Convolution Method for BSDEs

Polynice Oyono Ngou
(Joint with Cody B. Hyndman)

Department of Mathematics and Statistics
Université Concordia University

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FBSDE

A forward-backward stochastic differential equation (FBSDE) is a system of the form

$$\begin{cases} dX_t = a(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dW_t \\ -dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t^* dW_t \\ X_0 = x_0, \; Y_T = g(X_T) \end{cases} \tag{1.1}$$

on a (complete) filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where the coefficients a, σ, f and g are appropriate deterministic functions.

- X and Y are adapted and continuous processes with $\mathbf{E}\left[\sup_{t\in[0,T]} |X_t|^2 + \sup_{t\in[0,T]} |Y_t|^2\right] < \infty$.
- Z is an adapted process with $\mathbf{E}\left[\left(\int_0^T |Z_t|^2 dt\right)\right] < \infty$.

Properties

1. Existence and uniqueness (Pardoux and Tang [5]) under Lipschitz and monotonicity conditions.
2. Stability (Pardoux and Tang [5]) allows numerical methods.
3. Relationship to quasi-linear parabolic PDE (Pardoux and Peng [4] and Pardoux and Tang [5]) leads to PDE methods.
4. Path regularity in the decoupled case for the control process Z (Zhang [8]) leads to an error bound for time discretization schemes (Spatial discretization and Monte Carlo methods).

The Euler scheme

Given a solution of the forward process $\{X^{\pi}_{t_i}\}_{i=0}^{n}$ on the time mesh $\pi = \{t_0 = 0 < t_1 < ... < t_n = T\}$, the explicit Euler scheme is defined as (Zhang [8], Bouchard and Touzi [1])

$$\begin{cases} Z^{\pi}_{t_n} = 0, \; Y^{\pi}_{t_n} = \xi^{\pi} \\ Z^{\pi}_{t_i} = \frac{1}{\Delta_i}\mathbf{E}\left[Y^{\pi}_{t_{i+1}}\Delta W_i|\mathcal{F}_{t_i}\right] \\ Y^{\pi}_{t_i} = \mathbf{E}\left[Y^{\pi}_{t_{i+1}} + f(t_i, X^{\pi}_{t_i}, Y^{\pi}_{t_{i+1}}, Z^{\pi}_{t_i})\Delta_i|\mathcal{F}_{t_i}\right] \end{cases} \tag{1.2}$$

where $\Delta_i = t_{i+1} - t_i$. Alternatively, one can take

$$Y^{\pi}_{t_i} = \mathbf{E}\left[Y^{\pi}_{t_{i+1}}|\mathcal{F}_{t_i}\right] + f(t_i, X^{\pi}_{t_i}, \mathbf{E}\left[Y^{\pi}_{t_{i+1}}|\mathcal{F}_{t_i}\right], Z^{\pi}_{t_i})\Delta_i. \tag{1.3}$$

The Euler scheme yields a half ($\frac{1}{2}$) order error (in time).

The solution to the BSDE

$$Y_t = g(W_T) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s^* dW_s \tag{2.1}$$

with $W \in \mathbb{R}^d$, $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$, is given by (Pardoux and Peng [4])

$$Y_t = u(t, W_t) \tag{2.2}$$

$$Z_t = \nabla u(t, W_t). \tag{2.3}$$

where $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ solves

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} + f(t, u, \nabla u) = 0, \ (t, x) \in [0, T) \times \mathbb{R}^d \\ u(T, x) = g(x). \end{cases} \tag{2.4}$$

In the simple case of BSDEs:

- PDE and Monte Carlo based methods are time consuming.
- PDE based methods are mainly built for coupled problems and may be inaccurate for non-smooth drivers.
- The binomial method (Pend and Xu [6]) simulates the BSDE with an approximation of the Wiener process and gives a partial solution to the PDE. There is a contraction of the space grid through time steps!!

The convolution method, and the FFT algorithm solves some of those problems:

- FFT algorithm is efficient with $\mathcal{O}(n\log(n))$ operations given n interpolation points.
- Resolution on a equidistant and flexible space grid that suits simulation.
- The underlying trigonometric interpolation works well for non-smooth functions.

