Mean-Variance Hedging on uncertain time horizon in a market with a jump

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Young Researchers Meeting on BSDEs, Numerics and Finance, Oxford 2012

Joint work with Idris Kharroubi and Armand Ngoupeyou

[1]Supported by the "Chaire Risque de Crédit", Fédération Bancaire Française.

Introduction

- Progressive enlargement of filtrations and BSDEs with jumps (with I. Kharroubi), forthcoming in Journal of Theoritical Probability.

- A decomposition approach for the discrete-time approximation of FBSDEs with a jump I: the Lipschitz case (with I. Kharroubi).

- A decomposition approach for the discrete-time approximation of FBSDEs with a jump II: the quadratic case (with I. Kharroubi).

- Mean-Variance Hedging on uncertain time horizon in a market with a jump (with I. Kharroubi and A. Ngoupeyou).

Mean-variance hedging in literature

$$\inf_{\pi} \mathbb{E}\left[\left(x + \int_0^T \pi_s dS_s - \xi\right)^2\right].$$

There exist two approaches to solve mean-variance hedging problem with a deterministic finite horizon:

- martingale theory and projection arguments: Delbaen-Schachermayer, Gouriéroux-Laurent-Pham, Schweizer, ... for the continuous case, and Arai for the semimartingale case,
- quadratic stochastic control and BSDE: Lim-Zhou, Lim, ... for the continuous case and the discontinuous case (driven by a Brownian motion and a Poisson process).

Jeanblanc-Mania-Santacroce-Schweizer combine tools from both approaches which allows them to work in a general semimartingale model.

Mean-variance hedging with random horizon

For some financial products (e.g. insurance, credit-risk) the horizon of the problem is not deterministic

$$\inf_{\pi} \mathbb{E}\Big[\Big(x + \int_0^{T \wedge \tau} \pi_s dS_s - \xi\Big)^2\Big] .$$

We use a BSDE approach as in Lim and provide a solution to the mean-variance hedging problem with

- random horizon,
- dependent jump and continuous parts.

Theoretical issue: no result for our BSDEs in this framework.

Outline

1. Preliminaries and market model
   - The probability space
   - Financial model
   - Mean-variance hedging
2. Solution of the mean-variance problem by BSDEs
   - Martingale optimality principle
   - Related BSDEs
   - A verification Theorem
3. How to solve the BSDEs

Outline

Settings

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space equipped with

- W a standard Brownian motion with its natural filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$,
- τ a random time (we define the process H by $H_t := \mathbf{1}_{\tau \leq t}$).

τ not always an $\mathbb{F}$-stopping time.

$\Rightarrow$ $\mathbb{G}$ smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time: $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ where

$$\mathcal{G}_t := \bigcap_{\varepsilon>0} \widetilde{\mathcal{G}}_{t+\varepsilon} ,$$

for all $t \geq 0$, with $\widetilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma\big(\mathbb{1}_{\tau\leq u} , u \in [0, s]\big)$, for all $s \geq 0$.

Assumption on W and τ

(H) The process W remains a $\mathbb{G}$-Brownian motion.

(Hτ) The process H admits an $\mathbb{F}$-compensator of the form $\int_0^{.\wedge\tau} \lambda_s ds$, i.e. $H - \int_0^{.\wedge\tau} \lambda_s ds$ is a $\mathbb{G}$-martingale, where λ is a bounded $\mathcal{P}(\mathbb{F})$-measurable process. We then denote by M the $\mathbb{G}$-martingale defined by

$$M_t \quad := \quad H_t - \int_0^{t\wedge\tau} \lambda_s ds \; = \; H_t - \int_0^t \lambda_s^{\mathbb{G}} ds \; ,$$

for all $t \geq 0$, with $\lambda_t^{\mathbb{G}} := (1 - H_t)\lambda_t$.

