whether or not to hold the derivative which is treated as another investment instrument produce two optimal portfolio problems. The profit tendency of rationaleconomic men makes utilities of the above two selections equivalent, and the derivative price is derived from solving the utility equivalence equation.

Related Literature

- Leung, Sircar and Zariphopoulou(2007): Indifference valuation for defaultable bonds. Structural model of Black-Cox-type. GBM. The term-structure of the yield spread. Comparison with the Black-Cox model in the complete markets
- Liang and Jiang(2009): Indifference valuation for defaultable bonds. Default happening at the maturity and at the first-passage-time. Recovery rate.
- Sircar and Zariphopoulou(2006): Indifference valuation for credit derivatives in single and two-name cases. Intensity-based model.

Assumptions

- Two investible assets in the market: risk-free asset and risky asset
- No transaction cost
- Financing and shorting are forbidden.
- The investor has a utility function.

Investible Assets

Risk-free asset

bank account B_t

$$\frac{dB_t}{B_t} = rdt$$

Risky asset

stock S_t

$$\frac{dS_t}{S_t} = \tilde{\mu}dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{N}(dt, dz)$$

- $\tilde{\mu}$ is the drift coefficient.
- W_t is standard Brownian motion.
- $\tilde{N}(dt, dz) = N(dt, dz) - v(dz)dt$, $N(dt, dz)$ is random Poisson measure, and $v(dz)$ is the corresponding Levy measure.

Wealth

Wealth process

The investor's wealth $\tilde{X}_t$ and the self-financing trading strategy $\pi_t \in [0,1]$

$$d\tilde{X}_t = \pi_t \tilde{X}_t \frac{dS_t}{S_t} + (1-\pi_t)\tilde{X}_t \frac{dB_t}{B_t}$$

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Discount wealth process

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Discount wealth process

$X_t = e^{-rt}\tilde{X}_t$, $\mu = \tilde{\mu} - r$

$$dX_t = \mu \pi_t X_t dt + \sigma \pi_t X_t dW_t + \int_{-1}^{\infty} \pi_t X_t z \tilde{N}(dt, dz)$$

Defaultable Bond

- At the maturity time $T < \infty$, the investor gets 1 Dollar if no default occurs, or 0 if default occurs.
- The default time τ is the first jump time of an exogenous Poisson process. For simplicity, we assume τ is an exponential random variable with parameter β.
- The investor could choose whether or not to hold the bond in his investing period.

Two Opportunities for Investment

Not holding the defaultable bond

The value function

$$M(t,x) = \sup_{\pi(\cdot)\in\mathcal{A}} E\left[U(X_T)|X_t = x\right]$$

Holding the defaultable bond

The value function

$$V(t,x) = \sup_{\pi(\cdot)\in\mathcal{A}} E\left[U(X_{t+\tau})I_{0\leq\tau\leq T-t} + U(X_T+1)I_{\tau\geq T-t}|X_t = x\right] \quad (1)$$

$$= \sup_{\pi(\cdot)\in\mathcal{A}} E_{t,x}\left[\int_t^T e^{-\beta(s-t)}\beta U(X_s)ds + e^{-\beta(T-t)}U(X_T+1)\right] \quad (2)$$

Dynamic Programming Equations

- Not holding bond

$$\begin{cases} \dfrac{\partial M}{\partial t} + \sup\limits_{\pi \in [0,1]} \{\mathcal{L}^\pi M + \mathcal{B}^\pi M\} = 0, (t,x) \in [0,T) \times (0,+\infty) \\ M(T,x) = U(x), x \in (0,+\infty) \end{cases}$$

- Holding bond

$$\begin{cases} \dfrac{\partial V}{\partial t} + \sup\limits_{\pi \in [0,1]} \{\mathcal{L}^\pi V + \mathcal{B}^\pi V\} - \beta V + \beta U = 0, \\ (t,x) \in [0,T) \times (0,+\infty) \\ M(T,x) = U(x+1), x \in (0,+\infty) \end{cases} \tag{3}$$

$$\mathcal{L}^\pi \varphi = \left(\mu - \int_{-1}^{\infty} z v(dz)\right) \pi x \frac{\partial \varphi}{\partial x} + \frac{\sigma^2}{2}\pi^2 x^2 \frac{\partial^2 \varphi}{\partial x^2}$$

$$\mathcal{B}^\pi \varphi = \int_{-1}^{\infty} \Big[\varphi(t, x(1+\pi z)) - V(t,x)\Big] v(dz)$$

The Bond Price

If we can obtain the value functions M, V, then the utility indifference price p_0 of the defaultable bond is a solution of the following equation

$$M(0, x) = V(0, x - p_0)$$

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Viscosity Solution

Consider the general nonlinear integro-differential parabolic equation

$$-\frac{\partial u}{\partial t} + F(t, x, u, \mathcal{B}u, Du, D^2u) = 0, \ (t, x) \in \mathcal{O}_T = (0, T) \times \mathcal{O},$$

where

$$\mathcal{B}u = \int_{\mathbb{R}^N} [u(t, x + z) - u(t, x)] \, dp_{t,x}(z).$$

In addition, an elliptical condition is assumed for F, u is quasi-monotone and decreasing in the nonlocal item $\mathcal{B}u$.