From the explicit Euler scheme

$$\begin{cases} Z^\pi_{t_n} = 0, \; Y^\pi_{t_n} = \xi^\pi \\ Z^\pi_{t_i} = \frac{1}{\Delta_i}\mathbf{E}\left[Y^\pi_{t_{i+1}}\Delta W_i | \mathcal{F}_{t_i}\right] \\ Y^\pi_{t_i} = \mathbf{E}\left[Y^\pi_{t_{i+1}} | \mathcal{F}_{t_i}\right] + f(t_i, \mathbf{E}\left[Y^\pi_{t_{i+1}} | \mathcal{F}_{t_i}\right], Z^\pi_{t_i})\Delta_i \end{cases} \tag{2.5}$$

on the time mesh $\pi = \{t_0 = 0 < t_1 < ... < t_n = T\}$, we define the approximate solution u_i and the approximate gradient $\dot{u}_i$ as

$$u_i(x) = \tilde{u}_i(x) + \Delta_i f(t_i, \tilde{u}_i(x), \dot{u}_i(x)) \tag{2.6}$$

$$\dot{u}_i(x) = \frac{1}{\Delta_i}\int_{-\infty}^{\infty}(y-x)u_{i+1}(y)h(y-x)dy \tag{2.7}$$

for $i = 0, 1, ..., n-1$, where

$$\tilde{u}_i(x) = \int_{-\infty}^{\infty} u_{i+1}(y)h(y-x)dy \tag{2.8}$$

and $u_n(x) = g(x)$. The function h is the Gaussian density.

For any $\alpha \in \mathbb{R}$ and any real function η

1. we define $\eta^{\alpha}(x) = e^{-\alpha x}\eta(x)$.
2. $\mathfrak{F}[\eta](\nu) = \int_{-\infty}^{\infty} e^{-\mathbf{i}\nu x}\eta(x)dx$ is the Fourier transform of η and $\mathfrak{F}^{-1}$ is the inverse Fourier operator.

Then, the convolution theorem leads to

$$\tilde{u}_i(x) = e^{\alpha x}\mathfrak{F}^{-1}\left[\mathfrak{F}[u^{\alpha}_{i+1}](\nu)\phi(\nu - \mathbf{i}\alpha)\right](x) \quad (2.9)$$

$$\dot{u}_i(x) = e^{\alpha x}\mathfrak{F}^{-1}\left[(\alpha + \mathbf{i}\nu)\mathfrak{F}[u^{\alpha}_{i+1}](\nu)\phi(\nu - \mathbf{i}\alpha)\right](x) \quad (2.10)$$

where

$$\phi(\nu) = \exp\left(-\frac{1}{2}\Delta_i \nu^2\right). \quad (2.11)$$

- The expressions of equations (2.9) and (2.10) are identical in the multidimensional setting.
- Lord *et* al. [3] use a very similar approach in the context of American option pricing under Lévy processes.

The convolution method sums up in computing values of the form

$$\theta(x) = \mathfrak{F}^{-1}\left[\mathfrak{F}[\eta^{\alpha}](\nu)\psi(\nu - \mathbf{i}\alpha)\right](x) \qquad (2.12)$$

for some real valued function η and some complex function ψ.

- We solve on the restricted real interval $[x_0, x_N]$ with an even number N of nodes

$$x_j = x_0 + j\Delta x \ , \ j = 1, ..., N \text{ and } \Delta x = \frac{x_N - x_0}{N}. \qquad (2.13)$$

- The Fourier space is discretized on $[-\frac{L}{2}, \frac{L}{2}]$ with nodes

$$\nu_i = \nu_0 + i\Delta\nu \ , \ j = 1, ..., N \text{ and } \nu_0 = -\frac{L}{2}. \qquad (2.14)$$

- The Nyquist relation imposes $\Delta\nu \cdot \Delta x = \frac{2\pi}{N}$.
- Assumptions : $\eta^{\alpha}(x_0) = \eta^{\alpha}(x_N)$ and $\frac{\partial\eta^{\alpha}}{\partial x}(x_0) = \frac{\partial\eta^{\alpha}}{\partial x}(x_N)$.