Financial market

Financial market is composed by

- a riskless bond B with zero interest rate: $B_t = 1$,
- a risky asset S modeled by the stochastic differential equation

$$S_t = S_0 + \int_0^t S_{u^-}(\mu_u du + \sigma_u dW_u + \beta_u dM_u), \quad t \geq 0,$$

where μ, σ and β are $\mathcal{P}(\mathbb{G})$-measurable processes satisfying **(HS)**

(i) μ, σ and β are bounded,

(ii) there exists a constant $c > 0$ s.t.

$$\sigma_t \geq c, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$$

(iii) $-1 \leq \beta_t, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$

Admissible strategies

We consider the set $\mathcal{A}$ of investment strategies which are $\mathcal{P}(\mathbb{G})$-measurable processes π such that

$$\mathbb{E}\Big[\int_0^{T\wedge\tau}|\pi_t|^2dt\Big] \quad < \quad \infty \; .$$

We then define for an initial amount $x \in \mathbb{R}$ and a strategy π, the wealth $V^{x,\pi}$ associated with (x,π) by the process

$$V_t^{x,\pi} \quad = \quad x + \int_0^t \frac{\pi_r}{S_{r^-}}dS_r \; , \quad t \in [0, T\wedge\tau] \; .$$

Problem

For $x \in \mathbb{R}$, the problem of mean-variance hedging consists in computing the quantity

$$\inf_{\pi \in \mathcal{A}} \mathbb{E}\left[\left| V_{T\wedge\tau}^{x,\pi} - \xi \right|^2\right] , \tag{1}$$

where ξ is a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$\xi \quad = \quad \xi^b \mathbb{1}_{T<\tau} + \xi^a_\tau \mathbb{1}_{\tau \leq T} \ ,$$

where ξ^b is a bounded $\mathcal{F}_T$-measurable random variable and ξ^a is a continuous $\mathbb{F}$-adapted process satisfying

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\xi^a_t| \quad < \quad +\infty \ .$$

Outline

Sufficient conditions for optimality

We look for a family of processes $\{(J_t^\pi)_{t\in[0,T]} \ : \ \pi \in \mathcal{A}\}$ satisfying

(i) $J^\pi_{T\wedge\tau} = \left|V^{x,\pi}_{T\wedge\tau} - \xi\right|^2$, for all $\pi \in \mathcal{A}$.

(ii) $J_0^{\pi_1} = J_0^{\pi_2}$, for all $\pi_1, \pi_2 \in \mathcal{A}$.

(iii) J^π is a $\mathbb{G}$-submartingale for all $\pi \in \mathcal{A}$.

(iv) There exists some $\pi^* \in \mathcal{A}$ such that J^{π^*} is a $\mathbb{G}$-martingale.

Under these conditions, we have for any $\pi \in \mathcal{A}$

$$\mathbb{E}(J^{\pi^*}_{T\wedge\tau}) \ = \ J_0^{\pi^*} \ = \ J_0^\pi \ \leq \ \mathbb{E}(J^\pi_{T\wedge\tau})\ .$$

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Therefore, we get

$$J_0^{\pi^*} \;=\; \mathbb{E}\Big[\left|V^{x,\pi^*}_{T\wedge\tau} - \xi\right|^2\Big] \;=\; \inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\left|V^{x,\pi}_{T\wedge\tau} - \xi\right|^2\Big]\,.$$

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- $\mathcal{S}^\infty_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued càd-làg $\mathbb{G}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty} \ := \ \Big\| \sup_{t\in[0,T]} |Y_t| \Big\|_\infty \ < \ \infty \ .$$

- $\mathcal{S}^{\infty,+}_{\mathbb{G}}$ is the subset of $\mathcal{S}^\infty_{\mathbb{G}}$ of processes $(Y_t)_{t\in[0,T]}$ valued in $(0,\infty)$, such that

$$\Big\| \frac{1}{Y} \Big\|_{\mathcal{S}^\infty} \ < \ \infty \ .$$

- $L^2_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2} \ := \ \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt \Big] \Big)^{\frac{1}{2}} \ < \ \infty \ .$$

- $L^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\lambda)} \ := \ \Big(\mathbb{E}\Big[\int_0^{T\wedge\tau} \lambda_s |U_s|^2 ds \Big] \Big)^{\frac{1}{2}} \ < \ \infty \ .$$

