BSDEs with partially nonpositive jumps and Bellman IPDEs

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Introduction

Classical stochastic control problem:

$$V_0 \quad = \quad \sup_\alpha \mathbb{E}\Big[g(X_T^\alpha) + \int_0^T f(X_s^\alpha, \alpha_s)ds\Big] \,,$$

with X^α a controlled process. In the Markov case,

$$v(t,x) \quad = \quad \sup_\alpha \mathbb{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds\Big] \,,$$

with $X^{t,x,\alpha}$ a controlled diffusion of the form

$$X_s^{t,x,\alpha} \quad = \quad x + \int_t^s b(X_r^{t,x,\alpha}, \alpha_r)dr + \int_t^s \sigma(X_r^{t,x,\alpha}, \alpha_r)dW_r \,, \quad s \geqslant t \,.$$

The function v is a solution in the viscosity sens of the PDE

$$-\partial_t v - \sup_a \Big\{ \mathcal{L}^a v - f(.,a) \Big\} \;\; = \;\; 0 \, ,$$

with $\mathcal{L}$ the second order local operator associated to the diffusion X and defined by

$$\mathcal{L}^a v(t,x) \;\; = \;\; b(x,a).D_x v(t,x) + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top(x,a)D_x^2 v(t,x)\big]$$

for all (t,x,a).

Probabilistic representation of Bellman PDEs

• Convex case: Pontriaguine maximum principle provides an optimal strategy.

• General case: Second order BSDE Introduced by [Cheredito, Soner, Touzi and Victoir].
Reformulation by [Soner, Touzi and Zhang] as an equation over a family of singular probability measures, to ensure well posedness.

Our approach

Follows the ideas of [Pardoux, Pradeilles and Rao], and [K., Ma, Pham and Zhang].
Introduce a random measure μ on $\mathbb{R}_+ \times A$ where A is the set of control values.
Consider the constrained BSDE with jumps: find Y minimal s.t. (Y, Z, U, K) solves

$$\begin{aligned} Y_t &= g(X_T^I) + \int_t^T f(X_s^I, I_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_A U_s(a)\mu(da, ds) + K_t - K_t \\ U_t &\leqslant 0 \end{aligned}$$

with (X^I, I) defined by

$$\begin{aligned} dX_t^I &= b(X_t^I, I_t)dt + \sigma(X_t^I, I_t)dW_t \\ dI_t &= \int_A (a - I_{t-})\mu(da, dt) \end{aligned}$$

Our approach

Expected Markov Property $\Rightarrow$ $Y_t = v(t, X_t^I, I_t)$ for some deterministic function v.
Formally we have from Itô's formula

- $U_t(a) = v(t, X_t^I, a) - v(t, X_t^I, I_{t^-}) \geqslant 0$ $\Rightarrow$ v not dependent on a.
- v minimal supersolution (and therefore solution) to

$$-\partial_t v - \sup_a \left\{ \mathcal{L}^a v(t,x) - f(x,a) \right\} \quad = \quad 0$$

Interesting aspects

- Covers the general case.
- Solution of an equation under a single probability measure.
- Addition of a nonlocal term in the Bellman PDE.

Outline

1. BSDEs with partially nonpositive jumps
   - Definition
   - Existence of a minimal solution
2. Nonlinear IPDE representation
   - Markov BSDE and Bellman IPDE
   - Penalized BSDEs and IPDEs
   - An intermediary IPDE for v
   - Bellman IPDE for v
3. Conclusion

Outline

Settings

$(\Omega, \mathcal{G}, \mathbb{P})$ complete probability space equipped with

- W Standard Brownian Motion valued in $\mathbb{R}^d$.
- E is a Borelian subset of $\mathbb{R}^q$ and μ a Poisson random measure on $\mathbb{R}_+ \times E$, with compensator $\lambda(de)dt$ for some σ-finite measure λ on $(E, \mathcal{B}(E))$ s.t.

$$\int_E 1 \wedge |e|^2 \lambda(de) \quad < \quad +\infty \, .$$

We suppose that W and μ are independent, and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ the natural complete càd filtration of W and μ.

Data of the BSDE

We then are given three objects:

- a terminal $\mathcal{F}_T$-measurable random variable ξ,
- a generator functions $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L}^2(\lambda) \to \mathbb{R}$, which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{L}^2(\lambda))$-measurable,
- a borelian subset B of E such that $\lambda(B) < +\infty$.

