BSDEs with terminal conditions that have bounded Malliavin derivative

...and driver that have arbitrary growth

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Outline

1. Motivation
   - The Problem We Studied
   - Previous Work
2. Our Results
   - Main Results
   - Sketch of Proof
   - Semilinear parabolic PDEs
   - Lipschitzness and Bounded Malliavin derivative
   - Semilinear Parabolic PDEs with Neumann BC or with Dirichlet BC

Outline

Question

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

Let W be n dimensional Brownian motion on $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ where $(\mathcal{F}_t)$ be the filtration generated by W. ξ is $\mathcal{F}_T$ measurable $\mathbb{L}^2(d\mathbb{P})$ random variable. f is progressively measurable and $f(\cdot, 0, 0) \in \mathbb{H}^2$.

- (Pardoux and Peng) f is globally Lipschitz in $(y, z) \Rightarrow \exists!(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^n)$.
- (Kobylanski) ξ is bounded and $f(t, y, z) = a_0(t, y, z)y + f_0(t, y, z)$ where a_0 is bounded andf_0 has growth like $b + \rho(|y|)|z|^2$ for an increasing function $\rho \Rightarrow \exists(Y, Z) \in \mathbb{H}^\infty \times \mathbb{H}^2$

Can we assume superquadratic growth in z with/without superlinear growth in y?

$\Rightarrow$ Under "regularity" condition on ξ, yes.

Outline

Previous Works

- Lipschitz Driver: Pardoux & Peng(1990)
- Quadratic growth in z
  - Kobylanski(2000): bounded ξ
  - Briand & Hu(2006,2008), Delbaen et al.(2011): unbounded ξ
- Superquadratic growth in z
  - Delbaen et al.(2010): convex driver under Markovian framework and bdd ξ.
  - Cheridito & Stadje(2011): convex driver and Lipschitz ξ.
  - Richou(2011, preprint): Markovian framework.
- Superlinear growth in y
  - Lepeltier & San Martin(1997): $|y|\sqrt{log|y|}$ like growth.
  - Briand & Carmona(2000): monotonicity condition
- Quadratic in z and Polynomial in y
  - Kobylanski(2000): bounded ξ
  - Frei & Dos Reis(2012): bounded ξ and $|f(s,y,z)| \leq C(1+|y|+(1+|y|^k)|z|^2)$

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Main Theorem

We assume the following conditions.

(A1) $\xi \in \mathbb{D}^{1,2}$ with $|D_r^i \xi| \le C_i \; dr \otimes d\mathbb{P}$ a.e.

(A2) $f(\cdot, 0, 0) \in \mathbb{H}^4$

(A3) For a nondecreasing function ρ,
$|f(s,y,z) - f(s,y',z)| \le B|y - y'|$
$|f(s,y,z) - f(s,y,z')| \le \rho(|z| \vee |z'|)|z - z'|$

(A4) $f(\cdot, y, z) \in \mathbb{L}_a^{1,2}$ and $|D_r f(s,y,z)| \le \phi_i(s), dr \otimes d\mathbb{P}$-a.e. where ϕ_i are Borel measurable with $\int_0^T \phi_i^2(t)dt < \infty$

(A5) $|D_r f(s,y,z) - D_r f(s,y',z')| \le K_{r,s}^R (|y - y'| + |z - z'|)$ for all $|z|, |z'| \le R$ where for a.a r, $K_{r,\cdot}^R \in \mathbb{H}^4$ such that $\int_0^T \left\| K_{r,\cdot}^R \right\|_{\mathbb{H}^4}^4 dr < \infty$

Then, our BSDE has a unique solution $(Y, Z) \in \mathbb{S}^4 \times \mathbb{H}^\infty(\mathbb{R}^n)$.
Moreover, the bound of Z_t^i is $\left(C_i + \int_t^T \phi_i(s)ds\right) e^{B(T-t)}$.

(cf)If $K_{r,s}^R$ is bounded, we can replace (A2) to $f(\cdot, 0, 0) \in \mathbb{H}^2$ and $Y \in \mathbb{S}^2$ instead of $\mathbb{S}^4$.

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Main Corollary

If we assume stronger solution than (A1) and (A2), we can generalize (A3).