Construction of J^π using BSDEs

To construct such a family $\{(J_t^\pi)_{t\in[0,T]},\ \pi \in \mathcal{A}\}$, we set

$$J_t^\pi \quad := \quad Y_t |V_{t\wedge\tau}^{x,\pi} - \mathcal{Y}_t|^2 + \Upsilon_t \ , \quad t \geq 0 \ ,$$

where (Y, Z, U), $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$ and (Υ, Ξ, Θ) are solution in $\mathcal{S}_{\mathbb{G}}^\infty \times L_{\mathbb{G}}^2 \times L^2(\lambda)$ to

$$Y_t \quad = \quad 1 + \int_{t\wedge\tau}^{T\wedge\tau} \mathfrak{f}(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dM_s \ , \tag{2}$$

$$\mathcal{Y}_t \quad = \quad \xi + \int_{t\wedge\tau}^{T\wedge\tau} \mathfrak{g}(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{U}_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} \mathcal{Z}_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \mathcal{U}_s dM_s \ , \tag{3}$$

$$\Upsilon_t \quad = \quad \int_{t\wedge\tau}^{T\wedge\tau} \mathfrak{h}(s, \Upsilon_s, \Xi_s, \Theta_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} \Xi_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \Theta_s dM_s \ , \tag{4}$$

for all $t \in [0, T]$.

We are bounded to choose three functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ for which

- J^π is a submartingale for all $\pi \in \mathcal{A}$,
- there exists $\pi^* \in \mathcal{A}$ such that J^{π^*} is a martingale.

For that we would like to write J^π as the sum of a martingale M^π and a nondecreasing process K^π that is constant for some $\pi^* \in \mathcal{A}$.

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From Itô's formula, we get

$$dJ_t^\pi \quad = \quad dM_t^\pi + dK_t^\pi \ ,$$

where M^π is a local martingale and K^π is given by

$$dK_t^\pi \quad := \quad K_t(\pi_t)dt \ = \ \big(A_t|\pi_t|^2 + B_t\pi_t + C_t\big)dt \ ,$$

with

$$\begin{aligned}
A_t &:= |\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t + Y_t) \ , \\
B_t &:= 2(V_{t\wedge\tau}^\pi - \mathcal{Y}_t)(\mu_t Y_t + \sigma_t Z_t + \lambda_t^{\mathbb{G}}\beta_t U_t) - 2\sigma_t Y_t \mathcal{Z}_t - 2\lambda_t^{\mathbb{G}}\beta_t \mathcal{U}_t(Y_t + U_t) \ , \\
C_t &:= -\mathfrak{f}(t)|V_{t\wedge\tau}^\pi - \mathcal{Y}_t|^2 + 2X_t^\pi(Y_t\mathfrak{g}(t) - Z_t\mathcal{Z}_t - \lambda_t^{\mathbb{G}}U_t\mathcal{U}_t) + Y_t|\mathcal{Z}_t|^2 \\
&\quad + \lambda_t^{\mathbb{G}}|\mathcal{U}_t|^2(U_t + Y_t) - \mathfrak{h}(t) \ .
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&\quad +\lambda_t^{\mathbb{G}}|\mathcal{U}_t|^2(U_t + Y_t) - \mathfrak{h}(t) .
\end{aligned}$$

In order to obtain a nondecreasing process K^π for any $\pi \in \mathcal{A}$ and that is constant for some $\pi^* \in \mathcal{A}$ it is obvious that K_t has to satisfy $\min_{\pi\in\mathbb{R}} K_t(\pi) = 0$:

$$\underline{K}_t \;:=\; \min_{\pi\in\mathbb{R}} K_t(\pi) \;=\; C_t - \frac{|B_t|^2}{4A_t} \,.$$

We then obtain from the expressions of A, B and C that

$$\underline{K}_t \;=\; \mathfrak{A}_t|V^\pi_{t\wedge\tau} - \mathcal{Y}_t|^2 + \mathfrak{B}_t(V^\pi_{t\wedge\tau} - \mathcal{Y}_t) + \mathfrak{C}_t \,,$$

with

$$\mathfrak{A}_t \;:=\; -\mathfrak{f}(t) - \frac{|\mu_t Y_t + \sigma_t Z_t + \lambda^{\mathbb{G}}_t \beta_t U_t|^2}{|\sigma_t|^2 Y_t + \lambda^{\mathbb{G}}_t |\beta_t|^2 (U_t + Y_t)} \,,$$