Viscosity Solution

Definition

$u(t,x)$ is the viscosity subsolution(supersolution) of the above equation if $u(t,x)$ is upper(lower)-semicontinuous and one of the following equivalent items holds:

- $-q + F\left(t, x, u(t,x), \mathcal{B}u, p, X\right) \leq 0(\geq 0)$, for $\forall(q,p,X) \in P^{2,+}_{\mathcal{O}}(t,x)(P^{2,-}_{\mathcal{O}}(t,x))$, where $P^{2,+}_{\mathcal{O}}, P^{2,-}_{\mathcal{O}}$ are the parabolic semijets.
- $-\frac{\partial \phi}{\partial t}(\bar{t},\bar{x}) + F\left(\bar{t}, \bar{x}, u(\bar{t},\bar{x}, \mathcal{B}\phi, D\phi(\bar{t},\bar{x}), D^2\phi(\bar{t},\bar{x})\right) \leq 0(\geq 0)$, for each smooth function ϕ such that $u - \phi$ has a local maximum(resp., a minimum) at $(\bar{t},\bar{x})$.
- $-\frac{\partial \phi}{\partial t}(\bar{t},\bar{x}) + F\left(\bar{t}, \bar{x}, u(\bar{t},\bar{x}, \mathcal{B}\phi, D\phi(\bar{t},\bar{x}), D^2\phi(\bar{t},\bar{x})\right) \leq 0(\geq 0)$, for each smooth function ϕ such that $u - \phi$ has a global strict maximum(resp., a minimum) at $(\bar{t},\bar{x})$.

Value Function is Viscosity Solution

We mainly consider the equation (3) in the case of holding the defaultable bond. Using the dynamic programming principle and $It\hat{o}$ lemma, we can prove the following theorem.

Theorem

Assume the value function defined by (1) is continuous, and then it is a viscosity solution of the HJB equation (3).

The dynamic programming principle is as follows

$$V(t,x) = \sup_{\pi(\cdot)\in\mathcal{A}} E\left[\int_t^{t+h} e^{-\beta(s-t)}\beta U(X_s)ds + e^{-\beta h}V(t+h, X_{t+h})\right],$$

for $\forall 0 < h < T-t$.

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Change Variable

For simplicity, we consider the Levy process as jump-diffusion process, i.e. $v(dz) = \lambda p(z)dz$, where $p(z)$ is the density function of jump amplitude which satisfies $\int_{-1}^{\infty} zp(z)dz = \kappa$. The utility function $U(x) = -e^{-\gamma x}$, thus the value function is bounded. Let us make a change of variable $y = \ln x \in (-\infty, +\infty), W(t, y) = V(t, x)$, and now the HJB equation (3) becomes

$$\begin{cases} \frac{\partial W}{\partial t} + H\left(\frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, W\right) - \beta W + \beta U(e^y) = 0, \\ W(T, y) = U(e^y + 1), y \in (-\infty, +\infty) \end{cases} \tag{4}$$

$$H(p, A, W) = \sup_{\pi \in [0,1]} \left\{ \left[(\mu - \lambda\kappa)\pi - \frac{\sigma^2}{2}\pi^2\right] p + \frac{\sigma^2}{2}\pi^2 A \right.$$
$$\left. + \lambda \int_{-1}^{\infty} [W(t, y + \ln(1 + \pi z)) - W(t, y)] p(z) dz \right\}$$

Comparison Principle

For equation (4), we can prove the following comparison principle.

Theorem

For density function of jump amplitude, we assume $\sup_{\pi\in[0,1]} \int_{-1} \infty |\ln(1+\pi z)|^2 p(z)dz < \infty$. *If* u, v *are respectively bounded viscosity subsolution and viscosity supersolution of equation (3) such that* $u(T,y) \leq v(T,y)$*, then in* $[0,T) \times \mathbb{R}$ *we also have* $u(t,y) \leq v(t,y)$.

Using the comparison principle, we can show that the value function is the unique viscosity solution for equation (3).