Assumptions on the coefficients

Assumption (H0)

- *Square integrability*: $\mathbb{E}\big[|\xi|^2\big] + \mathbb{E}[\int_0^T |F(t,0,0,0)|^2 dt] < \infty$,
- *Lipschitz continuity*: *there exists a constant* L *such that*

$$|F(t,y,z,u) - F(t,y',z',u')| \leqslant L\big(|y-y'| + |z-z'| + |u-u'|_{\mathbf{L}^2(\lambda)}\big) ,$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$ *and* $u, u' \in \mathbf{L}^2(\lambda)$.

- *Monotonicity*: *there exist a predictable map* $\gamma : [0,T] \times \Omega \times E \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L}^2(\lambda) \times \mathbf{L}^2(\lambda) \to \mathbb{R}$, *two constants* $C_1 \geqslant C_2 > -1$ *such that*

$$C_1(1 \wedge |e|) \geqslant \gamma(t,e,y,z,u,u') \geqslant C_2(1 \wedge |e|) , \quad e \in E ,$$

$$F(t,y,z,u) - F(t,y,z,u') \leqslant \int_E \gamma(t,e,y,z,u,u')(u(e) - u'(e))\lambda(de) ,$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$, *and* $u, u' \in \mathbf{L}^2(\lambda)$.

BSDE with partially nonpositive jumps

Find a $(Y, Z, U, K) \in \mathbf{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$ satisfying

$$\begin{aligned} Y_t &= \xi + \int_t^T F(s, Y_s, Z_s, U_s)ds + K_T - K_t \qquad (1) \\ &\quad - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds, de), \quad 0 \leqslant t \leqslant T, \ a.s. \end{aligned}$$

with

$$U_t(e) \quad \leqslant \quad 0 \,, \qquad d\mathbb{P} \otimes dt \otimes \lambda(de) \ \ a.e. \text{ on } \Omega \times [0, T] \times B, \quad (2)$$

and for any other $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathbf{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$ satisfying (1)-(2), we have

$$Y_t \quad \leqslant \quad \tilde{Y}_t, \quad 0 \leqslant t \leqslant T, \ a.s.$$

We say that Y is the minimal solution to (1)-(2).

Penalized BSDE

For each $n \geqslant 1$, we introduce the penalized BSDE with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T F(s, Y_s^n, Z_s^n, U_s^n)ds + K_T^n - K_t^n \\ &\quad - \int_t^T Z_s^n dW_s - \int_t^T \int_E U_s^n(e)\tilde{\mu}(ds, de), \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{3}$$

where

$$K_t^n = n \int_0^t \int_B [U_s^n(e)]^+ \lambda(de)ds, \quad 0 \leqslant t \leqslant T,$$

and $[u]^+ = \max(u, 0)$ is the negative part function.

Comparison results

Lemma

The sequence $(Y^n)_n$ *is nondecreasing, i.e. for all* $n \in \mathbb{N}$, $Y_t^n \leqslant Y_t^{n+1}$, $0 \leqslant t \leqslant T$, *a.s.*

Lemma

For any quadruple $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}, \widetilde{K}) \in \mathbf{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\widetilde{\mu}) \times \mathbf{A}^2$ *satisfying* (1)-(2), *and for all* $n \in \mathbb{N}$, *we have*

$$Y_t^n \quad \leqslant \quad \widetilde{Y}_t, \quad 0 \leqslant t \leqslant T, \; a.s. \tag{4}$$

Provides the convergence of (Y^n) as soon as we have a supersolution to the constrained BSDE.

Convergence of the penalized BSDEs

Assumption (H1)

There exists $(\widetilde{Y}, \widetilde{Z}, \widetilde{K}, \widetilde{U}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\widetilde{\mu}) \times \mathbf{A^2}$ *satisfying* (1)-(2).

Theorem

Under **(H1)**, *there exists a unique minimal solution* $(Y, Z, U, K) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\widetilde{\mu}) \times \mathbf{A^2}$ *with* K *predictable, to* (1)-(2). Y *is the increasing limit of* (Y^n) *and also in* $\mathbf{L^2(0,T)}$ *and* $\mathbf{L^2(W)}$, K *is the weak limit of* (K^n) *in* $\mathbf{L^2(0,T)}$, *and for any* $p \in [1, 2)$,

$$\|Z^n - Z\|_{\mathbf{L^p(W)}} + \|U^n - U\|_{\mathbf{L^p}(\widetilde{\mu})} \longrightarrow 0,$$

as n *goes to infinity.*

Sketch of the proof

• From **(H1)** and Comparison Lemmata, $Y^n \uparrow Y$ as $n \uparrow \infty$.