$$\mathfrak{B}_t \;:=\; 2\Big\{ \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda^{\mathbb{G}}_t \beta_t U_t)(\lambda^{\mathbb{G}}_t \beta_t \mathcal{U}_t (Y_t + U_t) + \sigma_t Y_t \mathcal{Z}_t)}{|\sigma_t|^2 Y_t + \lambda^{\mathbb{G}}_t |\beta_t|^2 (U_t + Y_t)} + \mathfrak{g}(t) Y_t - Z_t \mathcal{Z}_t - \lambda^{\mathbb{G}}_t U_t \mathcal{U}_t \Big\} \,,$$

$$\mathfrak{C}_t \;:=\; -\mathfrak{h}(t) + |\mathcal{Z}_t|^2 Y_t + \lambda^{\mathbb{G}}_t (U_t + Y_t)|\mathcal{U}_t|^2 - \frac{|\sigma_t Y_t \mathcal{Z}_t + \lambda^{\mathbb{G}}_t \beta_t \mathcal{U}_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda^{\mathbb{G}}_t |\beta_t|^2 (U_t + Y_t)} \,.$$

Expressions of the generators

For that the family $(J^\pi)_{\pi\in\mathcal{A}}$ satisfies the conditions (iii) and (iv) we choose $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ such that

$$\mathfrak{A}_t = 0\,,\ \mathfrak{B}_t = 0 \quad \text{and} \quad \mathfrak{C}_t = 0\,,$$

for all $t\in[0,T]$.

$$\begin{cases} \mathfrak{f}(t,Y,Z,U) = -\frac{(\mu_t Y+\sigma_t Z+\lambda_t\beta_t U)^2}{|\sigma_t|^2 Y+\lambda_t|\beta_t|^2(U+Y)}\,, \\ \mathfrak{g}(t,\mathcal{Y},\mathcal{Z},\mathcal{U}) = \frac{1}{Y_t}\Big[Z_t\mathcal{Z}+\lambda_t U_t\mathcal{U}-\frac{(\mu_t Y_t+\sigma_t Z_t+\lambda_t\beta_t U_t)(\sigma_t Y_t\mathcal{Z}+\lambda_t\beta_t(U_t+Y_t)\mathcal{U})}{|\sigma_t|^2 Y_t+\lambda_t\beta_t^2(U_t+Y_t)}\Big]\,, \\ \mathfrak{h}(t,\Upsilon,\Xi,\Theta) = |\mathcal{Z}_t|^2 Y_t+\lambda_t(U_t+Y_t)|\mathcal{U}_t|^2-\frac{|\sigma_t Y_t\mathcal{Z}_t+\lambda_t\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2 Y_t+\lambda_t|\beta_t|^2(U_t+Y_t)}\,. \end{cases}$$

$\Rightarrow$ Nonstandard Decoupled BSDEs

Theorem

The BSDEs (2)-(3)-(4) admit solutions (Y,Z,U), $(\mathcal{Y},\mathcal{Z},\mathcal{U})$ and (Υ,Ξ,Θ) in $\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$. Moreover $Y\in\mathcal{S}^{\infty,+}_{\mathbb{G}}$.

Expressions of the generators

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Optimal strategy-SDE of the optimal value portfolio

A candidate to be an optimal strategy is

$$\pi_t^* \quad = \quad \arg\min_{\pi \in \mathbb{R}} K_t(\pi) \ , \qquad (5)$$

which gives the implicit equation in π^*

$$\pi_t^* \quad = \quad \big(\mathcal{Y}_{t^-} - V_{t^-}^{x,\pi^*}\big) D_t + E_t \ ,$$

with $D_t := \frac{\mu_t Y_{t-} + \sigma_t Z_t + \lambda_t^{\mathbb{G}} \beta_t U_t}{|\sigma_t|^2 Y_{t-} + \lambda_t^{\mathbb{G}} |\beta_t|^2 (U_t + Y_{t-})}$ and $E_t := \frac{\sigma_t Y_{t-} \mathcal{Z}_t + \lambda_t^{\mathbb{G}} \beta_t \mathcal{U}_t (Y_{t-} + U_t)}{|\sigma_t|^2 Y_{t-} + \lambda_t^{\mathbb{G}} |\beta_t|^2 (U_t + Y_{t-})}$.

