Optimal Simulation Schemes for Lévy driven SDEs

Salvador Ortiz-Latorre
Imperial College London

(joint work with A. Kohatsu-Higa and P. Tankov)

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Outline

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Problem

We are interested in the numerical approximation of $\mathbb{E}[f(X_1)]$, where

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s + \int_0^t h(X_{s-})\, dZ_s.$$

- $b : \mathbb{R}^d \to \mathbb{R}^d$, $h : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are C^1 with bounded derivatives.
- $B = \{B_t\}_{t\in[0,1]}$ is a k-dimensional Brownian motion.
- $Z = \{Z_t\}_{t\in[0,1]}$ is a **one dimensional** Lévy process (independent of B) with the following representation

$$Z_t = \int_0^t \int_{|y|\le 1} y\widetilde{N}(dy, ds) + \int_0^t \int_{|y|>1} yN(dy, ds),$$
$$\widetilde{N}(dy, ds) = N(dy, ds) - \nu(dy)\, ds,$$

where ν is an infinite activity Lévy measure, that is $\nu(\mathbb{R}) = +\infty$, and N is a Poisson random measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dy) \times dt$.

Previous works.

Case $b \equiv \sigma \equiv 0$, that is,

$$X_t = x + \int_0^t h\left(X_{s-}\right) dZ_s.$$

- **Euler scheme**: *Protter and Talay* (1997). Two difficulties:
  - No available algorithm to simulate the increments of Z.
  - A large jump between discretization times can lead a large discretization error.
- **Approximated Euler scheme**: *Jacod et al.* (2005).
- **Jump adapted schemes**:
  - *Rubenthaler* (2003). Replace Z with a compound Poisson approximation and place the discretization points at its jump times.
  - *Kohatsu-Higa and Tankov* (2010). Replace Z with a compound Poisson process plus a Brownian motion (similar to *Asmussen-Rosinski* (2001) approach).
  - *Tankov* (2011). Replace Z with Z_ε a finite intensity Lévy process incorporating all the jumps bigger than ε and an additional compound Poisson term matching a given number of moments of Z.

Definitions

- Let $\bar{X} = \{\bar{X}_t\}_{t\in[0,1]}$ be the family of approximating processes, which is the solution of the family of SDEs

$$\bar{X}_t = x + \int_0^t b(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)dB_s + \int_0^t h(\bar{X}_{s-})d\bar{Z}_s,$$

where $\bar{Z} = \{\bar{Z}_t\}_{t\in[0,1]}$ is a family of Lévy processes (independent of B) with the following representation

$$\bar{Z}_t = \bar{\mu}t + \bar{\sigma}W_t + \int_0^t \int_{|y|\leq 1} y\tilde{\bar{N}}(dy, ds) + \int_0^t \int_{|y|>1} y\bar{N}(dy, ds),$$

$$\tilde{\bar{N}}(dy, ds) = \bar{N}(dy, ds) - \bar{\nu}(dy)\, ds,$$

where $\bar{\lambda} = \int_{\mathbb{R}} \bar{\nu}(dy) < \infty$, $\bar{\sigma}^2 \geq 0$ and $\bar{N}$ is a Poisson random measure on $\mathbb{R} \times [0,\infty)$ with intensity $\bar{\nu}(dy) \times ds$ and $W = \{W_t\}_{t\in[0,1]}$ is a standard k-dimensional Brownian motion independent of all the other processes.

- We assume that $(\bar{\mu}, \bar{\nu}, \bar{\sigma})$ belongs to a set of possible approximation parameters denoted by $\mathcal{A}$.

Definitions

- We consider the following stopping times

$$\bar{T}_i \triangleq \inf\{t > \bar{T}_{i-1} : \bar{N}(\mathbb{R}, (\bar{T}_{i-1}, t]) \neq 0\}, \quad i \in \mathbb{N},$$
$$\bar{T}_0 \triangleq 0.$$

and the associated **jump** operators

$$(\bar{S}^i f)(x) \triangleq \mathbb{E}[f(x + h(x)\Delta\bar{Z}_{\bar{T}_i})], \quad i \in \mathbb{N}$$
$$(\bar{S}^0 f)(x) \triangleq f(x).$$

- Define the process

$$\bar{Y}_s(t,x) \triangleq x + \int_t^s \bar{b}(\bar{Y}_u(t,x))du + \int_t^s \sigma(\bar{Y}_u(t,x))dB_u + \bar{\sigma}\int_t^s h(\bar{Y}_u(t,x))dW_u,$$

and consider its associated semigroup $(\bar{P}_t f)(x) \triangleq \mathbb{E}[f(\bar{Y}_t(0,x))]$.