Convergence of $(Z^n, U^n, K^n)_n$?

• Under **(H1)**, there exists some constant C such that

$$\|Y^n\|_{\mathbf{S}^2} + \|Z^n\|_{\mathbf{L}^2(\mathbf{W})} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} + \|K^n\|_{\mathbf{S}^2} \quad \leqslant \quad C, \qquad \forall n \geqslant 1.$$

• Weak convergence method initiated by [Peng 99], gives convergence of $(Z^n, U^n, K^n)_n$ in L^p, see also [Royer 06].

• Uniform bound on $(K^n)_n$ gives at the limit

$$\mathbb{E}\Big[\int_0^T \int_B [U_s(e)]_+ \lambda(de) ds\Big] \;=\; 0\,.$$

Outline

Random measure

- $E = L \cup A$ where L and A are two borelian subsets of $\mathbb{R}^q$ with Borel σ-fields $\mathcal{B}(L)$ and $\mathcal{B}(A)$ and such that $L \cap A = \emptyset$.
- μ is of the form $\mu = \vartheta + \pi$, where ϑ and π are two independent Poisson random measures defined respectively on $\mathbb{R}_+ \times L$ and $\mathbb{R}_+ \times A$.
- ϑ and π have respective intensity measures $\lambda_\vartheta(d\ell)dt$ and $\lambda_\pi(d\ell)dt$ where λ_ϑ and λ_π are two measures defined respectively on $(L, \mathcal{B}(L))$ and $(A, \mathcal{B}(A))$ s.t.

$$\int_L (1 \wedge |\ell|^2)\lambda_\vartheta(d\ell) \ < \ \infty \qquad \text{and} \quad \int_A \lambda_\pi(da) \ < \ \infty \, .$$

- We denote by $\widetilde{\vartheta}(dt, d\ell) = \vartheta(dt, d\ell) - \lambda_\vartheta(d\ell)dt$ and $\widetilde{\pi}(dt, da) = \pi(dt, da) - \lambda_\pi(da)dt$ the compensated measures.

Studied FBSDE

- Forward equation:

$$\begin{aligned} dX_s &= b(X_s, I_s)ds + \sigma(X_s, I_s)dW_s + \int_L \gamma(X_{s-}, I_{s-}, \ell)\widetilde{\vartheta}(ds, d\ell), \\ dI_s &= \int_A (a - I_{s-})\pi(ds, da). \end{aligned}$$

- Backward equation:

$$\begin{aligned} Y_t &= g(X_T, I_T) + \int_t^T f\Big(X_s, I_s, Y_s, Z_s, \int_L U_s(\ell)\beta(X_s, I_s, \ell)\lambda_\vartheta(d\ell)\Big)ds \\ &\quad + K_T - K_t - \int_t^T Z_s.dW_s - \int_t^T\int_L U_s(\ell)\widetilde{\vartheta}(dt, d\ell) - \int_t^T\int_A V_s(a)\widetilde{\pi}(dt, da), \end{aligned}$$

with

$$V_t(e) \leqslant 0 \,, \quad d\mathbb{P} \otimes dt \otimes \lambda(de) \ \ a.e. \text{ on } \Omega \times [0, T] \times A \,,$$

Assumption on forward coefficients

Assumption (HFC)

(i) There exists a constant C such that

$$|b(x,a)-b(x',a)|+|\sigma(x,a)-\sigma(x',a')| \quad \leqslant \quad C\big(|x-x'|+|a-a'|\big) \ ,$$

for all $x,x'\in\mathbb{R}^d$ and $a,a'\in A$.

(ii) There exists a constant C such that

$$\begin{aligned}\big|\gamma(x,a,\ell)\big| \quad &\leqslant \quad C\big(1\wedge|\ell|\big),\\ \big|\gamma(x,a,\ell)-\gamma(x',a,\ell)\big| \quad &\leqslant \quad C\big(|x-x'|+|a-a'|\big)\big(1\wedge|\ell|\big),\end{aligned}$$

for all $x,x'\in\mathbb{R}^d$, $a\in A$ and $\ell\in L$.