Integrating each side of this equality w.r.t. $\frac{dS_t}{S_{t^-}}$ leads to the following SDE

$$V_t^* \quad = \quad x + \int_0^t \big(\mathcal{Y}_{r^-} - V_{r^-}^*\big) D_r \frac{dS_r}{S_{r^-}} + \int_0^t E_r \frac{dS_r}{S_{r^-}} \ , \quad t \in [0, T] \ . \qquad (6)$$

Nonstandard SDE since D and E are not bounded.

Optimal strategy-SDE of the optimal value portfolio

Proposition

The SDE (6) *admits a solution* V^* *which satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]}|V_t^*|^2\Big] \;<\; \infty\,.$$

From Itô's formula, we get

$$dJ_t^{\pi} = dM_t^{\pi} + dK_t^{\pi} ,$$

where M^{π} is a local martingale and K^{π} is given by

$$dK_t^{\pi} := K_t(\pi_t)dt = \big(A_t|\pi_t|^2 + B_t\pi_t + C_t\big)dt ,$$

with

$$\begin{aligned} A_t &:= |\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t + Y_t) , \\ B_t &:= 2(V_{t\wedge\tau}^{\pi} - \mathcal{Y}_t)(\mu_t Y_t + \sigma_t Z_t + \lambda_t^{\mathbb{G}}\beta_t U_t) - 2\sigma_t Y_t \mathcal{Z}_t - 2\lambda_t^{\mathbb{G}}\beta_t \mathcal{U}_t(Y_t + U_t) , \\ C_t &:= -\mathfrak{f}(t)|V_{t\wedge\tau}^{\pi} - \mathcal{Y}_t|^2 + 2X_t^{\pi}(Y_t\mathfrak{g}(t) - Z_t\mathcal{Z}_t - \lambda_t^{\mathbb{G}}U_t\mathcal{U}_t) + Y_t|\mathcal{Z}_t|^2 \\ &\quad +\lambda_t^{\mathbb{G}}|\mathcal{U}_t|^2(U_t + Y_t) - \mathfrak{h}(t) . \end{aligned}$$

Verification theorem

Theorem

The strategy π^* *given by* (5) *belongs to the set* $\mathcal{A}$ *and is optimal for the mean-variance problem* (1)

$$\mathbb{E}\left[\left|V_{T\wedge\tau}^{x,\pi^*} - \xi\right|^2\right] = \min_{\pi\in\mathcal{A}} \mathbb{E}\left[\left|V_{T\wedge\tau}^{x,\pi} - \xi\right|^2\right].$$

Outline

A decomposition Approach: Data

We consider a BSDE of the form

$$Y_t = \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s, U_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s \ , \quad (7)$$

- terminal condition

$$\xi = \xi^b \mathbb{1}_{T<\tau} + \xi^a_\tau \mathbb{1}_{\tau\leq T} \ ,$$

where ξ^b is an $\mathcal{F}_T$-measurable bounded r.v. and $\xi^a \in \mathcal{S}^\infty_{\mathbb{F}}$,

- generator: F is a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable map and

$$F(t, y, z, u)\mathbb{1}_{t\leq\tau} = F^b(t, y, z, u)\mathbb{1}_{t\leq\tau} \ , \quad t \geq 0 \ ,$$

where F^b is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable map.
We then introduce the following BSDE

$$Y^b_t = \xi^b + \int_t^T F^b(s, Y^b_s, Z^b_s, \xi^a_s - Y^b_s) ds - \int_t^T Z^b_s dW_s \ . \quad (8)$$

A decomposition Approach: Theorem

Theorem

Assume that BSDE (8) *admits a solution* $(Y^b, Z^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$. *Then BSDE*

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s, U_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s \;,$$

$t \in [0, T]$, *admits a solution* $(Y, Z, U) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$ *given by*

$$\begin{aligned} Y_t &= Y_t^b \mathbb{1}_{t<\tau} + \xi^a_\tau \mathbb{1}_{t\geq\tau} \;, \\ Z_t &= Z_t^b \mathbb{1}_{t\leq\tau} \;, \\ U_t &= \left(\xi^a_t - Y^b_t\right) \mathbb{1}_{t\leq\tau} \;, \end{aligned}$$

for all $t \in [0, T]$.