- In general, we do not know the exact solution of $\bar{Y}_s(t,x)$ and we have to use and approximation.

Assumptions

- One can prove that

$$\mathbb{E}[\mathbf{1}_{\{1<\bar{T}_1\}} f(\bar{X}_1)] = \mathbb{E}[\mathbf{1}_{\{1<\bar{T}_1\}} \bar{S}^0 \bar{P}_1 f(x)],$$

$$\mathbb{E}[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}} f(\bar{X}_1)] = \mathbb{E}[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}} \bar{S}^0 \bar{P}_{\bar{T}_1\wedge 1} \bar{S}^1 \bar{P}_{\bar{T}_2-\bar{T}_1} \cdots \bar{S}^i \bar{P}_{1-\bar{T}_i} f(x)],$$

- **Assumption** $(\mathcal{SR})$. There exists a process $\hat{X} = \{\hat{X}_t\}_{t\in[0,1]}$ satisfying

$$\mathbb{E}[\mathbf{1}_{\{1<\bar{T}_1\}} f(\hat{X}_1)] = \mathbb{E}[\mathbf{1}_{\{1<\bar{T}_1\}} \bar{S}^0 \hat{P}_1 f(x)],$$

$$\mathbb{E}[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}} f(\hat{X}_1)] = \mathbb{E}[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}} \bar{S}^0 \hat{P}_{\bar{T}_1\wedge 1} \bar{S}^1 \hat{P}_{\bar{T}_2-\bar{T}_1} \cdots \bar{S}^i \hat{P}_{1-\bar{T}_i} f(x)],$$

  for $i \in \mathbb{N}$, where $\hat{P}_t$ is a linear operator.
- **Assumption** $(\mathcal{H}_n)$. $\int |y|^{2n} \nu(dy) < \infty$, $\sup_{\bar{\nu}\in\mathcal{A}} \int |y|^{2n} \bar{\nu}(dy) < \infty$ and $h, b, \sigma \in C_b^n$.

Assumptions

- For $t \in [0,1]$, let $\{\bar{P}_t^i\}_{n\in\mathbb{N}}$ and $\{Q_t^i\}_{n\in\mathbb{N}}$ be two families of linear operators from $\cup_{p\geq 0} C_p$ to $\cup_{p\geq 0} C_p$.
- **Assumption** $(\mathcal{M}_0)$. For all $i \in \mathbb{N}$, if $f \in C_p$ with $p \geq 2$, then $Q_t^i f \in C_p$ and

$$\sup_{t\in[0,1]} \left\| Q_t^i f \right\|_{C_p} \leq K \left\| f \right\|_{C_p},$$

  for some constant $K(\mathcal{A}) > 0$. Furthermore, we assume $0 \leq Q_t^i f(x) \leq Q_t^i g(x)$ whenever $0 \leq f \leq g$ and $Q_t^i \mathbf{1}_{\mathbb{R}}(x) = \mathbf{1}_{\mathbb{R}}(x)$.
- **Assumption** $(\mathcal{M})$. For all $i \in \mathbb{N}$, Q_t^i satisfies $(\mathcal{M}_0)$ and for each $f_p(x) := |x|^p \, (p \in \mathbb{N})$,

$$Q_t^i f_p(x) \leq (1 + Kt) f_p(x) + K' t$$

  for some positive constants K and K'.

Assumptions

- **Assumption** $(\mathcal{R}(m))$. For all $i \in \mathbb{N}$, define $\mathrm{Err}_t^i = \bar{P}_t^i - Q_t^i$. For each $p \geq 2$, there exists a constant $q = q(m, p)$ such that if $f \in C_p^{m^*}$ with $m^* \geq 2m + 2$ then

$$\left\| \mathrm{Err}_t^i f \right\|_{C_q} \leq K t^{m+1} \left\| f \right\|_{C_p^{m^*}},$$

for all $t \in [0, 1]$.

