QUADRATIC 2BSDEs AND APPLICATIONS

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joint work with Anis Matoussi and Chao Zhou

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Outline

Standard BSDEs

$(\Omega, \mathcal{F}, \mathbb{P})$, W Brownian motion, $\{\mathcal{F}_t, t \geq 0\}$ corresponding filtration. Pardoux and Peng introduced the BSDE :

$$Y_t = \xi - \int_t^T F_t(Y_t, Z_t)dt + \int_t^T Z_t dW_t$$

and proved that for

$$\xi \in \mathbb{L}^2(\mathbb{P}), \quad F \text{ unif. Lipschitz in } (y, z) \quad \text{and } F_\cdot(0,0) \in \mathbb{H}^2$$

there is a unique solution $(Y, Z) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$:

$$\|Y\|_{\mathbb{D}^2} := \mathbb{E}\Big[\sup_{t\in[t,T]} |Y_t|^2\Big] \quad \text{and} \quad \|Z\|_{\mathbb{H}^2} := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]$$

BSDEs and semilinear PDES

The Markov case corresponds to

$$F_t(\omega, y, z) = f(t, X_t(\omega), y, z) \quad \text{and} \quad \xi(\omega) = g\big(X_T(\omega)\big)$$

where $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$

In this context, under the same conditions as before, we have

$$Y_t \quad = \quad V(t, X_t)$$

Moreover, if $V \in C^{1,2}$, then V is a classical solution of the semilinear PDE

$$\partial_t V + b \cdot DV + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^{\mathrm{T}} D^2 V\big] \quad = \quad f\big(., V, \sigma^{\mathrm{T}} DV\big)$$

Extension to the second order

- Cheridito, Soner, Touzi and Victoir 2007
- L. Denis and C. Martini 2006 : Quasi-sure analysis
- Peng 2007 : $G-$Brownian motion
- M. Soner, N. Touzi and J. Zhang (2010a,2010b,2010c,2010d)
- Nutz 2010, Kervarec Bion-Nadal 2010, Kervarec Denis (2010), Nutz, Soner 2011, DP 2011, DP, Matoussi, Zhou 2011...

Intuition from PDEs

Let V be a solution of

$$-\partial_t V - H(., V, DV, D^2 V) = 0 \quad \text{and} \quad V(T,.) = g$$

and suppose

$$H(x, r, p, \gamma) \quad = \quad \sup_{a \geq 0} \left\{ \frac{1}{2} a\gamma - F(x, r, p, a) \right\}$$

Then $V = \sup_a V^a$ where V^a is a solution of

$$-\partial_t V^a - \frac{1}{2} a D^2 V + F(., V^a, DV^a, a) = 0 \quad \text{and} \quad V^a(T,.) = g$$

a semilinear PDE which corresponds to a BSDE.

Link with the Quasi-sure stochastic analysis

This suggest to introduce

$$" Y_t = \sup_a \mathcal{Y}_t^a "$$

$$\mathcal{Y}_t^a = g(X_T^a) + \int_t^T f(., X_s^a, \mathcal{Y}_s^a, \mathcal{Z}_s^a, a_s)ds - \int_t^T \mathcal{Z}_s^a dX_s^a,$$

where $dX_s^a = a_s^{\frac{1}{2}} dW_s$.

This is similar to stochastic control theory, since we end up with a family of processes $\{\mathcal{Y}^a\}$. Then, changing a amounts to changing the underlying probability measure.

Nondominated family of measures on canonical space (dimension 1 for simplification)

$\Omega := C([0,T],\mathbb{R})$, B : coordinate process, $\mathbb{P}_0$: Wiener measure
$\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$: filtration generated by B, $\widehat{a}_t$ density of $< B >_t$, defined pathwise.

For every positive and integrable α, define

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,T], \mathbb{P}_0 - \text{a.s.}$$

$\overline{\mathcal{P}}_S$: collection of all such $\mathbb{P}^\alpha$

Then every $\mathbb{P} \in \overline{\mathcal{P}}_S$

- satisfies the Blumenthal zero-one law
- and the martingale representation property

Generator $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times D_H \to \mathbb{R}$

- Convex conjugate :

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a\gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{R}_+^*;$$
$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

Assumption : The domain of F is independent of (ω, y, z) and F is uniformly continuous in ω.

We assume for simplicity that $\hat{F}^0$ is bounded, and then we consider

$$\mathcal{P}_H = \left\{ \mathbb{P} \in \overline{\mathcal{P}}_S : \hat{a}, \hat{a}^{-1} \text{ bdd and } \hat{a} \in \mathrm{Dom(F)} \right\}$$

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Definition (Denis-Martini 06)

$\mathcal{P}_H$−q.s. means $\mathbb{P}$−a.s. for all $\mathbb{P} \in \mathcal{P}_H$

We introduce the following norms and spaces

$$\|Y\|^2_{\mathbb{D}^2_H} := \sup_{\mathbb{P}\in\mathcal{P}^H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big], \ \|Z\|^2_{\mathbb{H}^2_H} := \sup_{\mathbb{P}\in\mathcal{P}^H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\hat{a}_s^{1/2} Z_s\right|^2 ds\right].$$

