Limit theorems under sublinear expectations and probabilities

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Outline

- Introduction
- Law of large numbers
- Central limit theorem

Introduction

- Capacity and Choquet expectation

Marinacci, M. (1999) Limit laws for non-additive probabilities and their frequentist interpretation. J. Econom. Theory, 84, 145-195.

Maccheroni, F. and Marinacci, M. (2005) A strong law of large numbers for capacities. The Annals of Probability, 33, 1171-1178.

etc.

- Sublinear expectation

Peng, S. (2008) A new central limit theorem under sublinear expectations. in arXiv:0803.2656v1.

Chen, Z. (2010) Strong laws of large numbers for capacities. in arXiv:1006.0749v1.

etc.

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{M}$ be the set of all probabilities on Ω.

For each non-empty subsets $\mathcal{P} \subset \mathcal{M}$, we can define:

- Upper probability $V(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{F}$
- Lower probability $v(A) := \inf_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{F}$
- Sublinear expectation $\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X]$, i.e.

(1) $\hat{\mathbb{E}}[X] \leq \hat{\mathbb{E}}[Y]$ if $X \leq Y$
(2) $\hat{\mathbb{E}}[c] = c$, $\forall c \in \mathbb{R}$
(3) $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$
(4) $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, $\forall \lambda \geq 0$

Definition (Independence)

- X_n is said to be independent of $(X_1, \cdots, X_{n-1})$, if for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ (all bounded and Lipschitz functions on $\mathbb{R}^n$),

$$\hat{\mathbb{E}}[\varphi(X_1, \cdots, X_n)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x_1, \cdots, x_{n-1}, X_n)]|_{(x_1, \cdots, x_{n-1})=(X_1, \cdots, X_n)}].$$

- X_n is said to be product independent of $(X_1, \cdots, X_{n-1})$ if for each nonnegative bounded Lipschitz function φ_k,

$$\hat{\mathbb{E}}[\prod_{k=1}^{n} \varphi_k(X_k)] = \prod_{k=1}^{n} \hat{\mathbb{E}}[\varphi_k(X_k)].$$

- X_n is said to be sum independent of $(X_1, \cdots, X_{n-1})$ if for each $\varphi \in C_{b,Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(\sum_{k=1}^{n} X_k)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x + X_n)]|_{x=\sum_{k=1}^{n-1} X_k}].$$

Example:

We consider X and Y such that

$$-\hat{\mathbb{E}}[-X] < \hat{\mathbb{E}}[X] = 0 \quad \text{and} \quad -\hat{\mathbb{E}}[-Y] < \hat{\mathbb{E}}[Y] = 0.$$

Independent case:

$$\begin{aligned}
\hat{\mathbb{E}}[XY] &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[xY]|_{x=X}] \\
&= \hat{\mathbb{E}}[(x^+\hat{\mathbb{E}}[Y] + x^-\hat{\mathbb{E}}[-Y])|_{x=X}] \\
&= \hat{\mathbb{E}}[X\hat{\mathbb{E}}[Y] + X^-(\hat{\mathbb{E}}[Y] + \hat{\mathbb{E}}[-Y])] \\
&> 0
\end{aligned}$$

Definition (Identical distribution)

X_2 is said to be identically distributed with X_1, denoted by $X_1 \sim X_2$, if for each $\varphi \in C_{b,Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(X_1)] = \hat{\mathbb{E}}[\varphi(X_2)].$$

Definition (G-normal distribution)

ξ is said to be G-normal distributed, denoted by $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, where $\overline{\sigma}^2 = \hat{\mathbb{E}}[\xi^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-\xi^2]$, if

$$\forall a, b \geq 0, a\xi + b\bar{\xi} \sim \sqrt{a^2 + b^2}\xi,$$

where $\bar{\xi}$ is independent of ξ and $\xi \sim \bar{\xi}$.

Remark

If $\xi + \bar{\xi} \sim \sqrt{2}\xi$, then ξ is G-normal distributed.

Proposition

If ξ is G-normal distributed with $\overline{\sigma}^2 = \hat{\mathbb{E}}[\xi^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-\xi^2]$, for each $\varphi \in C_{b,Lip(\mathbb{R})}$, we define $u(t,x) = \hat{\mathbb{E}}[\varphi(x+\sqrt{t}\xi)], (t,x) \in [0,\infty] \times \mathbb{R}$, then $u(t,x)$ is the unique viscosity solution of the following G-heat PDE:

$$\partial_t u - G(\partial^2_{xx} u) = 0, u|_{t=0} = \varphi,$$

where $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-), \alpha \in \mathbb{R}$.