Assumption on backward coefficients

Assumption (HBC)

(i) $\sup_{x\in\mathbb{R}^d, a\in A} \frac{|g(x,a)|+f(x,a,0,0,0)|}{1+|x|} < \infty$.
(ii) $r \mapsto f(x,y,z,r)$ *is nondecreasing for all* $(x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.
(iii) There exists some constant C *s.t.*

$$|g(x,a) - g(x',a')| + |f(x,a,y,z,r) - f(x',a',y',z',r')| \leqslant C\big(|x-x'| + |a-a'| + |y-y'| + |z-z'| + |r-r'|\big),$$

for all $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $r, r' \in \mathbb{R}$ *and* $a, a' \in A$.
(iv) There exists a constant C *such that*

$$\begin{aligned} 0 \leqslant \beta(x,a,\ell) &\leqslant C\big(1 \wedge |\ell|\big), \\ |\beta(x,a,\ell) - \beta(x',a,\ell)| &\leqslant C\big(|x-x'| + |a-a'|\big)\big(1 \wedge |\ell|\big), \end{aligned}$$

for all $x, x' \in \mathbb{R}^d$, $a \in A$ *and* $\ell \in L$.

Markov property

Assumption

For any initial condition (t, x, a) for the processes (X, I), the constrained FBSDE admits a solution $(\widetilde{Y}^{t,x,a}, \widetilde{Z}^{t,x,a}, \widetilde{V}^{t,x,a}, \widetilde{K}^{t,x,a})$ $\in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\widetilde{\vartheta}) \times \mathbf{A^2}$ with

$$\sup_{(t,x,a)\in[0,T]\times\mathbb{R}^d} \frac{|\widetilde{Y}_t^{t,x,a}|}{1+|x|} \quad < \quad \infty$$

we get from Section 1 the existence of a unique minimal solution $(Y, Z, V, K) \in \mathbf{S}^2 \times \mathbf{L^2(W)} \times \mathbf{L^2}(\widetilde{\vartheta}) \times \mathbf{A^2}$

Markov property: $Y_t = v(t, X_t, I_t)$, for some deterministic function $v : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}$ defined by:

$$v(t, x, a) \quad := \quad Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times A, \qquad (5)$$

Bellman IPDE

$$
\begin{aligned}
-\frac{\partial v}{\partial t} - \sup_{a\in A}\Big[\mathcal{L}^a v + f\big(.,a,v,\sigma^\top(.,a)D_x v,\mathcal{M}^a v\big)\Big] &= 0, \quad \text{on } [0,T)\times\mathbb{R}^d,\\
v(T,x) &= \sup_{a\in E} g(x,a), \quad x\in\mathbb{R}^d,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}^a v(t,x) &= b(x,a).D_x v(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x,a)D_x^2 v(t,x))\\
&\quad + \int_L \big[v(t,x+\gamma(x,a,\ell)) - v(t,x) - \gamma(x,a,\ell).D_x v(t,x)\big]\lambda_\vartheta(d\ell),\\
\mathcal{M}^a v(t,x) &= \int_L \big(v(t,x+\gamma(x,a,\ell)) - v(t,x)\big)\beta(x,a,\ell)\lambda_\vartheta(d\ell)\ ,
\end{aligned}
$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$.

Functions associated to the Penalized BSDEs

We define the function

$$v_n(t,x,a) \quad := \quad Y_t^{n,t,x,a}, \quad (t,x,a) \in [0,T] \times \mathbb{R}^d \times A, \qquad (6)$$

where $\{(Y_r^{n,t,x,a}, Z_r^{n,t,x,a}, U_r^{n,t,x,a}(.)), t \leqslant r \leqslant T\}$ is the unique solution to the Markov penalized BSDE:

$$\begin{aligned} Y_r^n \quad &= \quad g(X_T, I_T) + \int_r^T f(X_s, I_s, Y_s^n, Z_s^n, U_s^n) ds + n \int_r^T \int_A [V_s^n(a)]^+ \lambda(da) ds \\ &\quad - \int_r^T Z_s^n . dW_s - \int_r^T \int_L U_s^n(\ell) \tilde{\vartheta}(dt, d\ell) - \int_r^T \int_L V_s^n(a) \tilde{\pi}(dt, da). \end{aligned}$$

with $(X_r, I_r) = (X_r^{t,x,a}, I_r^{t,a})$, for $r \in [t,T]$.