Expressions of the generators

$$\begin{cases} \mathfrak{f}(t, Y, Z, U) & = & -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U + Y)} \,, \\ \mathfrak{g}(t, \mathcal{Y}, \mathcal{Z}, \mathcal{U}) & = & \frac{1}{Y_t}\Big[Z_t \mathcal{Z} + \lambda_t U_t \mathcal{U} - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t \mathcal{Z} + \lambda_t \beta_t (U_t + Y_t)\mathcal{U})}{|\sigma_t|^2 Y_t + \lambda_t \beta_t^2 (U_t + Y_t)} \Big] \,, \\ \mathfrak{h}(t, \Upsilon, \Xi, \Theta) & = & |\mathcal{Z}_t|^2 Y_t + \lambda_t (U_t + Y_t) |\mathcal{U}_t|^2 - \frac{|\sigma_t Y_t \mathcal{Z}_t + \lambda_t \beta_t \mathcal{U}_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \,. \end{cases}$$

Solution to BSDE $(\mathfrak{f}, 1)$

According to the general existence Theorem, we consider for coefficients $(\mathfrak{f}, 1)$ the BSDE in $\mathbb{F}$: find $(Y^b, Z^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\begin{cases} dY_t^b & = & \Big\{ \dfrac{|(\mu_t - \lambda_t\beta_t)Y_t^b + \sigma_t Z_t^b + \lambda_t\beta_t|^2}{|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2} - \lambda_t + \lambda_t Y_t^b \Big\} dt \\ & & + Z_t^b dW_t \;, \\ Y_T^b & = & 1 \;. \end{cases}$$

The generator of this BSDE can be written under the form

$$\Big\{ \frac{|\mu_t - \lambda_t\beta_t|^2}{|\sigma_t|^2} Y_t^b - \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4} |\mu_t - \lambda_t\beta_t|^2 - \lambda_t + \lambda_t Y_t^b + \frac{2(\mu_t - \lambda_t\beta_t)}{|\sigma_t|^2} (\sigma_t Z_t^b + \lambda_t\beta_t) \\ + \frac{\big|\sigma_t Z_t^b + \lambda_t\beta_t + (\lambda_t\beta_t - \mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2} \Big\} \;.$$

Introduction of a modified BSDE

Let $(Y^{\varepsilon}, Z^{\varepsilon})$ be the solution in $\mathcal{S}^{\infty}_{\mathbb{F}} \times L^2_{\mathbb{F}}$ to the BSDE

$$\begin{cases} dY^{\varepsilon}_t &= \Big\{ \frac{|\mu_t - \lambda_t\beta_t|^2}{|\sigma_t|^2} Y^{\varepsilon}_t - \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t - \lambda_t\beta_t|^2 + \frac{2(\mu_t - \lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_t Z^{\varepsilon}_t + \lambda_t\beta_t) \\ & \quad -\lambda_t + \lambda_t Y^{\varepsilon}_t + \frac{\left|\sigma_t Z^{\varepsilon}_t + \lambda_t\beta_t + (\lambda_t\beta_t - \mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\right|^2}{|\sigma_t|^2(Y^{\varepsilon}_t \vee \varepsilon) + \lambda_t|\beta_t|^2} \Big\} dt + Z^{\varepsilon}_t dW_t \,, \\ Y^{\varepsilon}_T &= 1 \,, \end{cases}$$

where ε is a positive constant such that

$$\exp\Big(- \int_0^T \Big(\lambda_t + \frac{|\mu_t - \lambda_t\beta_t|^2}{|\sigma_t|^2}\Big) dt \Big) \quad \geq \quad \varepsilon \,, \quad \mathbb{P} - a.s.$$

Question: $Y^{\epsilon} \geq \epsilon$?

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where ε is a positive constant such that

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Question: $Y^\epsilon \geq \epsilon$?