Proposition

Let $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, if φ is a convex function, then $\hat{\mathbb{E}}[\varphi(\xi)] = E_Z[\varphi(\overline{\sigma}Z)]$, where $Z \sim \mathcal{N}(0,1)$.

Definition (Maximal distribution)

η is said to be maximal distributed, if

$$\forall a, b \geq 0, a\eta + b\bar{\eta} \sim (a+b)\eta.$$

Remark

If $\eta + \bar{\eta} \sim 2\eta$, then η is maximal distributed.

Proposition

If η is maximal distributed with $\overline{\mu} = \hat{\mathbb{E}}[\eta]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-\eta]$, then for each $\varphi \in C_{b,Lip(\mathbb{R})}$,

$$\hat{\mathbb{E}}[\varphi(\eta)] = \max_{\underline{\mu} \leq \mu \leq \overline{\mu}} \varphi(\mu).$$

Law of large numbers

Weak law of large numbers (Peng)

Let $\{X_n\}$ be a sequence of i.i.d random variables with finite means $\overline{\mu} = \hat{\mathbb{E}}[X_1]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-X_1]$. Suppose $\hat{\mathbb{E}}[|X_1|^{1+\alpha}] < \infty$ for some $\alpha > 0$, let $S_n = \sum_{k=1}^{n} X_k$, then

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{S_n}{n})] = \max_{\underline{\mu}\leq\mu\leq\overline{\mu}} \varphi(\mu), \quad \forall\varphi \in C_{b,Lip}(\mathbb{R}).$$

Strong law of large numbers (Chen)

Let $\{X_n\}$ be a sequence of i.i.d random variables with finite means $\overline{\mu} = \hat{\mathbb{E}}[X_1]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-X_1]$. Suppose $\hat{\mathbb{E}}[|X_1|^{1+\alpha}] < \infty$ for some $\alpha > 0$, let $S_n = \sum_{k=1}^{n} X_k$, then

(I) $v(\underline{\mu} \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq \overline{\mu}) = 1.$

(II) Furthermore, if V is upper continuous, i.e., $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$, then $V(\limsup_{n\to\infty} \frac{S_n}{n} = \overline{\mu}) = 1, \ \ V(\liminf_{n\to\infty} \frac{S_n}{n} = \underline{\mu}) = 1.$

(III) Suppose that V is upper continuous, and $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in $\mathbb{R}$, i.e.,

$C(\{x_n\}) = \{x|$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x\}$, then $V(C(\{\frac{S_n}{n}\}) = [\underline{\mu}, \overline{\mu}]) = 1.$

Law of large numbers

We assume $\mathcal{P}$ is weakly compact. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of random variables satisfying: $\sup_{k\geq 1}\hat{\mathbb{E}}[|X_k|^{1+\alpha}] < \infty$, for some $\alpha >$ 0, and $\hat{\mathbb{E}}[X_k] \equiv \overline{\mu}, -\hat{\mathbb{E}}[-X_k] \equiv \underline{\mu}, k = 1, 2, \cdots$. Set $S_n = \sum_{k=1}^{n} X_k$.

(i) If $\{X_k\}_{k=1}^{\infty}$ is product independent, then

$$v(\underline{\mu} \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq \overline{\mu}) = 1.$$

(ii) If $\{X_k\}_{k=1}^{\infty}$ is product and sum independent, then

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{S_n}{n})] = \max_{\underline{\mu}\leq\mu\leq\overline{\mu}} \varphi(\mu).$$

(iii) If $\{X_k\}_{k=1}^{\infty}$ is sum independent, and $V(\cdot)$ is upper continuous, then

$$\forall \mu \in [\underline{\mu}, \overline{\mu}], \qquad V(\lim_{n\to\infty} \frac{S_n}{n} = \mu) = 1.$$

Lemma

We assume that $\mathcal{P}$ is weakly compact. Let P be a probability measures such that for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$,

$$E_P[\varphi(X_1, \cdots, X_n)] \leq \hat{\mathbb{E}}[\varphi(X_1, \cdots, X_n)], \qquad (*)$$

Then $(*)$ also holds for each bounded measurable function φ.

Proof: $(*)$ holds for bounded u.s.c. function φ since $\exists \varphi_k \in C_{b,Lip}(\mathbb{R}^n)$, s.t. $\varphi_k \downarrow \varphi$ ($\varphi_k = \sup_{y \in \mathbb{R}^n}\{\varphi(y) - k||x - y||\}$). $\mathcal{P}$ is weakly compact, we have $\hat{\mathbb{E}}[\varphi_k(X_1, \cdots, X_n)] \downarrow \hat{\mathbb{E}}[\varphi(X_1, \cdots, X_n)]$ (Theorem 31 in Denis, Hu, Peng(2008)), then $E_P[\varphi(X_1, \cdots, X_n)] = \lim_{k\to\infty} E_P[\varphi_k(X_1, \cdots, X_n)] \leq \hat{\mathbb{E}}[\varphi(X_1, \cdots, X_n)]$.