- **Assumption** $(\mathcal{M}_P)$. If $f \in C_p^m$ one has that for $k = 1, ..., n-1$

$$\sup_{(t_{k+1},\ldots,t_n)\in[0,1]^{n-k}} \left\| \prod_{i=k+1}^{n} \bar{P}_{t_i}^i f \right\|_{C_p^m} \leq C \left\| f \right\|_{C_p^m}.$$

Outline

Main result

Theorem

Assume $(\mathcal{H}_{n+1})$, *that* $\hat{X} = \{\hat{X}_t\}_{t\in[0,1]}$ *satisfies* $(\mathcal{SR})$, *and that the operators* $\bar{P}^i_t \triangleq \bar{S}^{i-1}\bar{P}_t$ *and* $Q^i_t \triangleq \bar{S}^{i-1}\hat{P}_t$ *satisfy assumptions* $(\mathcal{M})$, $(\mathcal{M}_P)$ *and* $(\mathcal{R}(m))$, $m \geq 2$. *Then, if* $f \in C_p^{2(m+1)} \cap C_b^{n+1}$, $n \geq 2$, $p \geq 2$ *there exist some positive constants* K *and* C_i, $i = 1, ..., n+1$ *such that*

$$
\begin{aligned}
&\left|\mathbb{E}[f(X_1)] - \mathbb{E}[f(\hat{X}_1)]\right| \\
&\leq C_1 \left|\int_{|y|>1} y(\nu - \bar{\nu})(dy) - \bar{\mu}\right| + C_2 \left|\int_{\mathbb{R}} y^2 (\nu - \bar{\nu})(dy) - \bar{\sigma}^2\right| \\
&+ \sum_{i=3}^{n} C_i \left|\int_{\mathbb{R}} y^i (\nu - \bar{\nu})(dy)\right| \\
&+ C_{n+1} \int_{\mathbb{R}} |y|^{n+1} |\nu - \bar{\nu}|(dy) + K \|f\|_{C_p^{2(m+1)}} \bar{\lambda}^{-m}.
\end{aligned}
$$

A simple example

- Parametrize the set $\mathcal{A}$ by a parameter $\varepsilon \in (0,1]$ so that:

$$\bar{\mu} \triangleq \mu_\varepsilon = \int_{|y|>1} y(\nu - \nu_\varepsilon)(dy),$$
$$\bar{\sigma}^2 \triangleq \sigma_\varepsilon^2 = \int_{\mathbb{R}} y^2(\nu - \nu_\varepsilon)(dy),$$
$$\bar{\nu}(dy) \triangleq \nu_\varepsilon(dy) = \mathbf{1}_{\{|y|>\varepsilon\}}\nu(dy),$$

- Set $\hat{P}_t \triangleq \hat{P}_t^\varepsilon$ as the operator associated with a one step Euler scheme.
- Then $\hat{X}^\varepsilon$ is an Euler scheme between jumps and it jumps with the law of a compound Poisson process which has ν_ε as its associated Lévy measure.
- The above result reads

$$\left|\mathbb{E}[f(X_1)] - \mathbb{E}[f(\hat{X}_1^\varepsilon)]\right| \leq C_3 \int_{|y|\leq\varepsilon} |y|^3 \nu(dy) + K \|f\|_{C_p^4} \lambda_\varepsilon^{-1}.$$

- In the particular case of α-tempered stable Lévy measures one obtains that the best convergence rate is λ_ε^{-1} for $\alpha \leq 1$ and the worse case is $\lambda_\varepsilon^{-1/2}$ for $\alpha \to 2$.

Idea of the proof

- We can expand the error as follows

$$\left|\mathbb{E}[f(X_1)] - \mathbb{E}[f(\hat{X}_1)]\right| \leq |\mathbb{E}[f(X_1)] - \mathbb{E}[f(\bar{X}_1)]| + |\mathbb{E}[f(\bar{X}_1)] - \mathbb{E}[f(\hat{X}_1)]| \triangleq \mathcal{D}_1 + \hat{\mathcal{D}}_1.$$

- Note that $\mathbb{E}[f(X_1)] - \mathbb{E}[f(\bar{X}_1)] = \mathbb{E}[u(0,x)] - \mathbb{E}[u(1,\bar{X}_1)]$, where $u(t,x)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= -Lu(t,x), \\ u(1,x) &= f(x) \end{aligned}$$

  and L is the generator of $P_t f(x) = \mathbb{E}[f(X_t(0,x))]$.
- To bound $\mathcal{D}_1$ we use the Itô formula and a Taylor expansion.