(A1') (A1) and $|\xi| \leq E$

(A2') $|f(s,y,z)| \leq G(1+|y|) + \psi(|y|,|z|)|z|$ for nondecreasing function ψ

(A3') For nondecreasing functions ρ_1 and ρ_2,
$|f(s,y,z) - f(s,y',z)| \leq \rho_1(|y| \vee |y'|)|y-y'|$
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(A5') $|D_r f(s,y,z) - D_r f(s,y',z')| \leq K^R_{r,s}(|y-y'| + |z-z'|)$ for all $|y|,|y'|,|z|,|z'| \leq R$ where for a.a r, $K^R_{r,\cdot} \in \mathbb{H}^4$

In addition, we assume (A4). Then, our BSDE has a unique solution $(Y,Z) \in \mathbb{S}^\infty \times \mathbb{H}^\infty(\mathbb{R}^n)$. Moreover,
$|Y_t| \leq (E+1)e^{G(T-t)} - 1 := d(t)$ for all t a.s.
$|Z^i_t| \leq \left(C_i + \int_t^T \phi_i(s)ds\right) e^{\rho_1(d(t))(T-t)} =: e_i(t)$ for almost all t a.s.

ex) $f(y,z) = |z|^3 + \frac{|z|}{1+|z|}|y|^2$

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(A5') $|D_r f(s,y,z) - D_r f(s,y',z')| \leq K^R_{r,s}(|y-y'| + |z-z'|)$ for all $|y|,|y'|,|z|,|z'| \leq R$ where for a.a r, $K^R_{r,\cdot} \in \mathbb{H}^4$

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Outline

Sketch of Proof

1. Assume that (A1-(A4) hold and (A5) holds with K independent of R, f is differentiable in y and z,and ρ being constant. Then,

Theorem (El Karoui et al(1997))

We have a unique solution (Y, Z) *in* $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. *Moreover,* $(Y, Z) \in \mathbb{L}_a^{1,2}(\mathbb{R}^{n+1})$ *and for all* $i = 1, \ldots, n$, $(D_r^i Y_t, D_r^i Z_t) = (U_t^r, V_t^r)$ $dr \otimes dt \otimes d\mathbb{P}$*-a.e. and* $Z_t^i = U_t^t$ $dt \otimes d\mathbb{P}$*-a.e. where* $U_t^r = 0$ *and* $V_t^r = 0$ *for* $0 \leq t < r \leq T$, *and, for each fixed* r, $(U_t^r, V_t^r)_{r \leq t \leq T}$ *is the unique pair in* $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ *solving the BSDE*

$$U_t^r = D_r^i \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U_s^r + \partial_z f(s, Y_s, Z_s) V_s^r + D_r^i f(s, Y_s, Z_s)] ds - \int_t^T V_s^r dW_s.$$

Sketch of Proof

2 Using Ito formula on $e^{Bt}|U_t^r|^2$, we can bound $D_r Y_t$ and Z_t.

3 Show the solution exists uniquely for small time with Z bounded by big enough S. <= cutoff argument.

4 By comparison theorem, we get better bound for Z which is not blowing up.

5 Iterate the procedure to time 0.

6 Using mollification, we can show the claim for Lipschitz driver.

7 Generalize to K^R using the explicit bound of Z.

8 Under (A1')-(A3'), (A4) and (A5'), by comparison theorem, we can bound Y and prove the corollary.

Outline

Markovian BSDEs and semilinear parabolic PDEs

For some $B > 0$, consider

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r \qquad (1)$$

$$\text{where} \quad X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r)dW_r$$

where $|b(s,x) - b(s,y)| \leq B|x-y|$, $|b(s,x)| \leq B(1+|x|)$, and $|\sigma| \leq B$.

(B1) $h : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz with coefficient B.

(B2) $\int_0^T |g(s,0,0,0)|^2 ds < \infty$.

(B3) For a nondecreasing function ρ,

$$|g(s,x,y,z) - g(s,x,y',z)| \leq B|y-y'|$$
$$|g(s,x,y,z) - g(s,x,y,z')| \leq \rho\left(|z| \vee |z'|\right)|z-z'|$$

(B4) $|g(s,x,y,z) - g(s,x',y,z)| \leq B|x-x'|$

(B5) $|g(s,x,y,z) - g(s,x',y,z) - g(s,x,y',z') + g(s,x',y',z')| \leq B^R|x-x'|\left(|y-y'| + |z-z'|\right)$ for all $|z|, |z'| \leq R$

(B1') (B1) with $|h| \le B$

(B2') $|g(s,x,y,z)| \le B(1+|y|) + \psi(|y|,|z|)|z|$.