If φ is a bounded measurable function, then

$$\begin{aligned} & E_P[\varphi(X_1, \cdots, X_n)] \\ = & \sup\{E_P[\overline{\varphi}(X_1, \cdots, X_n)] : \overline{\varphi} \text{ is bounded upper semi-continuous and } \overline{\varphi} \leq \varphi\} \\ \leq & \hat{\mathbb{E}}[\varphi(X_1, \cdots, X_n)]. \end{aligned}$$

Lemma

If for each nonnegative bounded Lipschitz function φ_i, $i = 1, \cdots, n$,

$$\hat{\mathbb{E}}[\prod_{i=1}^{n} \varphi_i(X_i)] = \prod_{i=1}^{n} \hat{\mathbb{E}}[\varphi_i(X_i)],$$

then for each nonnegative bounded measurable function φ_i, $i = 1, \cdots, n$,

$$\hat{\mathbb{E}}[\prod_{i=1}^{n} \varphi_i(X_i)] = \prod_{i=1}^{n} \hat{\mathbb{E}}[\varphi_i(X_i)].$$

Sketch proof of main theorem:

(i) Similar to Proof of Chen (2010)

Consider $Y_k = X_k I_{\{|X_k| \le k^\beta\}} - \hat{\mathbb{E}}[X_k I_{\{|X_k| \le k^\beta\}}]$, where $\frac{1}{1+\alpha} < \beta < 1$.

$$e^x \le 1 + x + |x|^{1+\alpha} e^{2|x|}, x \in \mathbb{R}, 0 < \alpha < 1.$$

$$e^{\frac{Y_k}{n^\beta}} \le 1 + \frac{Y_k}{n^\beta} + \frac{|Y_k|^{1+\alpha}}{n^{\beta(1+\alpha)}} e^{2|\frac{Y_k}{n^\beta}|} \le 1 + \frac{Y_k}{n^\beta} + \frac{|Y_k|^{1+\alpha}}{n} e^4.$$

$$\hat{\mathbb{E}}[e^{\frac{Y_k}{n^\beta}}] \le 1 + \frac{C}{n}.$$

$$\hat{\mathbb{E}}[\prod_{k=1}^n e^{\frac{Y_k}{n^\beta}}] = \prod_{k=1}^n \hat{\mathbb{E}}[e^{\frac{Y_k}{n^\beta}}] \le (1 + \frac{C}{n})^n \le e^C.$$

$$V(\frac{\sum_{k=1}^n Y_k}{n} \ge \varepsilon) = V(e^{\frac{\sum_{k=1}^n Y_k}{n^\beta}} \ge e^{\varepsilon n^{1-\beta}}) \le e^{-\varepsilon n^{1-\beta}} \hat{\mathbb{E}}[\prod_{k=1}^n e^{\frac{Y_k}{n^\beta}}] \le e^{-\varepsilon n^{1-\beta}} e^C.$$

Corollary

Let $\{X_n\}$ be a sequence of bounded and sum independent random variables satisfying: $\sup_{k\geq 1}\hat{\mathbb{E}}[|X_k|^{1+\alpha}]<\infty$, for some $\alpha>0$, and $\hat{\mathbb{E}}[X_k]\equiv\overline{\mu}, -\hat{\mathbb{E}}[-X_k]\equiv\underline{\mu}, k=1,2,\cdots$. Set $S_n=\sum_{k=1}^n X_k$.

(i) $v(\underline{\mu}\leq\liminf_{n\to\infty}\frac{S_n}{n}\leq\limsup_{n\to\infty}\frac{S_n}{n}\leq\overline{\mu})=1$.

(ii) $\lim_{n\to\infty}\hat{\mathbb{E}}[\varphi(\frac{S_n}{n})]=\max_{\underline{\mu}\leq\mu\leq\overline{\mu}}\varphi(\mu)$.

(iii) Furthermore, if V is upper continuous, then

$$\forall\mu\in[\underline{\mu},\overline{\mu}],\qquad V(\lim_{n\to\infty}\frac{S_n}{n}=\mu)=1.$$

Central limit theorem

Central limit theorem (Peng)

Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0$ and $\hat{\mathbb{E}}[|X_1|^q] < \infty$ for some $q > 2$. Let $S_n = \sum_{k=1}^n X_k$. Then for all $\varphi \in C(\mathbb{R})$ with quadratic growth condition,

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{S_n}{\sqrt{n}})] = \hat{\mathbb{E}}[\varphi(\xi)],$$

where $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ with $\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2]$.