Idea of the proof

- Using Assumption $(\mathcal{SR})$ we can prove that

$$\begin{aligned}
\hat{\mathcal{D}}_1 &= \left|\mathbb{E}[f(\bar{X}_1)] - \mathbb{E}[f(\hat{X}_1)]\right| \\
&\leq \sum_{i=0}^{\infty} \left|\mathbb{E}\left[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}}\left(\prod_{k=1}^{i+1} \bar{P}^k_{\bar{T}_k\wedge 1-\bar{T}_{k-1}} - \prod_{k=1}^{i+1} Q^k_{\bar{T}_k\wedge 1-\bar{T}_{k-1}}\right) f(x)\right]\right| \\
&\leq K \left\|f\right\|_{C_p^{2(m+1)}} \sum_{i=0}^{\infty}\sum_{k=1}^{i+1} \mathbb{E}\left[\mathbf{1}_{\{\bar{T}_i<1<\bar{T}_{i+1}\}}\left(\bar{T}_k\wedge 1-\bar{T}_{k-1}\right)^{m+1}\right], \\
&\leq K \left\|f\right\|_{C_p^{2(m+1)}} \bar{\lambda}^{-m}
\end{aligned}$$

- Assumptions $(\mathcal{R}(m)), (\mathcal{M})$ and $(\mathcal{M}_P)$ are used to bound

$$\begin{aligned}
&\prod_{i=1}^{n} \bar{P}^i_{t_i-t_{i-1}} f(x) - \prod_{i=1}^{n} Q^i_{t_i-t_{i-1}} f(x) \\
&= \sum_{k=1}^{n}\left(\prod_{i=1}^{k-1} Q^i_{t_i-t_{i-1}}\left(\bar{P}^k_{t_k-t_{k-1}} - Q^k_{t_k-t_{k-1}}\right)\prod_{i=k+1}^{n} \bar{P}^i_{t_i-t_{i-1}}\right) f(x).
\end{aligned}$$

Outline

Motivation of the optimization problem

- We want to choose $(\bar{\mu}, \bar{\sigma}, \bar{\nu})$ which makes the first four terms of the error expansion small. That is, we want to minimize

$$
\begin{aligned}
&C_1 \left| \int_{|y|>1} y(\nu - \bar{\nu})(dy) - \bar{\mu} \right| + C_2 \left| \int_{\mathbb{R}} y^2 (\nu - \bar{\nu})(dy) - \bar{\sigma}^2 \right| \\
&+ \sum_{i=3}^{n} C_i \left| \int_{\mathbb{R}} y^i (\nu - \bar{\nu})(dy) \right| + C_{n+1} \int_{\mathbb{R}} |y|^{n+1} |\nu - \bar{\nu}|(dy).
\end{aligned}
$$

- Our approach is to take

$$
\bar{\mu} = \int_{|y|>1} y(\nu - \bar{\nu})(dy) \quad \text{and} \quad \bar{\sigma} = 0
$$

so that it becomes

$$
\sum_{i=2}^{n} C_i \left| \int_{\mathbb{R}} y^i (\nu - \bar{\nu})(dy) \right| + C_{n+1} \int_{\mathbb{R}} |y|^{n+1} |\nu - \bar{\nu}|(dy).
$$

Optimization problem

Problem ($\Omega_{n,\Lambda}$)

Let ν *be a Lévy measure on* $\mathbb{R}$ *admitting the first* n *moments, where* $n \geq 2$, *and define* $m_k = \int_{\mathbb{R}} y^k \nu(dy), 1 \leq k \leq n$. *For any* $\bar{\nu} \in \mathcal{M}$ *define the functional*

$$J(\bar{\nu}) \triangleq \int_{\mathbb{R}} |y|^n |\nu - \bar{\nu}|(dy).$$

The problem $\Omega_{n,\Lambda}, n \geq 2$, *consists in finding*

$$\mathcal{E}_n(\Lambda) \triangleq \min_{\bar{\nu} \in \mathcal{M}} J(\bar{\nu})$$

under the constraints

$$\int_{\mathbb{R}} \bar{\nu}(dy) = \Lambda \quad \text{and} \quad \int_{\mathbb{R}} y^k \bar{\nu}(dy) = m_k, \ k = 2, \ldots, n-1,$$

where $\Lambda \geq \min_{\bar{\nu} \in M_{n-1}} \bar{\nu}(\mathbb{R})$, *where we set by convention* $\min_{\bar{\nu} \in M_1} \bar{\nu}(\mathbb{R}) = 0$.