(B3') For nondecreasing functions ρ_1 and ρ_2,

$$|g(s,x,y,z) - g(s,x,y',z)| \le \rho_1(|y| \vee |y'|)|y-y'|$$
$$|g(s,x,y,z) - g(s,x,y,z')| \le \rho_2\left(|y|, |z| \vee |z'|\right)|z-z'|$$

(B5') $|g(s,x,y,z) - g(s,x',y,z) - g(s,x,y',z') + g(s,x',y',z')| \le B^R|x-x'|\left(|y-y'| + |z-z'|\right)$ for all $|y|, |y'|, |z|, |z'| \le R$

Theorem

If (B1)-(B5) hold, then BSDE (1) *has a unique solution such that $X^{t,x}$ is a square integrable strong solution and $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}^4 \times \mathbb{H}^\infty(\mathbb{R}^n)$.*

Theorem

If (B1')-(B3'),(B4), and (B5') hold, then BSDE (1) *has a unique solution such that $X^{t,x}$ is a square integrable strong solution, $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}^\infty \times \mathbb{H}^\infty(\mathbb{R}^n)$.*

Sketch of Proof

It is easy to check conditions of main theorem and corollary using the following two lemmas.

Lemma

$X^{t,x} \in \mathbb{D}^{1,2}$ with $|D_r X^{t,x}| \leq B e^{BT}$.

Proof.

Differentiability is proved in the reference and we can use Gronwall's inequality to Malliavin derivative of above SDE of $X^{t,x}$. $\square$

Lemma (Proposition 1.1.4 of Nualart(2006))

If ϕ is L-Lipschitz and $X \in \mathbb{D}^{1,2}$, then $\phi(X) \in \mathbb{D}^{1,2}$ and there exists random vector R bounded by L such that $D\phi(X) = R(DX)$.

Viscosity Solution

Theorem

Assume either (B1)-(B5) or (B1')-(B3'), (B4), (B5'). Then, $u(t,x) := Y_t^{t,x}$ *is a unique viscosity solution of*

$$u_t(t,x) + L_t u(t,x) + g(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0 \qquad (2)$$
$$u(T,x) = g(x)$$

where

$$L_t = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(t)\partial_{x_i}\partial_{x_j} + \sum_i b_i(t,x)\partial_{x_i}.$$

Proof.

By standard result, it is known u is a viscosity solution. By Ishii and Lions(1990), under our conditions, viscosity solution is unique. □

Classical Solution

Assume either (i) or (ii) holds.

(i) Assume (B1)–(B5). Also, suppose that $b(s,x)$, $h(x)$, and $g(s,x,y,z)$ are $\mathcal{C}^3$ in (x,y,z) and they have bounded first order, second order, and third order derivatives in

$$\{(s,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq M_z\}$$

where M_z is the bound of $Z^{t,x}$ in our previous theorem.

(ii) Assume (B1')–(B3'), (B4), (B5'). Also, suppose that $b(s,x)$, $h(x)$, and $g(s,x,y,z)$ are $\mathcal{C}^3$ in (x,y,z) and they have bounded first order, second order, and third order derivatives in

$$\{(s,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq M'_y, |z| \leq M'_z\}$$

where M'_y and M'_z are the bounds of $Y^{t,x}$ and $Z^{t,x}$ in our previous theorem, respectively.

Theorem

Then, $u(t,x)$ in previous theorem is actually of class $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^m;\mathbb{R})$ which solves (2). Moreover, if (i) holds, $\nabla u(t,x)\sigma(t)$ is bounded for almost every t. If (ii) holds, then $u(t,x)$ is bounded for all t and $\nabla u(t,x)\sigma(t)$ is bounded for almost every t.

Proof.

Apply the standard result in El Karoui et al.(1997) with appropriate bound. □

cf) When $g(z)=a|z|^d$ with $d>1$, these results closely related with Amour and Ben-Artzi(1998), Gilding et al.(2003) which deals with classical solutions. For classical solution, Amour and Ben-Artzi assumed bounded terminal condition with $\mathcal{C}_b^2$. Gilding et al. assumed the boundedness of terminal condition and fixed $a=1$.