Change of probability

Define the process L^ε by

$$L_t^\varepsilon \quad := \quad 2\frac{(\mu_t - \lambda_t\beta_t)}{\sigma_t} + 2\frac{\sigma_t\big(\lambda_t\beta_t + \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}(\lambda_t\beta_t - \mu_t)\big)}{|\sigma_t|^2(Y_t^\varepsilon \vee \varepsilon) + \lambda_t|\beta_t|^2} + \frac{|\sigma_t|^2 Z_t^\varepsilon}{|\sigma_t|^2(Y_t^\varepsilon \vee \varepsilon) + \lambda_t|\beta_t|^2} \ .$$

Since $L^\varepsilon \in BMO(\mathbb{P})$, we can apply Girsanov theorem:

$$\bar{W}_t \quad := \quad W_t + \int_0^t L_s^\varepsilon ds \ ,$$

is a Brownian motion under the probability $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} \quad := \quad \mathcal{E}\Big(\int_0^T L_t^\epsilon dW_t\Big) \ .$$

Comparison under $\mathbb{Q}$

$$\begin{cases} -dY_t^\varepsilon &= \Big\{ \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^4} |\mu_t - \lambda_t \beta_t|^2 - \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} Y_t^\varepsilon - 2\lambda_t \beta_t \frac{(\mu_t - \lambda_t \beta_t)}{|\sigma_t|^2} \\ & \quad + \lambda_t - \lambda_t Y_t^\varepsilon - \frac{\left| \lambda_t \beta_t + (\lambda_t \beta_t - \mu_t) \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^2} \right|^2}{|\sigma_t|^2 (Y_t^\varepsilon \vee \varepsilon) + \lambda_t |\beta_t|^2} \Big\} dt - Z_t^\varepsilon d\bar{W}_t \, , \\ Y_T^\varepsilon &= 1 \, . \end{cases}$$

We remark that

$$generator \geq -\lambda_t y - \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} y \, .$$

Therefore, we get from a comparison theorem that

$$Y_t^\varepsilon \;\geq\; \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(- \int_t^T \big(\lambda_s + \frac{|\mu_s - \lambda_s \beta_s|^2}{|\sigma_s|^2}\big) ds \Big) \Big| \mathcal{F}_t \Big] \;\geq\; \epsilon \, .$$

Moreover $Z^\epsilon \in BMO(\mathbb{P})$.

Expressions of the generators

$$\left\{\begin{array}{lcl}\mathfrak{f}(t,Y,Z,U) & = & -\frac{(\mu_t Y+\sigma_t Z+\lambda_t\beta_t U)^2}{|\sigma_t|^2 Y+\lambda_t|\beta_t|^2(U+Y)}\ ,\\ \mathfrak{g}(t,\mathcal{Y},\mathcal{Z},\mathcal{U}) & = & \frac{1}{Y_t}\left[Z_t\mathcal{Z}+\lambda_t U_t\mathcal{U}-\frac{(\mu_t Y_t+\sigma_t Z_t+\lambda_t\beta_t U_t)(\sigma_t Y_t\mathcal{Z}+\lambda_t\beta_t(U_t+Y_t)\mathcal{U})}{|\sigma_t|^2 Y_t+\lambda_t\beta_t^2(U_t+Y_t)}\right]\ ,\\ \mathfrak{h}(t,\Upsilon,\Xi,\Theta) & = & |\mathcal{Z}_t|^2 Y_t+\lambda_t(U_t+Y_t)|\mathcal{U}_t|^2-\frac{|\sigma_t Y_t\mathcal{Z}_t+\lambda_t\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2 Y_t+\lambda_t|\beta_t|^2(U_t+Y_t)}\ .\end{array}\right.$$

Solution to BSDE $(\mathfrak{g}, \xi)$

We consider the associated decomposed BSDE in $\mathbb{F}$: find $(\mathcal{Y}^b, \mathcal{Z}^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\begin{cases} d\mathcal{Y}^b_t &= \Big\{ \dfrac{((\mu_t - \lambda_t\beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t\beta_t)(\sigma_t Y^b_t \mathcal{Z}^b_t + \lambda_t\beta_t\xi^a_t - \lambda_t\beta_t\mathcal{Y}^b_t)}{Y^b_t(|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2)} \\ & \quad - \frac{Z^b_t}{Y^b_t}\mathcal{Z}^b_t - \dfrac{\lambda_t}{Y^b_t}\xi^a_t + \dfrac{\lambda_t}{Y^b_t}\mathcal{Y}^b_t \Big\} dt + \mathcal{Z}^b_t dW_t \,, \\ \mathcal{Y}^b_T &= \xi^b \,. \end{cases}$$