Optimization results

Theorem

The problem $\Omega_{n,\Lambda}$ admits a solution. The measure $\bar{\nu}$ is a solution of $\Omega_{n,\Lambda}$ if and only if it satisfies the constraints, and there exist a piecewise polynomial function $P(y) = a_0 + \sum_{i=2}^{n-1} a_i y^i + |y|^n$ such that $P(y) \geq 0$ for all $y \in \mathbb{R}$, a function $\alpha : \mathbb{R} \mapsto [0,1]$ and a positive measure τ on $\mathbb{R}$ such that

$$\bar{\nu}(dy) = \nu(dy)\mathbf{1}_{\{P(y)<2|y|^n\}} + \alpha(y)\nu(dy)\mathbf{1}_{\{P(y)=2|y|^n\}} + (\tau(dy)+\nu(dy))\mathbf{1}_{\{P(y)=0\}}.$$

Remark

If the measure ν is absolutely continuous with respect to Lebesgue's measure, the previous expression for $\bar{\nu}$ simplifies to

$$\bar{\nu}(dy) = \nu(dy)\mathbf{1}_{\{P(y)<2|y|^n\}} + \tau(dy)\mathbf{1}_{\{P(y)=0\}}.$$

Moreover, in the case $n = 2q, q \in \mathbb{N}$, $P(y)$ is a polynomial and the measure τ may always be taken to be an atomic measure with at most q atoms.

Optimal schemes

Case $n = 2$
An optimal solution is given by

$$\bar{\nu}_{\varepsilon}(dy) = \mathbf{1}_{\{y^2 > \varepsilon\}} \nu(dy),$$

where $\varepsilon = \varepsilon(\Lambda)$ solves

$$\nu(\{y^2 > \varepsilon\}) = \Lambda.$$

The approximation error $\mathcal{E}_2(\Lambda)$ is given by

$$\mathcal{E}_2(\Lambda) = J(\bar{\nu}_{\varepsilon(\Lambda)}) = \int_{y^2 \leq \varepsilon(\Lambda)} y^2 \nu(dy),$$

which can go to zero at an arbitrarily slow rate as $\Lambda \to \infty$.

Optimal schemes

Case $n=3$
An optimal solution is given by

$$\bar{\nu}_{\varepsilon}(dy)=\mathbf{1}_{\{|y|>\varepsilon\}}\nu(dy)+\alpha_1\delta_{-2\varepsilon}+\alpha_2\delta_{2\varepsilon},$$

where $\varepsilon=\varepsilon(\Lambda)$ solves

$$\int_{\{|y|>\varepsilon\}}\nu(dy)+\frac{1}{4\varepsilon^2}\int_{\{|y|\leq\varepsilon\}}y^2\nu(dy)=\Lambda,$$

and

$$\alpha_1+\alpha_2=\frac{1}{4\varepsilon^2}\int_{\{|y|\leq\varepsilon\}}y^2\nu(dy).$$

The worst case approximation error $\mathcal{E}_3(\Lambda)$ satisfies $\mathcal{E}_3(\Lambda)=o(\Lambda^{-1/2})$ as $\Lambda\to\infty$.

Optimal schemes

Case $n = 4$

An optimal solution is given by

$$\bar{\nu}_{\varepsilon}(dy) = \nu(dy)\mathbf{1}_{\{|y|>\varepsilon\sqrt{\sqrt{2}-1}\}} + \alpha_1\delta_{-\varepsilon} + \alpha_2\delta_{\varepsilon},$$

where $\varepsilon = \varepsilon(\Lambda)$ solves

$$\int_{\{|y|>\varepsilon\sqrt{\sqrt{2}-1}\}} \nu(dy) + \frac{1}{\varepsilon^2}\int_{\{|y|\leq\varepsilon\sqrt{\sqrt{2}-1}\}} y^2\nu(dy) = \Lambda,$$

and the constants α_1 and α_2 satisfy

$$\alpha_1 = \frac{1}{2\varepsilon^3}\left(-\int_{\{|y|\leq\varepsilon\sqrt{\sqrt{2}-1}\}} y^3\nu(dy) + \varepsilon\int_{\{|y|\leq\varepsilon\sqrt{\sqrt{2}-1}\}} y^2\nu(dy)\right),$$

$$\alpha_2 = \frac{1}{2\varepsilon^3}\left(\int_{\{|y|\leq\varepsilon\sqrt{\sqrt{2}-1}\}} y^3\nu(dy) + \varepsilon\int_{\{|y|\leq\varepsilon\sqrt{\sqrt{2}-1}\}} y^2\nu(dy)\right),$$