Viscosity property of the penalized functions

We then introduce the IPDE associated to the penalized BSDEs:

$$\begin{aligned} -\frac{\partial v_n}{\partial t} - \mathcal{L}^a v_n - f\left(., a, v, \sigma^\top(., a) D_x v, \mathcal{M}^a v\right) & \\ - \int_A [v_n(., a') - v_n(., a)] \lambda_\pi(da') & \\ -n \int_A [v_n(., a') - v_n(., a)]^+ \lambda_\pi(da') &= 0, \quad \text{on } [0, T) \times \mathbb{R}^d \times A, \\ v_n(T, x, a) &= g(x, a), \quad (x, a) \in \mathbb{R}^d \times A. \end{aligned}$$

Theorem (Barles, Buckdahn and Pardoux)

The function v_n is a continuous (viscosity) solution of the penalized IPDE.

Stability arguments leads to the variational IPDE

$$\min\Big\{ -\frac{\partial v}{\partial t} - \mathcal{L}^a v - f\big(.,a,v,\sigma^\top(.,a)D_x v, \mathcal{M}^a v\big) - \int_A [v(.,a') - v(.,a)]\lambda_\pi(da') \, , - \int_A \big[v(.,a') - v(.,a)\big]^+ \lambda_\pi(da')\Big\} \quad = \quad 0,$$

on $[0,T) \times \mathbb{R}^d \times A$ and $v(T,.) = g(.)$.

Proposition

The function v is the unique (viscosity) solution to the variational IPDE. v is therefore continuous.

Uniqueness based on Ishii's Lemma for nonlocal IPDE proved in [Barles and Imbert].

The nondependence of v in a.

Theorem

The function v does not depend on the variable $a \in A$ on $[0, T) \times \mathbb{R}^d$.

Based on:

- the identification

$$V_s^{t,x,a}(a') \quad = \quad v(s, X_{s^-}^{t,x,a}, a') - v(s, X_{s^-}^{t,x,a}, I_{s^-})$$

- the constraint satisfied by v

$$V^{t,x,a'} \quad \leqslant \quad 0 \,, \qquad a.e.$$

- the continuity of the function v

Supersolution Property

Recall Bellman nonlocal IPDE

$$\begin{aligned} -\frac{\partial v}{\partial t} - \sup_{a\in A}\Big[\mathcal{L}^a v + f\big(.,a,v,\sigma^\top(.,a)D_x v,\mathcal{M}^a v\big)\Big] &= 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \\ v(T^-,x) &= \sup_{a\in E} g(x,a), \quad x\in\mathbb{R}^d, \end{aligned}$$

Proposition

The function v is a viscosity supersolution of the Bellman nonlocal IPDE.

Direct consequence of the viscosity property for the intermediary IPDE and the nondependence of v in $a \in A$.

Subsolution Property

Proposition

The function v is a viscosity subsolution of the Bellman nonlocal IPDE.

Based on the following dynamic characterization of the function v.

Lemma

For any stopping time θ valued in $[t, T]$, $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, K_s^{t,x})_{s\in[t,\theta]}$ is also a minimal solution to :

$$\begin{aligned} Y_s &= v(\theta, X_\theta) + \int_s^\theta f(X_r, I_r, Y_r, Z_r, \int U_r \beta_r d\lambda_\vartheta) dr + K_\theta^{t,x} - K_s^{t,x} \\ &\quad - \int_s^\theta Z_r . dW_r - \int_s^\theta \int_L U_r(\ell) \widetilde{\vartheta}(dr, d\ell) - \int_t^T \int_A V_r(a) \widetilde{\pi}(dr, da) \\ V_s(a) &\leqslant 0 \quad d\mathbb{P} \otimes dt \otimes \lambda_\pi(da) \quad a.e.\ on\ \Omega \times [t, \theta] \times A. \end{aligned}$$

Outline

Summary

• Define a new class of BSDE with a constraint on a part of the jump component.
• Link it with nonlinear IPDE of HJB type and with nonlocal terms.

Perpectives

• Speed of convergence of the penalized BSDEs
• Numerical approximation of the function v by BSDE discretization methods.

Thank You!