Change of probability

Define the process ρ by $\rho_t := \frac{Z_t^b}{Y_t^b} - \frac{\sigma_t\left((\mu_t - \lambda_t\beta_t)Y_t^b + \sigma_t Z_t^b + \lambda_t\beta_t\right)}{|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2}$. Since $\rho \in BMO(\mathbb{P})$, we can apply Girsanov theorem

$$\widetilde{W}_t := W_t - \int_0^t \rho_s ds$$

is a $\widetilde{\mathbb{Q}}$-Brownian motion, where $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}|_{\mathcal{F}_T} := \mathcal{E}(\int_0^T \rho_t dW_t)$.
Hence, BSDE can be written

$$\begin{cases} d\mathcal{Y}_t^b = a_t(\mathcal{Y}_t^b - \xi_t^a)dt + \mathcal{Z}_t^b d\widetilde{W}_t \,, \\ \mathcal{Y}_{T\wedge\tau}^b = \xi^b \,, \end{cases}$$

with $a_t := \frac{\lambda_t|\sigma_t|^2 Y_t^b - \lambda_t\beta_t((\mu_t - \lambda_t\beta_t)Y_t^b + \sigma_t Z_t^b)}{Y_t^b(|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2)}$.
We can prove that $\mathcal{Y}^b$ defined by

$$\mathcal{Y}_t^b := \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[\exp\Big(-\int_t^T a_u du\Big)\xi^b + \int_t^T \exp\Big(-\int_t^s a_u du\Big) a_s \xi_s^a ds \Big| \mathcal{F}_t\Big]$$

is solution of this BSDE.

Expressions of the generators

$$\left\{ \begin{array}{rcl} \mathfrak{f}(t, Y, Z, U) & = & -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U + Y)} \,, \\ \mathfrak{g}(t, \mathcal{Y}, \mathcal{Z}, \mathcal{U}) & = & \frac{1}{Y_t}\left[Z_t \mathcal{Z} + \lambda_t U_t \mathcal{U} - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t \mathcal{Z} + \lambda_t \beta_t (U_t + Y_t)\mathcal{U})}{|\sigma_t|^2 Y_t + \lambda_t \beta_t^2 (U_t + Y_t)} \right] \,, \\ \mathfrak{h}(t, \Upsilon, \Xi, \Theta) & = & |\mathcal{Z}_t|^2 Y_t + \lambda_t (U_t + Y_t) |\mathcal{U}_t|^2 - \frac{|\sigma_t Y_t \mathcal{Z}_t + \lambda_t \beta_t \mathcal{U}_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \,. \end{array} \right.$$

Solution to BSDE $(\mathfrak{h}, 0)$

We consider the associated decomposed BSDE in $\mathbb{F}$: find $(\Upsilon^b, \Theta^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\Upsilon^b_t = \int_t^T \Big(|\mathcal{Z}^b_t|^2 Y^b_t + \lambda_t |\xi^a_t - \mathcal{Y}^b_t|^2 - \frac{|\sigma_t Y^b_t \mathcal{Z}^b_t + \lambda_t \beta_t (\xi^a_t - \mathcal{Y}^b_t)|^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2} - \lambda_s \Upsilon_s \Big) ds - \int_{t\wedge\tau}^{T\wedge\tau} \Xi^b_s dW_s \ .$$

We can prove that Υ^b defined by

$$\Upsilon^b_t := \mathbb{E}\Big[\int_t^T \exp\Big(- \int_t^s \lambda_u du \Big) R_s ds \Big| \mathcal{F}_t \Big] \ ,$$

where $R_t := |\mathcal{Z}^b_t|^2 Y^b_t + \lambda_t |\xi^a_t - \mathcal{Y}^b_t|^2 - \dfrac{|\sigma_t Y^b_t \mathcal{Z}^b_t + \lambda_t \beta_t (\xi^a_t - \mathcal{Y}^b_t)|^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2}$,

is solution of this BSDE.

Thanks!