The worst case approximation error $\mathcal{E}_4(\Lambda)$ satisfies $\mathcal{E}_4(\Lambda) = o(\Lambda^{-1})$ as $\Lambda \to \infty$.

Outline

Approximation between jumps

- The solution between jumps $\bar{Y}_t(x)$ satisfies the following equation

$$\bar{Y}_t(x) = x + \int_0^t \bar{b}(\bar{Y}_s(x))ds + \int_0^t \sigma(\bar{Y}_s(x))dB_s,$$

where

$$\begin{aligned} \bar{b}(x) &= b(x) + \bar{\gamma} h(x), \\ \bar{\gamma} &= \int_{|y|>1} y(\nu - \bar{\nu})(dy). \end{aligned}$$

- One can consider different weak approximation schemes to solve $\bar{Y}_t(x)$.
- We have studied the weak Taylor methods of orders: $1, 2$ and 3.

Generic algorithm

Algorithm to generate a sample of $\hat{X}_1$

Requires:

An initial condition x_0.

An optimal Lévy measure $\bar{\nu}$.

A weak approximation method $\bar{Y}_t^{WA}(y)$, *to solve* $\bar{Y}_t(y)$

Compute $\bar{\lambda} = \bar{\nu}(\mathbb{R})$ *and* $\bar{\gamma} = \int_{|y|>1} y(\nu - \bar{\nu})(dy)$

Set $T_{last} = 0, x_{new} = x_0$

Simulate *the next jump time* $T \sim \mathrm{Exp}(\bar{\lambda})$

While $(T < 1 - T_{last})$ **do**

Compute $\bar{Y}_T^{WA}(x_{new})$

Simulate Δ, *a jump from the Poisson random measure with Lévy measure* $\bar{\nu}$

Set $x_{new} = \bar{Y}_T^{WA}(x_{new}) + h(\bar{Y}_T^{WA}(x_{new}))\Delta$

Set $T_{last} = T$

Simulate *the next jump time* $T \sim \mathrm{Exp}(\bar{\lambda})$

Compute $\bar{Y}_{1-T_{last}}^{WA}(x_{new})$

Return $\bar{Y}_{1-T_{last}}^{WA}(x_{new})$

Example

- Let Z be a tempered stable process which has the following Lévy measure

$$\nu(dy) = C\left\{\frac{e^{-\lambda_+ y}}{y^{1+\alpha}}\mathbf{1}_{\{y>0\}} + \frac{e^{-\lambda_-|y|}}{|y|^{1+\alpha}}\mathbf{1}_{\{y<0\}}\right\},$$

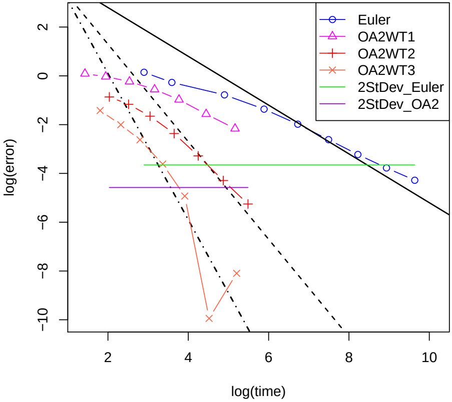
  with $C > 0, \lambda_+ > 0, \lambda_- > 0$ and $\alpha \in (0,2)$.
- We approximate $\mathbb{E}[X_t^2]$, where X_t is the solution of

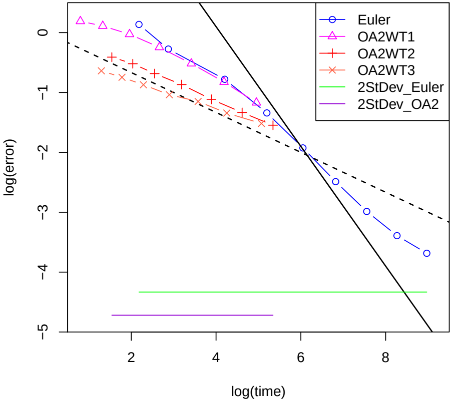
$$dX_t = h(X_t)\{\sigma dB_t + dZ_t\},$$

  whith $h(x) = x$. The exact solution is available.
- Euler scheme difficult to implement. Poirot-Tankov (2006) approach.
- Jump times of the truncated Poisson measure are easy to simulate.