We also define $\mathcal{L}^2_H$ as the closure of $UC_b(\Omega)$ under the norm $\mathbb{L}^2_H$

$$\|\xi\|_{\mathbb{L}^2_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} \left(\mathcal{E}_t^{\mathbb{P}}\left[|\xi|^\kappa\right]\right)^{\frac{2}{\kappa}}\right],$$

where $\kappa \in (1,2]$ and

$$\mathcal{E}_t^{\mathbb{P}}[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi].$$

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$$\mathcal{E}^{\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[\xi].$$

Definition

For $\mathcal{F}_T-$meas. ξ, consider the 2BSDE :

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies for $t \leq T$

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{ess inf}} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}'}],\ \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H$$

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2BSDEs with quadratic growth

Assumptions

- F is Lipschitz in y uniformly in (t, z, ω, a)

$$\left|F_t(y,z,a) - F_t(y',z,a)\right| \leq C\left|y - y'\right|.$$

- $z \longrightarrow F_t(y,z,a)$ is C^2 with

$$\left|D_z F_t(y,z,a)\right| \leq \theta_0 + \theta_1\left|a^{1/2}z\right| \text{ and } \left|D_{zz}F_t(y,z,a)\right| \leq \theta_1.$$

Theorem (P., Zhou 2011)

For all $\xi \in \mathcal{L}_H^\infty$, the 2BSDE has a unique solution in $\mathbb{D}_H^\infty \times \mathbb{H}_H^2$.

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Outline

The problem

A financial market consists of one bond with interest rate zero and 1 stock. The price process is :

$$dS_t = S_t(b_t dt + dB_t)$$

The wealth process of a trading strategy π with initial capital x satisfies the following equation :

$$X_t^\pi = x + \int_0^t \pi_s(dB_s + b_s ds)\ 0 \leq t \leq T,$$

The problem of the investor is then

$$V(x) := \sup_{\pi \in \tilde{\mathcal{B}}} \inf_{\mathbb{Q} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}[U(X_T^\pi - F)],$$

where $\widetilde{B}$ is some closed set.

The case of exponential utility

$$U(x) = -\exp(-\beta x),\ x \in \mathbb{R} \text{ for } \beta > 0.$$

In order to solve the problem, we follow the general martingale approach introduced by El Karoui and Rouge and generalized by Hu, Imkeller and Müller. We want to construct a family of processes R^π which satisfies

1. $R^\pi_T = \exp(-\beta(X^\pi_T - F))$ for all $\pi \in \tilde{\mathcal{B}}$
2. $R^\pi_0 = R_0$ is constant for all $\pi \in \tilde{\mathcal{B}}$
3. $\underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R^\pi_T] \geq R^\pi_t$ for $\forall \pi \in \tilde{\mathcal{B}}$
4. $R^{\pi^*}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R^{\pi^*}_T]$ for some $\pi^* \in \tilde{\mathcal{B}}$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$

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The case of exponential utility

Definition

A strategy π is admissible if and only if $\pi = (\pi_t)_{0 \leq t \leq T}$ and $\pi_t \in \tilde{B}$, $\lambda \otimes \mathbb{P} - p.s.$ and $\int_0^T \pi_s dB_s$ is in $\mathbb{B}MO(\mathcal{P}_H)$.

Then, we show that we can define

$$R_t^\pi = exp(-\beta(X_t^\pi - Y_t)) \ t \in [0, T], \ \pi \in \tilde{\mathcal{B}}.$$

where $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ the unique solution of the following 2BSDE with quadratic generator :

$$Y_t = F - \int_t^T Z_s dB_s - \int_t^T f(s, Z_s) ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \mathcal{P}_H - a.s.$$

where

$$f(\cdot, z) = -\frac{\beta}{2} dist^2(\hat{a}^{1/2} z + \frac{1}{\beta}\hat{\theta}, \bar{B}(\omega)) + z' \hat{a}^{1/2}\hat{\theta} + \frac{1}{2\beta}\left|\hat{\theta}\right|^2$$

Explicit calculations and examples

When the set of trading strategies is no longer constrained, the 2BSDEs can be solved explicitely, since their generators are linear in y and quadratic in z.

- Power utility and no constraints $\longrightarrow$ value function of the Merton problem with constant volatility equal to the upper bound of the volatility interval. Intuition from the PDE

$$-\frac{\partial v}{\partial t} - \sup_{\delta \in \tilde{A}} \inf_{\alpha \in [\underline{a}, \bar{a}]} \left[x\delta b \frac{\partial v}{\partial x} + \frac{1}{2} x^2 \delta^2 \alpha \frac{\partial^2 v}{\partial x^2} \right] = 0$$

together with the terminal condition
$v(T, x) = U(x), \ x \in \mathbb{R}_+$.

Explicit calculations and examples

- The optimal probability measure is not always of bang-bang type. With exponential utility, no constraints and a liability $\xi = -B_T^2$, depending on b, the optimal probability changes continuously with t in the volatility interval.
- This is a major difference between superreplication and indifference pricing under volatility uncertainty.

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THANK YOU FOR YOUR ATTENTION