Applying lower Riemann sums on the inverse Fourier transform integral and any classical quadrature rule with weights $\{w_i\}_{i=0}^{N}$ on the Fourier transform integral gives

$$\theta(x_k) = (-1)^k \mathfrak{D}^{-1}\left[\left\{\psi(\nu_j)\mathfrak{D}\left[\{(-1)^i \tilde{w}_i \eta^\alpha(x_i)\}_{i=0}^{N-1}\right]_j\right\}_{j=0}^{N-1}\right]_k$$
$$\text{for } k = 0, 1, ..., N-1 \text{ and } \theta(x_N) = \theta(x_0) \qquad (2.15)$$

where $\tilde{w}_0 = w_0 + w_N$ and $\tilde{w}_i = w_i$ if $i \neq 0$. For any set $\{x_j\}_{j=0}^{N-1}$ of numbers

$$\mathfrak{D}[\{x_j\}_{j=0}^{N-1}]_k = \frac{1}{N}\sum_{j=0}^{N-1} e^{-\mathbf{i}jk\frac{2\pi}{N}} x_j \qquad (2.16)$$

is the discrete Fourier transform (DFT) and

$$\mathfrak{D}^{-1}[\{x_j\}_{j=0}^{N-1}]_k = \sum_{j=0}^{N-1} e^{\mathbf{i}jk\frac{2\pi}{N}} x_j. \qquad (2.17)$$

If the generic function η^α does not satisfy the value and derivative assumptions, then we consider the transformation

$$\eta^\alpha_{\beta,\kappa}(x) = e^{-\alpha x}(\eta(x) + \beta x + \kappa) \tag{2.18}$$

which satisfies the conditions for optimal values of α, β and κ.
The transformation leads to

$$\begin{aligned} \theta(x) &= \mathfrak{F}^{-1}\left[\mathfrak{F}[\eta^\alpha](\nu)\psi(\nu - \mathbf{i}\alpha)\right](x) \\ &= \mathfrak{F}^{-1}\left[\mathfrak{F}[\eta^\alpha_{\beta,\kappa}](\nu)\psi(\nu - \mathbf{i}\alpha)\right](x) - H(x,\alpha,\beta,\kappa). \end{aligned} \tag{2.19}$$

We have:

- $H(x,\alpha,\beta,\kappa) = e^{-\alpha x}(\beta x + \kappa)$ if $\psi(\nu) = \phi(\nu)$.
- $H(x,\alpha,\beta,\kappa) = e^{-\alpha x}\beta$ if $\psi(\nu) = \mathbf{i}\nu\phi(\nu)$.

Time discretization

- The Euler scheme time discretization error is known to the half order. Zhang [8], Bouchard and Touzi [1].
- Using the usual *ansatz* of $u = e^{\mathbf{i}k\Delta x}$, for a space step Δx and a maximal time step of $|\pi|$, stability occurs if

$$|\pi| \sup_{t \in [0,T]} |f(t,0,0)| \leq 1 \tag{3.1}$$

when using the trapezoidal quadrature rule i.e with weights $w_0 = w_N = \frac{1}{2}$ and $w_i = 1,\ i = 1, 2, ..., N-1$.

Space discretization

We need smoothness for the BSDE coefficients (driver f and terminal condition g) to develop an error bound.
Existing results (under the trapezoidal rule, see Plato [7]):

1. The DFT computes Fourier coefficients with a second order $\mathcal{O}(\Delta x^2)$ accuracy.
2. The inverse DFT then recovers the function values with a global error of $\mathcal{O}(\Delta x^{\frac{3}{2}})$.

These rates of accuracy are improved if the quadrature rule is of a higher order and the coefficient f and g have the appropriate smoothness.

The (1-D) reflected BSDE

$$Y_t = g(W_T) + \int_t^T f(s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s \quad (3.2)$$

admits the triple solution (Y, Z, A) where $Y_t \geq B(t, W_t)$ for a lower barrier function $B : [0, T] \times \mathbb{R} \to \mathbb{R}$ and A is a continuous and increasing process such that $\int_0^T (Y_t - B(t, W_t))dA_t = 0$.
We have that

$$\begin{aligned} Y_t &= u(t, W_t) && (3.3) \\ Z_t &= \nabla u(t, W_t) && (3.4) \end{aligned}$$

where $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ solves a parabolic PDE with obstacle.

Starting from the Euler scheme, we have the numerical solution

$$u_i(x) = \tilde{u}_i(x) + \Delta_i f(t_i, \tilde{u}_i(x), \dot{u}_i(x)) + \Delta \bar{u}_i(x) \quad (3.5)$$

$$\dot{u}_i(x) = \frac{1}{\Delta_i} \int_{-\infty}^{\infty} (y - x) u_{i+1}(y) h(y - x) dy \quad (3.6)$$

$$\Delta \bar{u}_i(x) = [\tilde{u}_i(x) + \Delta_i f(t_i, \tilde{u}_i(x), \bar{u}_i(x)) - B(t_i, x)]^- \quad (3.7)$$

where

$$\tilde{u}_i(x) = \int_{-\infty}^{\infty} u_{i+1}(y) h(y - x) dy. \quad (3.8)$$

The conditional expectations of equation (3.6) and (3.8) are computed with the convolution method.