Error vs Computation Time

- Optimal comp. Poisson approx. $\mathbf{n} = \mathbf{2}$ + Weak Taylor (order $1, 2$ and 3)

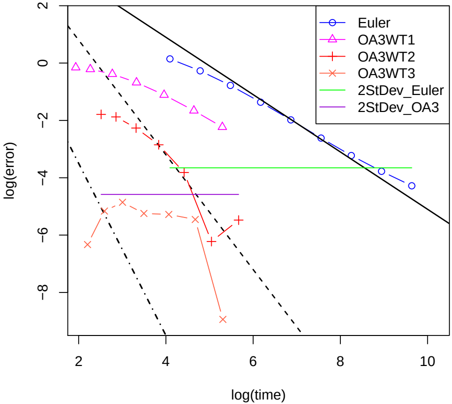


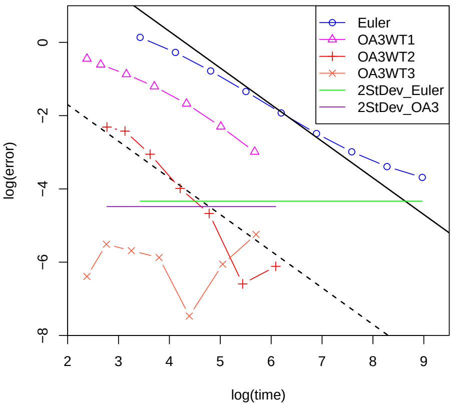


- **Left**: $C = 0.5, \alpha = 0.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3. (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-3})$
- **Right**: $C = 0.1, \alpha = 1.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3. (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-\frac{1}{3}})$

Error vs Computation Time

- Optimal comp. Poisson approx. $\mathbf{n = 3}$ + Weak Taylor (order 1, 2 and 3)

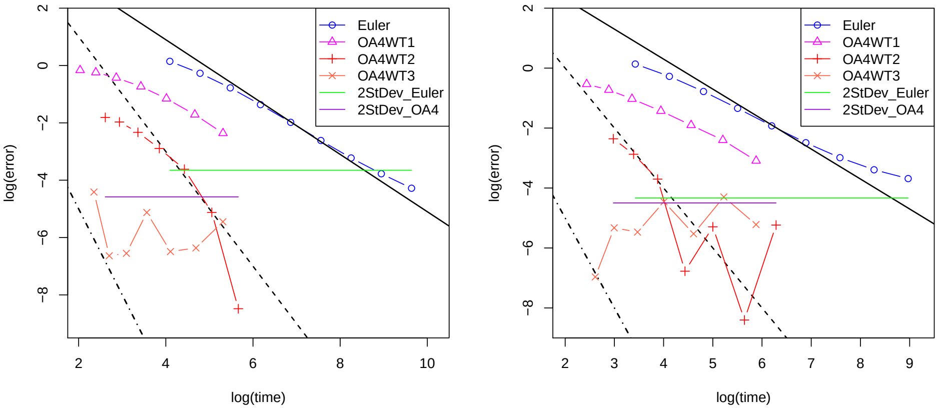




- **Left**: $C = 0.5, \alpha = 0.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3 (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-5})$
- **Right**: $C = 0.1, \alpha = 1.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3 (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-1})$

Error vs Computation Time

- Optimal comp. Poisson approx. $\mathbf{n} = \mathbf{4}$ + Weak Taylor (order 1, 2 and 3)



- **Left**: $C = 0.5, \alpha = 0.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3 (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-7})$
- **Right**: $C = 0.1, \alpha = 1.5, \lambda_+ = 3.5, \lambda_- = 2, \sigma = 0.3 (\mathcal{E}(\Lambda) \sim \Lambda^{1-\frac{n}{\alpha}} = \Lambda^{-\frac{5}{3}})$

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