Outline

Lipschitz⇒Bdd Malliavin derivative

We provide a sufficient condition for (A1).

Definition

We call a random variable ξ Lipschitz continuous in the Brownian motion W with constants $C_1, \dots, C_n \in \mathbb{R}_+$ if $\xi = \varphi(W)$ for a function $\varphi : \mathcal{C}([0,T];\mathbb{R}^n) \to \mathbb{R}$ satisfying

$$|\varphi(v) - \varphi(w)| \leq \sup_{0 \leq t \leq T} \sum_{i=1}^{n} C_i |v^i(t) - w^i(t)|.$$

Theorem

Let ξ be Lipschitz continuous in W with constants $C_1, \dots, C_n \in \mathbb{R}_+$. Then $\xi \in \mathbb{D}^{1,2}$ and $|D_t^i \xi| \leq C_i$ $dt \otimes d\mathbb{P}$-a.e. for all $i = 1, \dots, n$. Converse is not TRUE.

Example

Assume $T = n = 1$. Define

$$g(t) := \sum_{k=1}^{\infty} (-1)^{k-1} 2^k 1_{\{1-2^{1-k} < t \le 1-2^{-k}\}}, \quad h(t) := \int_0^t g(s) ds,$$

and set

$$\xi := \int_0^1 h(t) dW_t.$$

Then $D\xi = h$ is bounded by 1.
On the other hand, it follows from integration by parts that

$$\xi = -\lim_{k\to\infty} \int_0^{1-2^{-2k}} g(t) W_t dt.$$

Outline

Semilinear Parabolic PDEs with Neumann BC

Let $\mathcal{O}$ be a bounded open subset in $\mathbb{R}^m$ with smooth boundary. Let the forward process in FBSDE (1) is changed to

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s n(X_r^{t,x})d\kappa_r^{t,x}$$

$$\kappa_s^{t,x} = \int_t^s 1_{\{X_r^{t,x} \in \partial\mathcal{O}\}} d\kappa_r^{t,x}, \quad \kappa_\cdot^{t,x} \text{ is increasing}$$

Example

Assume $\mathcal{O}$ is an open bounded box, b is Lipschitz, σ is a constant. Then, $X^{t,x}$ is Lipschitz in underlying Brownian motion. Therefore, $X^{t,x} \in \mathbb{D}^{1,2}$ and $DX^{t,x}$ is bounded.

This gives a viscosity solution of semilinear parabolic PDE with Neumann boundary condition by Pardoux and Zhang(1998).

Semilinear Parabolic PDEs with Dirichlet BC

Consider the following Markovian BSDEs which is indexed by $(t, x) \in [0, T] \times \mathbb{R}^m$ with a random terminal time $\tau := \inf\{s \geq t : X_s^{t,x} \notin \mathcal{O}\} \wedge T$ for a bounded open set $\mathcal{O} \subset \mathbb{R}^p$.

$$Y_s^{t,x} = h(X_\tau^{t,x}) + \int_{s\wedge\tau}^{\tau} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_{s\wedge\tau}^{\tau} Z_r^{t,x} dW_r$$

$$\text{where } X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r$$

Example

Assume $\mathcal{O} = (a, b) \subset \mathbb{R}$. Let

$$X_s^{t,x} := x + \sigma(W_s - W_t)$$

$$\tau := \inf\left\{s \geq t : X_s^{t,x} \notin \mathcal{O}\right\} \wedge T$$

where σ is a constant. Then, $X_\tau^{t,x}$ is Lipschitz in underlying BM.

This gives a probabilistic representation of a classical solution of semilinear parabolic PDE.(Ladyzenskaja at al. 1968). Peng(1992) used strong version of Ladyzenskaja's theorem to prove this for Lipschitz driver. We used our result to recover part of original Ladyzenskaja's theorem.

Summary

- The regularity of the terminal condition gives the bound of Z (and Y) in our BSDE. Then we can apply most of standard BSDE results.
- The Lipschitzness is a sufficient, but not necessary condition for bounded Malliavin derivative and this is often easier to check than Malliavin boundedness.
- Comparison theorem is crucial in the global existence of solution. In multidimensional superquadratic BSDE, we have solution only for small time if we assume the terminal condition has bounded Malliavin derivative.

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