Other extensions

- The methods can be applied given any explicit scheme for (R)BSDEs: Euler scheme or the $\theta-$schemes of Zhao, Shen and Peng [9].
- $\theta-$schemes allow to enhance the time discretization error.
- An arithmetic Brownian motion $X_t = \mu t + \sigma W_t$ can be considered as the forward process. One just needs to adjust for the characteristic function.

We price an at-the-money American call option with one year maturity $T = 1$ on the stocks $S_t = e^{X_t}$ with return process X_t

$$dX_t = (\mu - \delta - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \tag{4.1}$$

We take an initial stock value of $S_0 = K = 100$, an expected return of $\mu = 0.05$, a volatility of $\sigma = 0.2$ and a dividend rate δ.
The option price solves a reflected BSDE with driver (El Karoui et al. [2])

$$f(t, y, z) = -ry - \left(\frac{\mu - r}{\sigma}\right) z + (R - r)\left(y - \frac{z}{\sigma}\right)^{-} \tag{4.2}$$

where $r = 0.01$ is the lending rate and R is the borrowing rate. The terminal condition is

$$g(x) = (e^x - K)^+ \tag{4.3}$$

and the barrier is given by $B(t, x) = g(x)$.

We set $\delta = 0$ and $R = r$:

- The European and American call options have the same price.
- The Black-Scholes formula and the convolution method return an option price of 8.433 and an option delta of 0.560.
- We use $n = 500$ time steps, $N = 2^{12}$ space steps on the restricted domain $[x_0, x_N] = X_0 + [-5, 5]$ for the convolution method.

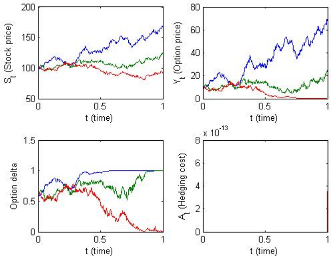
Table 4.1: American call option prices.

K (Strike)	n=500	n=1000	n=2000	n=5000
110	4.6097	4.6090	4.6100	4.6101
100	8.4328	8.4331	8.4332	8.4332
90	14.1925	14.1927	14.1928	14.1929

We set $\delta = 0$ and $R = 0.03$:

- The European and American call options have the same price but Black-Scholes formula doesn't apply.
- The convolution method return an option price of 9.413 and an option delta of 0.600.

Figure 4.1: Paths simulation for the American option



We set $\delta = 0.035$ and $R = 0.03$, the convolution method returns an option price of 7.561 and an option delta of 0.521.

Figure 4.2: Path simulation for the American option on dividend paying stock

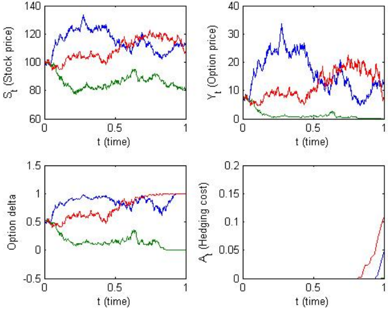
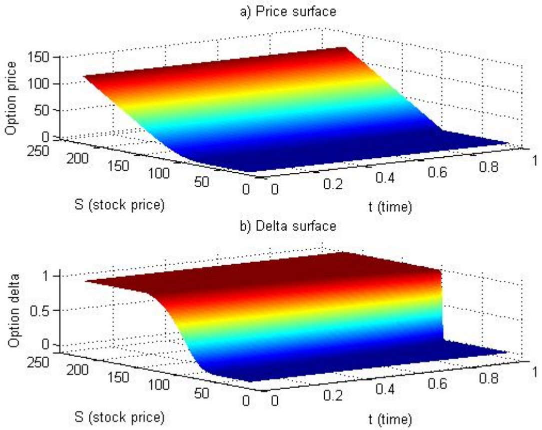


Figure 4.3: American option (dividend paying stock) price surface



- An explicit Euler scheme was used to develop a convolution method for BSDEs.
- The conditional expectations are computed with the FFT algorithm.
- We introduced a transformation that allows to take into account non-periodic problem.
- Reflected BSDEs were also considered.
- Error analysis and numerical examples in non-smooth and non-linear cases shows that the method is accurate.

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Thank You!!!