Setup

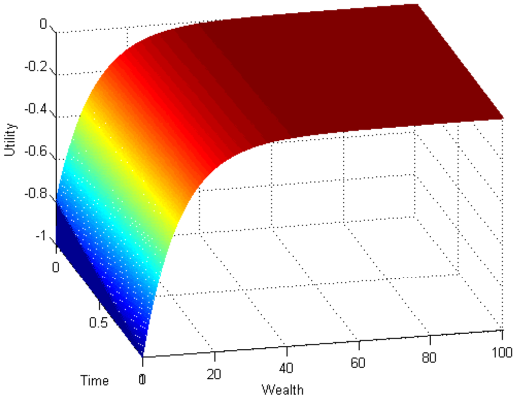
- The density function of jump amplitude of stock price

$$p(z) = \frac{1}{\sqrt{2\pi}(1+z)} \exp\left\{-\frac{\ln^2(1+z)}{2}\right\}$$

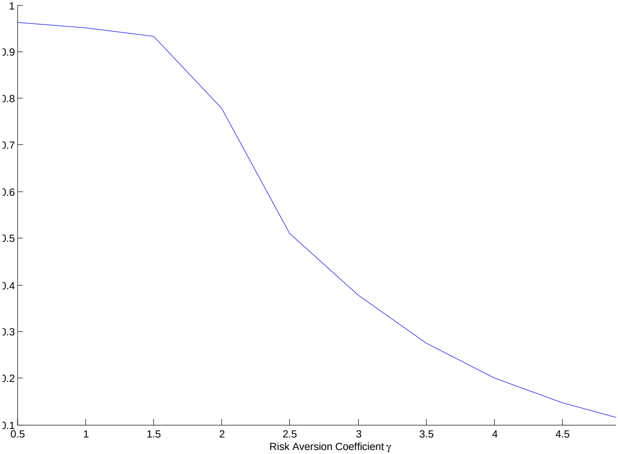
  i.e. $\ln(1+Z)$ has distribution $N(0,z)$.
- The utility function: $U(x) = -e^{-\gamma x}$
- Parameters:
  - drift $\mu = 0.07$
  - volatility $\sigma = 0.3$
  - intensity of Poisson process in jump-diffusion $\lambda = 0.1$
  - risk aversion coefficient $\gamma = 1$
  - default intensity $\beta = 0.03$

Value Function

The figure of the value function of not holding the defaultable bond:

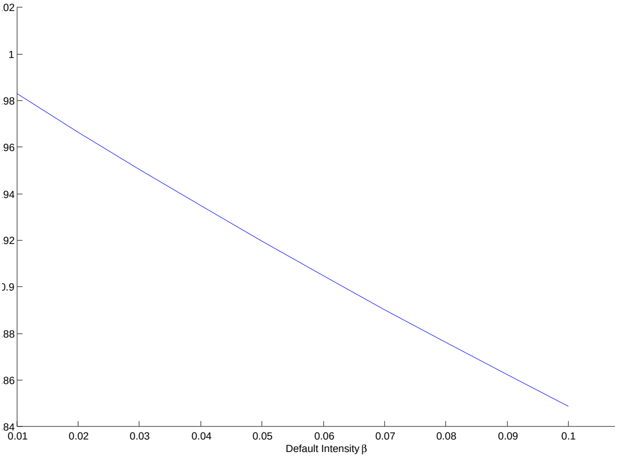


Bond Price vs Risk Aversion



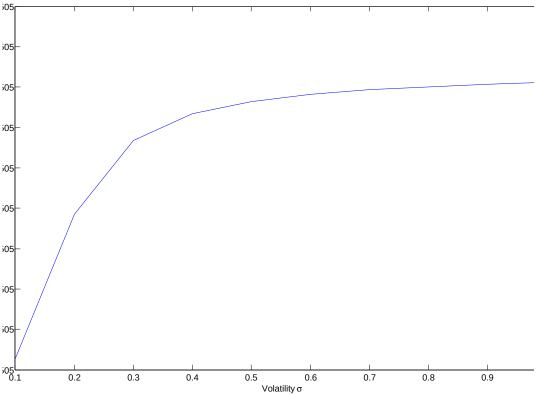
Bond price is decreasing in the risk aversion coefficient. Intuitively, the more the investor averse the credit risk, the lower the bond's value is for the investor.

Bond Price vs Default Intensity



Bond price is decreasing in the default intensity. Intuitively, the larger default intensity suggests higher default risk.

Bond Price vs Volatility



Bond price is increasing in the volatility of GBM. The larger σ, reflecting higher risk of the stock, makes the attraction of stock poorer to investors. In the same circumstances, investors prefer the bond to the stock, thus leading to a higher price of the bond.

The end

THANK YOU!