Let $\mathcal{M}_n^q([\underline{\sigma}^2, \overline{\sigma}^2], K)$ denote the set of n-stages martingale S with filtration $\mathcal{F}$, such that for all k, both relations hold:

- $E[S_k] = 0$
- $\underline{\sigma}^2 \leq E[|S_{k+1} - S_k|^2 | \mathcal{F}_k] \leq \overline{\sigma}^2$
- $E[|S_{k+1} - S_k|^q | \mathcal{F}_k] \leq K^q$

Let

$$V_n[\varphi] := \sup_{S \in \mathcal{M}_n^q([\underline{\sigma}^2, \overline{\sigma}^2], K)} E[\varphi(\frac{S_n}{\sqrt{n}})].$$

Theorem (Central limit theorem)

We assume that $q > 2$. Then for all $\varphi \in C(\mathbb{R})$ with quadratic growth condition, we have

$$\lim_{n \to \infty} V_n[\varphi] = \hat{\mathbb{E}}[\varphi(\xi)],$$

where $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$.

Maximal L^p variation problem

Let $\mathcal{M}_n(\mu)$ be the set of n-stage martingales whose terminal distribution is μ. We define the L^p-variation of length n of the martingale $(L_k)_{k=1,\cdots,n}$ as

$$\mathcal{V}_n^p(L) = E[\sum_{k=1}^{n}(E[|L_k - L_{k-1}|^p|(L_i, i \leq k-1)])^{\frac{1}{p}}].$$

The value function is denoted by

$$V_n(\mu) = \frac{1}{\sqrt{n}} \sup_{L \in \mathcal{M}_n(\mu)} \mathcal{V}_n^p(L).$$

Theorem

For $p \in [1, 2)$ and $\mu \in \Delta(K)$, where K is a compact subset of $\mathbb{R}$. We have

$$\lim_{n\to\infty} V_n(\mu) = E[f_\mu(Z)Z],$$

where $Z \sim N(0, 1)$ and $f_\mu : \mathbb{R} \to \mathbb{R}$ is a increasing function such that $f_\mu(Z) \sim \mu$, i.e., $f_\mu(x) = F_\mu^{-1}(F_{\mathcal{N}}(x))$ with $F_\mu^{-1} = \inf\{s : F_\mu(s) > y\}$.

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Sketch Proof of Theorem:

$$\sup_{L\sim\mu, S\in\mathcal{M}'_n} E[L\frac{S_n}{\sqrt{n}}] \leq V_n(\mu) \leq \sup_{L\sim\mu, S\in\mathcal{M}^q_n([0,1],2)} E[L\frac{S_n}{\sqrt{n}}].$$

$$\sup_{L\sim\mu, S_n} E[L\frac{S_n}{\sqrt{n}}] = \inf_{\varphi\in Conv(K)} E[\varphi(L) + \varphi^*(\frac{S_n}{\sqrt{n}})],$$

$$\begin{aligned}\sup_{L\sim\mu, S\in\mathcal{M}^q_n([0,1],2)} E[L\frac{S_n}{\sqrt{n}}] &= \sup_{S\in\mathcal{M}^q_n([0,1],2)} \inf_{\varphi\in Conv(K)} E[\varphi(L) + \varphi^*(\frac{S_n}{\sqrt{n}})] \\ &= \inf_{\varphi\in Conv(K)} E[\varphi(L) + \sup_{S\in\mathcal{M}^q_n([0,1],2)} E[\varphi^*(\frac{S_n}{\sqrt{n}})]].\end{aligned}$$

Since φ^* is a convex function, we have

$$\lim_{n\to\infty} \sup_{S\in\mathcal{M}_n^q([0,1],2)} E[\varphi^*(\frac{S_n}{\sqrt{n}})] = E[\varphi^*(Z)],\ \ Z \sim N(0,1).$$

$$\begin{aligned}\lim_{n\to\infty} V_n(\mu) &= \inf_{\varphi\in Conv(K)} E[\varphi(L) + \lim_{n\to\infty} \sup_{S\in\mathcal{M}_n^q([0,1],2)} E[\varphi^*(\frac{S_n}{\sqrt{n}})]] \\ &= \inf_{\varphi\in Conv(K)} E[\varphi(L) + \varphi^*(Z)] = E[f_\mu(Z)Z].\end{aligned}$$

Thank you for your attention!