Existence, Uniqueness and Representation of Second Order Backward SDEs With Jumps

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Outline

- We want to define a notion of model uncertainty in a model with jumps.
- One possible solution is to define 2nd order BSDEs with jumps.
- Another possibility is to work with G-Lévy processes as defined in *Hu, M. and Peng, S. (2009). G-Lévy Processes under Sublinear Expectations, preprint*.

The form of the equation

A backward SDE with jumps, in the standard Lipschitz case, takes the following form:

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x)(\mu_{B^d} - \nu)(ds, dx), \quad \mathbb{P}\text{-a.s} .$$

Tang S., Li X.(1994). Necessary condition for optimal control of stochastic systems with random jumps, SIAM JCO, 332:1447–1475.

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We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

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B is the canonical process defined on $\Omega = \mathbb{D}([0,T],\mathbb{R}^d)$.
For α and ν satisfying mild integrability conditions, let $\mathbb{P}^{\alpha,\nu}$ be a probability measure on $\mathbb{D}$ such that B is a semimartingale under $\mathbb{P}^{\alpha,\nu}$ with characteristics

$$\left(-\int_0^{\cdot}\int_E x\mathbf{1}_{|x|>1}\nu_s(dx)ds,\ \int_0^{\cdot}\alpha_s ds,\ \nu_s(dx)ds\right).$$

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$\mathbb{P}^{\alpha,\nu}$ is the solution to the **martingale problem** on $\mathbb{D}$ associated to (α, ν).

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$(\hat{\alpha}, \hat{\nu}) = (\alpha, \nu), \ \mathbb{P}^{\alpha,\nu}$-a.s $, \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}$.

Aggregation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given measurable space. Let $\mathcal{P}$ be a set of non necessarily dominated probability measures and let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of random variables indexed by $\mathcal{P}$.

Definition

An *aggregator* of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ is a random variable $\hat{X}$ such that

$$\hat{X} = X^{\mathbb{P}}, \ \mathbb{P} - \text{a.s, for every } \mathbb{P} \in \mathcal{P}.$$

Aggregation, a very simple example

Example

Let $\mathbb{P}_1$ be the Wiener measure, and let $\mathbb{P}_2$ the law of $\sqrt{2}B$ under $\mathbb{P}_1$. Then

$$\int_0^t B_s dB_s = B_t^2 - t, \quad \mathbb{P}_1\text{-a.s. and}$$

$$\int_0^t B_s dB_s = B_t^2 - 2t, \quad \mathbb{P}_2\text{-a.s.}$$

Aggregation

Cohen, S.N. (2011) Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces, preprint arXiv:1110.2592v2. gave general conditions on a set $\mathbb{P}$ of probability measures such that any consistent family of processes indexed by $\mathbb{P}$ has an aggregator.

Aggregation

Proposition

There exists a set $\mathcal{P}$ of probability measures such that

- *Every $\mathbb{P}$ in $\mathcal{P}$ satisfies the martingale representation property and the Blumenthal $0-1$ law.*
- *Every family of progressively measurable processes indexed by $\mathcal{P}$, and satisfying the consistency condition has a $\mathcal{P}$-q.s unique aggregator.*
- *$\mathcal{P}$ is stable by concatenation and bifurcation.*

The 2BSDEJ

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \hat{\alpha}, \hat{\nu})ds - \int_t^T Z_s dB_s^c$$
$$- \int_t^T \int_E U_s(x)(\mu_{B^d} - \hat{\nu})(ds, dx) + K_T - K_t, \quad \mathbb{P}^{\alpha,\nu}\text{-a.s} \ , \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}.$$

Assumptions

(i) The domains $D^1_{F_t(y,z,u)} = D^1_{F_t}$ and $D^2_{F_t(y,z,u)} = D^2_{F_t}$ are independent of (ω, y, z, u).

(ii) For fixed (y, z, a, ν), F is $\mathbb{F}$-progressively measurable in $D^1_{F_t} \times D^2_{F_t}$.

(iii) The following uniform Lipschitz-type property holds. For all $(y, y', z, z', u, t, a, \nu, \omega)$

$$\left|F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y', z', u, a, \nu)\right| \leq C\left(\left|y - y^{'}\right| + \left|a^{1/2}\left(z - z^{'}\right)\right|\right).$$

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(iv) For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes γ and γ' such that

$$F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \leq \int_E \left(u^1(e) - u^2(e)\right)\gamma_t(e)\nu(de),$$

$$\int_E \left(u^1(e) - u^2(e)\right)\gamma'_t(e)\nu(de) \leq F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \text{ and}$$

$c_1(1 \wedge |x|) \leq \gamma_t(x) \leq c_2(1 \wedge |x|)$ where $-1 < c_1 \leq 0,\ c_2 \geq 0,$
$c'_1(1 \wedge |x|) \leq \gamma'_t(x) \leq c'_2(1 \wedge |x|)$ where $-1 < c'_1 \leq 0,\ c'_2 \geq 0.$

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$$\left| F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y', z', u, a, \nu) \right| \leq C \left(\left| y - y' \right| + \left| a^{1/2} \left(z - z' \right) \right| \right).$$

(iv) For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes γ and γ' such that

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(v) F is uniformly continuous in ω for the $|| \cdot ||_\infty$ norm.

The form of the equation

Definition

We say $(Y, Z, U) \in \mathbb{D}^{2,\kappa} \times \mathbb{H}^{2,\kappa} \times \mathbb{J}^{2,\kappa}$ is a solution to a 2BSDEJ if

- $Y_T = \xi$, $\mathbb{P}$-a.s, $\forall \mathbb{P} \in \mathcal{P}$.
- For all $\mathbb{P} \in \mathcal{P}$ and $0 \leq t \leq T$, the process $K^{\mathbb{P}}$ defined below is predictable and has non-decreasing paths $\mathbb{P} - a.s.$

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s, U_s) ds + \int_0^t Z_s dB_s^c + \int_0^t \int_E U_s(x) \tilde{\mu}_{B^d}(ds, dx). \quad (1)$$

- The family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the minimum condition

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[K_T^{\mathbb{P}'} \right], \ 0 \leq t \leq T, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}. \quad (2)$$

A wellposedness result

Theorem

There exists a unique solution (Y, Z, U) to the previously defined 2BSDE with jumps. Moreover, for any $\mathbb{P} \in \mathcal{P}$ and $0 \leq t_1 < t_2 \leq T$,

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \ \mathbb{P} - a.s., \tag{3}$$

where, for any $\mathbb{P} \in \mathcal{P}$, $\mathbb{F}^+$-stopping time τ, and $\mathcal{F}_\tau^+$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, $(y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi))$ denotes the solution to the following standard BSDE on $0 \leq t \leq \tau$

$$y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, u_s^{\mathbb{P}})ds + \int_t^\tau z_s^{\mathbb{P}} dB_s^c + \int_t^\tau \int_E u_s^{\mathbb{P}}(x)\tilde{\mu}_{B^d}(ds,\ dx), \ \mathbb{P} - a.s. \tag{4}$$

Robust utility maximization problem

Robust utility maximization problem

- The market :

$$\frac{dS_t}{S_{t^-}} = b_t dt + dB_t^c + \int_E \beta_t(x)\mu_{B^d}(dt,dx), \mathbb{P}\text{-a.s. } \forall\, \mathbb{P} \in \mathcal{P}. \qquad (5)$$

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- The value function V of the maximization problem can be written as

$$V^{\xi}(x) := \sup_{\pi \in \mathcal{C}} \inf_{\mathbb{P} \in \overline{\mathcal{P}}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-\eta\left(X_T^{\pi} - \xi\right)\right)\right].$$

where

$$\mathcal{C} := \{(\pi_t) \text{ which are predictable and take values in } C\},$$

is our set of admissible strategies.

Robust utility maximization problem

Proposition

Assume that $\exp(\eta\xi) \in \overline{\mathcal{L}}_H^{2,\kappa}$. *Then the value function of the previous optimization problem is given by*

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where Y_0 *is defined as the initial value of the unique solution* $(Y, Z, U) \in \mathbb{D}^{2,\kappa} \times \mathbb{H}^{2,\kappa} \times \mathbb{J}^{2,\kappa}$ *of the following 2BSDEJ*

$$\begin{aligned} Y_t = e^{\eta\xi} + \int_t^T \widehat{F}_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s^c \\ - \int_t^T \int_E U_s(x)\widetilde{\mu}_{B^d}(ds, dx) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \end{aligned}$$

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where the generator is defined as follows

$$\widehat{F}_t(\omega, y, z, u) := F_t(\omega, y, z, u, \widehat{a}_t, \widehat{\nu}_t), \tag{6}$$

Robust utility maximization problem

Proposition

where

$$F_t(y,z,u,a,\nu) := \inf_{\pi \in C} \Big\{ (-\eta b_t + \frac{\eta^2}{2}\pi a)\pi y - \eta\pi a z + \int_E \left(e^{-\eta\pi\beta_t(x)} - 1 \right) (y + u(x))\nu(dx) \Big\}.$$

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$$F_t(y,z,u,a,\nu) := \inf_{\pi\in C}\Big\{(-\eta b_t + \frac{\eta^2}{2}\pi a)\pi y - \eta\pi a z + \int_E \left(e^{-\eta\pi\beta_t(x)} - 1\right)(y+u(x))\nu(dx)\Big\}.$$

Moreover, there exists an optimal trading strategy π^* realizing the supremum above.

Probabilistic counterpart of fully non-linear PIDEs

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$$v(T,x) = g(x).$$

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where h is the Fenchel-Legendre transform of the generator f in (a,ν):

$$h(t,x,y,z,u,\gamma,v) = \sup_{(a,\nu)\in\mathbb{S}_d\times D_2}\left\{\frac{1}{2}a:\gamma + \int_0^T\int_E \tilde{v}(e)\nu_s(de)ds - f(t,x,y,z,u,a,\nu)\right\}$$

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with

$$\tilde{v}(e) := v(e+x) - v(x) - \mathbf{1}_{\{|e|\leq 1\}}\ e.(\nabla v)(x).$$

Paper in preparation!

Thank you for your attention !